Maximally local-edge-connected graphs and digraphs

Angelika Hellwig and Lutz Volkmann [1]

Lehrstuhl II für Mathematik, RWTH Aachen, 52056 Aachen, Germany
e-mail: volkm@math2.rwth-aachen.de

Abstract

The *local-edge-connectivity* $\lambda(u,v)$ of two vertices u and v in a graph or digraph D is the maximum number of edge-disjoint u-v paths in D, and the *edge-connectivity* of D is defined as $\lambda(D) = \min\{\lambda(u,v) \mid u,v \in V(D)\}$. Clearly, $\lambda(u,v) \leq \min\{d^+(u), d^-(v)\}$ for all pairs u and v of vertices in D. We call a graph or digraph D *maximally local-edge-connected* when

$$\lambda(u,v) = \min\{d^+(u), d^-(v)\}$$

for all pairs u and v of vertices in D.

Recently, Fricke, Oellermann, and Swart have shown that some known sufficient conditions that guarantee equality of $\lambda(G)$ and minimum degree $\delta(G)$ for a graph G are also guarantee that G is maximally local-edge-connected.

In this paper we extend some results of Fricke, Oellermann, and Swart to digraphs and we present further sufficient conditions for graphs and digraphs to be maximally local-edge-connected.

Keywords: *Local-edge-connectivity; Edge-connectivity; Minimum degree*

[1] Corresponding author

1. Terminology and introduction

We consider finite graphs and digraphs without loops and multiple edges. If v is a vertex of a digraph D, then we denote the sets of *out-neighbors* and *in-neighbors* of v by $N^+(v)$ and $N^-(v)$, respectively. Furthermore, the *degree* $d(v)$ of v is defined as the minimum value of its *out-degree* $d^+(v) = |N^+(v)|$ and its *in-degree* $d^-(v) = |N^-(v)|$. The *local-edge-connectivity* $\lambda(u,v)$ of two vertices u and v in a digraph or graph D is the maximum number of edge-disjoint u-v paths in D, and the *edge-connectivity* of D, denoted by $\lambda(D)$, is defined as $\lambda(D) = \min\{\lambda(u,v) | u,v \in V(D)\}$. Clearly, $\lambda(u,v) \leq \min\{d^+(u), d^-(v)\}$ for all pairs u and v of vertices in D. We call a graph or digraph D *maximally local-edge-connected* when

$$\lambda(u,v) = \min\{d^+(u), d^-(v)\}$$

for all pairs u and v of vertices in D. For two vertex sets X, Y of a digraph or graph let (X,Y) be the set of arcs or edges from X to Y. If D is a digraph (graph) and $X \subseteq V(D)$, then let $D[X]$ be the subdigraph (subgraph) induced by X, and let $\bar{X} = V(D) - X$. For other graph theory terminology we follow Chartrand and Lesniak [3].

Sufficient conditions for equality of edge-connectivity $\lambda(D)$ and minimum degree $\delta(D)$ for a graph and a digraph D were given by several authors, for example: Chartrand [2], Lesniak [13], Plesník [14], Goldsmith and White [11], Bollobás [1], Goldsmith and Entringer [10], Soneoka, Nakada, Imase, and Peyrat [16], Plesník and Znám [15], Volkmann [18], [19], Fàbrega and Fiol [7], [8], Xu [21], Dankelmann and Volkmann [4], [5], [6], and Hellwig and Volkmann [12].

Recently, Fricke, Oellerman, and Swart [9] have shown that some known sufficient conditions that guarantee $\lambda(G) = \delta(G)$ for a graph G also guarantee that G is maximally local-edge-connected. The next observation shows that the results of Fricke, Oellermann, and Swart [9] generalize the corresponding known one.

Oservation 1.1 If a digraph or graph D is maximally local-edge-connected, then $\lambda(D) = \delta(D)$.

Proof. Since D is maximally local-edge-connected, we have $\lambda(u,v) = \min\{d^+(u), d^-(v)\}$ for all pairs u and v of vertices in D. Thus,

$$\lambda(D) = \min_{u,v \in V(D)} \{\lambda(u,v)\} = \min_{u,v \in V(D)} \{\min\{d^+(u), d^-(v)\}\} = \delta(D). \qquad \square$$

In this paper we extend some results of Fricke, Oellermann, and Swart [9] to digraphs, and we present further sufficient conditions for graphs and digraphs to be maximally local-edge-connected.

Our proofs are based on the following simple and well-known lemma.

Lemma 1.2 Let u and v be a pair of vertices in the digraph or graph D. Then $\lambda(u,v) \geq q$ if and only if $|(S,\bar{S})| \geq q$ for all subsets $S \subset V(D)$ such that $u \in S$ and $v \in \bar{S}$.

2. Main results

Theorem 2.1 If D is a digraph with diameter at most two, then

$$\lambda(u,v) = \min\{d^+(u), d^-(v)\}$$

for all pairs u and v of vertices in D.

Proof. Let u and v be any two vertices of D. As noted above, $\lambda(u,v) \leq \min\{d^+(u), d^-(v)\}$. Next we will show that $|(S,\bar{S})| \geq \min\{d^+(u), d^-(v)\}$ for all $S \subset V(D)$ such that $u \in S$ and $v \in \bar{S}$. Let S be such a set.

Case 1. $|N^+(x) \cap \bar{S}| \geq 1$ for all $x \in S$. This implies

$$\begin{aligned} \min\{d^+(u), d^-(v)\} &\leq |N^+(u)| = |N^+(u) \cap \bar{S}| + |N^+(u) \cap S| \\ &\leq |N^+(u) \cap \bar{S}| + \sum_{x \in N^+(u) \cap S} |N^+(x) \cap \bar{S}| \\ &\leq \sum_{x \in S} |N^+(x) \cap \bar{S}| = |(S,\bar{S})|. \end{aligned}$$

Case 2. There exists a vertex $x \in S$ with $|N^+(x) \cap \bar{S}| = 0$. Since the diameter of D is at most two, it follows that $|N^-(y) \cap S| \geq 1$ for all $y \in \bar{S}$. This leads to

$$\begin{aligned} \min\{d^+(u), d^-(v)\} &\leq |N^-(v)| = |N^-(v) \cap S| + |N^-(v) \cap \bar{S}| \\ &\leq |N^-(v) \cap S| + \sum_{y \in N^-(v) \cap \bar{S}} |N^-(y) \cap S| \\ &\leq \sum_{y \in \bar{S}} |N^-(y) \cap S| = |(S,\bar{S})|. \end{aligned}$$

Now it follows from Lemma 1.2 that $\lambda(u,v) \geq \min\{d^+(u), d^-(v)\}$, and the proof of Theorem 2.1 is complete. □.

Corollary 2.2 (Fricke, Oellermann, Swart [9] 2000) If G is a graph with diameter at most two, then $\lambda(u, v) = \min\{d(u), d(v)\}$ for all pairs u and v of vertices in G.

Proof. Define the digraph D on the vertex set $V(G)$ by replacing each edge of G by two arcs in opposite directions and apply Theorem 2.1. □

According to Observation 1.1, Theorem 2.1 also leads immediately to the following sufficient condition for equality of edge-connectivity and minimum degree.

Corollary 2.3 If D is a digraph of diameter at most two, then $\lambda(D) = \delta(D)$.

Corollary 2.4 (Plesník [14] 1975) If G is a graph of diameter at most two, then $\lambda(G) = \delta(G)$.

Corollary 2.5 Let D be a digraph of order n. If $d^+(x) + d^-(y) \geq n - 1$ for all pairs of nonadjacent vertices x and y, then $\lambda(D) = \delta(D)$.

Corollary 2.6 (Lesniak [13] 1974) Let G be a graph of order n. If $d(x) + d(y) \geq n - 1$ for all pairs of nonadjacent vertices x and y, then $\lambda(G) = \delta(G)$.

Corollary 2.7 Let D be a digraph of order n. If $n \leq 2\delta(D) + 1$, then $\lambda(D) = \delta(D)$.

Corollary 2.8 (Chartrand [2] 1966) Let G be a graph of order n. If $n \leq 2\delta(G) + 1$, then $\lambda(G) = \delta(G)$.

Recently, Fricke, Oellermann, and Swart [9] presented the following generalization of a 1989 result of Volkmann [19].

Theorem 2.9 (Fricke, Oellermann, Swart [9] 2000) Let G be a p-partite graph of order n and minimum degree δ with $p \geq 2$. If

$$n \leq 2\left\lfloor \frac{p\delta}{p-1} \right\rfloor - 1,$$

then $\lambda(u, v) = \min\{d(u), d(v)\}$ for all pairs u and v of vertices in G.

The next two theorems are improvements of Theorem 2.9, and their proofs are a little bit shorter than that of Fricke, Oellermann, and Swart [9].

Theorem 2.10 Let D be a p-partite digraph of order n and minimum degree δ with $p \geq 2$. If

$$n \leq 2\left\lfloor \frac{p\delta}{p-1} \right\rfloor - 1,$$

then $\lambda(u,v) = \min\{d^+(u), d^-(v)\}$ for all pairs u and v of vertices in D.

Proof. Let u and v be any two vertices of D. As observed earlier, $\lambda(u,v) \leq \min\{d^+(u), d^-(v)\}$. In view of Lemma 1.2, it is enough to show that $|(S,\bar{S})| \geq \min\{d^+(u), d^-(v)\}$ for all $S \subset V(D)$ such that $u \in S$ and $v \in \bar{S}$.

Case 1. Let $|S| \leq n/2$. Then, the hypothesis implies

$$1 \leq |S| \leq \left\lfloor \frac{p\delta}{p-1} \right\rfloor - 1 \leq \frac{p\delta}{p-1} - 1. \tag{1}$$

Since D is p-partite, the well known Theorem of Turán [17] (see also [20], p. 212) yields $|E(D[S])| \leq |S|^2(p-1)/p$, and hence we have

$$|(S,\bar{S})| \geq \sum_{y\in S} d^+(y) - \frac{p-1}{p}|S|^2 = \sum_{y\in S}(d^+(y) - \delta) + |S|\delta - \frac{p-1}{p}|S|^2. \tag{2}$$

If we define $g(x) = -x^2(p-1)/p + \delta x$, then, because of (1), we have to determine the minimum of g in the interval $I : 1 \leq x \leq p\delta/(p-1) - 1$. It is easy to see that

$$\min_{x\in I}\{g(x)\} = g(1) = g\left(\frac{p}{p-1}\delta - 1\right) = \delta - \frac{p-1}{p}.$$

This leads together with (2) to

$$|(S,\bar{S})| \geq \sum_{y\in S}(d^+(y)-\delta)+\delta-\frac{p-1}{p} \geq d^+(u)-\delta+\delta-\frac{p-1}{p} = d^+(u)-\frac{p-1}{p},$$

which yields the desired inequality $|(S,\bar{S})| \geq d^+(u) \geq \min\{d^+(u), d^-(v)\}$.

Case 2. Let $|\bar{S}| \leq n/2$. Analogously to Case 1, we then obtain $|(S,\bar{S})| \geq d^-(v) \geq \min\{d^+(u), d^-(v)\}$, and the proof of Theorem 2.10 is complete. □

Corollary 2.11 (Volkmann [19] 1989) Let D be a p-partite digraph of order n and minimum degree δ with $p \geq 2$. If $n \leq 2\lfloor p\delta/(p-1)\rfloor - 1$, then $\lambda(D) = \delta$.

Analogously to Theorem 2.10, one can prove the following common generalization of a result by Dankelmann and Volkmann [4] and Theorem 2.9.

Theorem 2.12 Let G be a graph of order n and minimum degree δ without a complete subgraph of order $p+1$. If

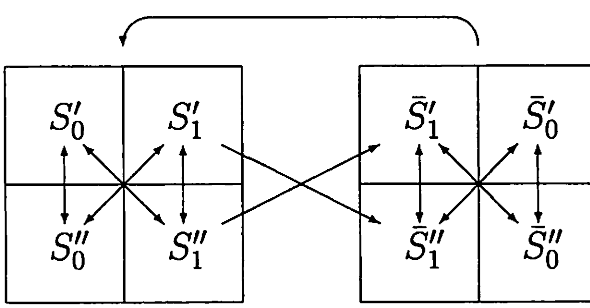
$$n \leq 2\left\lfloor \frac{p\delta}{p-1} \right\rfloor - 1,$$

then $\lambda(u,v) = \min\{d(u), d(v)\}$ for all pairs u and v of vertices in G.

Theorem 2.13 Let D be a bipartite digraph of order n and minimum degree $\delta \geq 2$ with the bipartition $V' \cup V''$. If $d(x) + d(y) \geq (n+1)/2$ for each pair of vertices $x, y \in V'$ and each pair of vertices $x, y \in V''$, then $\lambda(u,v) = \min\{d^+(u), d^-(v)\}$ for all pairs u and v of vertices in D.

Proof. If u and v are two vertices of D, then $\lambda(u,v) \leq \min\{d^+(u), d^-(v)\}$. Next we will show that $|(S, \bar{S})| \geq \min\{d^+(u), d^-(v)\}$ for all $S \subset V(D)$ such that $u \in S$ and $v \in \bar{S}$. Let S be such a set.

Now let $S_1 \subseteq S$ and $\bar{S_1} \subseteq \bar{S}$ be the set of vertices incident with an arc of $(S, \bar{S})$ and define $S_0 = S - S_1$ and $\bar{S_0} = \bar{S} - \bar{S_1}$. In addition, let $S'_0 = S_0 \cap V'$, $S'_1 = S_1 \cap V'$, $S''_0 = S_0 \cap V''$, $S''_1 = S_1 \cap V''$, $\bar{S}'_0 = \bar{S_0} \cap V'$, $\bar{S}'_1 = \bar{S_1} \cap V'$, $\bar{S}''_0 = \bar{S_0} \cap V''$, and $\bar{S}''_1 = \bar{S_1} \cap V''$ (see the figure).



Clearly, $|S_1|, |\bar{S_1}| \leq |(S, \bar{S})|$. We assume, without loss of generality, that $|V'| \leq |V''|$ and so $|V'| \leq n/2$.

It seems likely that there is a symmetry between S and $\bar{S}$; indeed, this is easily observed later. Consequently we may proceed under the assumption that $|S| \leq n/2$.

Firstly, we show that $\min\{|S'_0|, |S''_0|\} \leq 1$ and $\min\{|S''_0|, |\bar{S}''_0|\} \leq 1$.

Suppose that $|S'_0|, |S''_0| \geq 2$. Then it follows from the hypothesis that

$$|S''_0| + |S''_1| \geq |N^+(S'_0)| \geq \frac{n+1}{4} \quad \text{and} \quad |S'_0| + |S'_1| \geq |N^+(S''_0)| \geq \frac{n+1}{4}.$$

This leads to the contradiction $n/2 \geq |S| = |S_0'| + |S_1'| + |S_0''| + |S_1''| \geq (n+1)/2$.

Next suppose that $|S_0''|, |\bar{S_0''}| \geq 2$. Then it follows from the hypothesis that

$$|S_0'| + |S_1'| \geq |N^+(S_0'')| \geq \frac{n+1}{4} \quad \text{and} \quad |\bar{S_0'}| + |\bar{S_1'}| \geq |N^+(\bar{S_0''})| \geq \frac{n+1}{4}.$$

This leads to the contradiction $n/2 \geq |V'| = |S_0'| + |S_1'| + |\bar{S_0'}| + |\bar{S_1'}| \geq (n+1)/2$.

Thus, it remains to investigate the three cases $|S_0''| = 0$, $|S_0''| = 1$, and $|S_0''| \geq 2$ but $|S_0'| \leq 1$ and $|\bar{S_0''}| \leq 1$.

Case 1. Let $|S_0''| = 0$.

Subcase 1.1. Let $u \in S_0'$.

Then $N^+(u) \subseteq S_1''$, and so $|(S, \bar{S})| \geq d^+(u) \geq \min\{d^+(u), d^-(v)\}$.

Subcase 1.2. Let $u \in S_1'$.

Then $N^+(u) \subseteq S_1'' \cup \bar{S_1''}$, and so $|(S, \bar{S})| \geq d^+(u) \geq \min\{d^+(u), d^-(v)\}$.

Subcase 1.3. Let $u \in S_1''$.

If $|N^+(u) \cap S_0'| = 0$, or $|N^+(u) \cap S_0'| = 1$ and $|S_1''| \geq 2$, or $2 \leq |N^+(u) \cap S_0'| \leq |S_1''| - 1$, then it is a simple matter to verify that $|(S, \bar{S})| \geq d^+(u) \geq \min\{d^+(u), d^-(v)\}$.

If $|N^+(u) \cap S_0'| = 1$ and $|S_1''| \leq 1$, we contradict $\delta \geq 2$.

In the remaining case, $|N^+(u) \cap S_0'| \geq 2$ and $|N^+(u) \cap S_0'| \geq |S_1''|$, it follows from the hypothesis that $|S_1''| \geq (n+1)/4$ and therefore $|S_0'| \geq (n+1)/4$, a contradiction to $|S| \leq n/2$.

Case 2. Let $|S_0''| = 1$.

Subcase 2.1. Let $u \in S_0'$.

If $|N^+(u) \cap S_0''| = 0$, or $|N^+(u) \cap S_0''| = 1$ and $|S_1'| \geq 1$, or $|N^+(u) \cap S_0''| = 1$, $|S_1'| = 0$, and there exists a vertex $w \in S_1''$ such that $|N^+(w) \cap \bar{S_1'}| \geq 2$, or $|N^+(u) \cap S_0''| = 1$, $|S_1'| = 0$, $|N^+(x) \cap \bar{S_1'}| = 1$ for all $x \in S_1''$, and there exists a vertex $y \in S_1''$ such that $y \notin N^+(u)$, then it is a simple matter to verify that $|(S, \bar{S})| \geq d^+(u) \geq \min\{d^+(u), d^-(v)\}$.

There remains the case that $|N^+(u) \cap S_0''| = 1$, $|S_1'| = 0$, $|N^+(x) \cap \bar{S_1'}| = 1$ for all $x \in S_1''$, and $S_1'' - N^+(u) = \emptyset$.

Because of $\delta \geq 2$, we deduce that $|S_0'| \geq 2$, and hence the hypothesis implies $|S_1''| \geq (n+1)/4 - 1$. The same argument for $a \in S_0''$ and $b \in S_1''$ together with $|N^+(b) \cap \bar{S_1'}| = 1$ leads to $|S_0'| \geq (n+1)/4 - 1/2$. The assumption $|S| \leq n/2$ yields

$$|S_1''| = \frac{n+1}{4} - 1 \quad \text{and} \quad |S_0'| = \frac{n+1}{4} - \frac{1}{2},$$

however, since $|S_1''|$ and $|S_0'|$ are integers, this is impossible.

Subcase 2.2. Let $u \in S_0''$.

If $|N^+(u) \cap S_0'| \leq |S_1''|$, then $|(S, \bar{S})| \geq d^+(u) \geq \min\{d^+(u), d^-(v)\}$. In the remaining case, $|N^+(u) \cap S_0'| \geq |S_1''| + 1 \geq 2$, it follows by the hypothesis that $|S_1''| \geq (n+1)/4 - 1$ and so $|S_0'| \geq |N^+(u) \cap S_0'| \geq (n+1)/4$. This contradicts the assumption $|S| \leq n/2$.

Subcase 2.3. Let $u \in S_1'$.

If $|N^+(u) \cap S_0''| = 0$, or $|N^+(u) \cap S_0''| = 1$ and $|S_1'| \geq 2$, or $|N^+(u) \cap S_0''| = 1$, $|\bar{S}_1'| = 1$, and there exists a vertex $w \in S_1''$ such that $|N^+(w) \cap \bar{S}_1'| \geq 2$, or $|N^+(u) \cap S_0''| = 1$, $|\bar{S}_1'| = 1$, $|N^+(x) \cap \bar{S}_1'| = 1$ for all $x \in S_1''$, and there exists a vertex $y \in S_1''$ such that $y \notin N^+(u)$, then $|(S, \bar{S})| \geq d^+(u) \geq \min\{d^+(u), d^-(v)\}$.

There remains the case that $|N^+(u) \cap S_0''| = 1$, $|S_1'| = 1$, $|N^+(x) \cap \bar{S}_1'| = 1$ for all $x \in S_1''$, and $S_1'' - N^+(u) = \emptyset$.

Because of $\delta \geq 2$ and $|S_1'| = 1$, we observe that $S_1'' \neq \emptyset$. For $a \in S_0''$ and $b \in S_1''$ the hypothesis and $|N^+(b) \cap \bar{S}_1'| = 1$ lead to $|S_0'| \geq (n+1)/4 - 3/2$.

If $|S_0'| \geq 2$, then the hypothesis yields $|S_1''| \geq (n+1)/4 - 1$. Combining this with the assumption $|S| \leq n/2$, we obtain the contradiction

$$|S_1''| = \frac{n+1}{4} - 1 \quad \text{and} \quad |S_0'| = \frac{n+1}{4} - \frac{3}{2}.$$

In the remaining case, $|S_0'| = 1$, the inequality $1 = |S_0'| \geq (n+1)/4 - 3/2$ yields $n \leq 9$. In addition, it follows from $|S| \leq n/2$ and $|S| \geq 4$ that $|S| = 4$ and $n = 8$ or $n = 9$, and thus $|S_1''| = 1$. Consequently, the vertices in S_0' and S_0'' are of degree two. Since by the hypothesis there are no further vertices of degree two, we conclude that $d^+(u) \geq 3$, and so $|\bar{S}_1''| \geq 1$. From $|S_1''| = 1$ and $|N^+(x) \cap \bar{S}_1'| = 1$ for all $x \in S_1''$, we deduce that $|\bar{S}_1'| = 1$. Since there are no further vertices of degree two, we see that $\bar{S}_0' \neq \emptyset$ and hence $|\bar{S}_1'' \cup \bar{S}_0''| \geq 3$. This is a contradiction when $n = 8$. In the case $n = 9$, we obtain $|\bar{S}_0'| = 1$, $\bar{S}_0'' = \emptyset$, $|\bar{S}_1''| = 3$, and thus $|(S, \bar{S})| = 4 \geq d^-(v) \geq \min\{d^+(u), d^-(v)\}$.

Subcase 2.4. Let $u \in S_1''$.

If $|N^+(u) \cap S_0'| \leq |S_1''| - 1$, then $|(S, \bar{S})| \geq d^+(u) \geq \min\{d^+(u), d^-(v)\}$. In the remaining case, $|N^+(u) \cap S_0'| \geq |S_1''|$, we discuss the two cases $|S_0'| = 1$ and $|S_0'| \geq 2$.

If $|S_0'| = 1$, then the assumption $|S_0''| = 1$, leads to $|S_1'| \geq 1$ and thus $|S| \geq 4$ and $n \geq 8$. Furthermore, $1 = |S_0'| \geq |N^+(u) \cap S_0'| \geq |S_1''|$, shows that $|S_1''| = 1$. If there is a vertex $x \in S_1'$ such that $x \notin N^+(u)$ or $|N^+(x) \cap \bar{S}_1''| \geq 2$, then $|(S, \bar{S})| \geq d^+(u) \geq \min\{d^+(u), d^-(v)\}$.

In the remaining case, the hypothesis yields for $x \in S_1'$ and $y \in S_0'$ the inequality $5 \geq d(x) + d(y) \geq (n+1)/2$, and so $n \leq 9$. As above, we obtain the desired result.

If $|S_0'| \geq 2$, then $|S_1''| \geq (n+1)/4 - 1$, and so $|S_0'| \geq |N^+(u) \cap S_0'| \geq |S_1''| \geq (n+1)/4 - 1$. The assumption $|S| \leq n/2$ implies $S_1' = \emptyset$. Furthermore, $|S_1''| \geq 2$, because otherwise the vertices of S_0' are of degree at most two, a contradiction to the hypothesis and $n \geq 8$. Consequently, $|S| \geq 5$ and $n \geq 10$.

If there exists a vertex $x \in S_1'' - \{u\}$ with only one positive neighbor in $\bar{S}_1'$, then for $a \in S_0''$, it follows from the hypothesis that $2|S_0'| + 1 \geq d(x) + d(a) \geq (n+1)/2$, and so $|S_0'| \geq (n+1)/4 - 1/2$. As $|S| \leq n/2$, we obtain the contradiction

$$|S_1''| = \frac{n+1}{4} - 1 \text{ and } |S_0'| = \frac{n+1}{4} - \frac{1}{2}.$$

There remains the case that each vertex $x \in S_1'' - \{u\}$ has at least two positive neighbors in $\bar{S}_1'$. If $2|S_1''| - 2 \geq |S_0'|$, then

$$\begin{aligned} |(S, \bar{S})| &\geq |N^+(u) \cap \bar{S}_1'| + 2(|S_1''| - 1) \\ &\geq |N^+(u) \cap \bar{S}_1'| + |S_0'| \\ &\geq d^+(u) \geq \min\{d^+(u), d^-(v)\}. \end{aligned}$$

If $2|S_1''| - 2 \leq |S_0'| - 1$, then $|S_0'| \geq 2|S_1''| - 1 \geq (n+1)/2 - 3$, and hence

$$\frac{n}{2} \geq |S| = |S_0'| + |S_1''| + 1 \geq \frac{n+1}{2} - 3 + \frac{n+1}{4}.$$

However, this leads to the contradiction $n \leq 9$.

Case 3. Let $|S_0''| \geq 2$, $|S_0'| \leq 1$, and $|\bar{S}_0''| \leq 1$.

Because of $|S_0'| \leq 1$, this case is analogous to the Cases 1. and 2.

The assumption $|S| > n/2$ leads to $|\bar{S}| \leq n/2$. If we consider $d^-(v)$ instead of $d^+(u)$, then the case $|S| > n/2$ can be proved in a similar manner as the case $|S| \leq n/2$. □

Corollary 2.14 Let G be a bipartite graph with the bipartition $V' \cup V''$ of order n and minimum degree $\delta \geq 2$. If $d(x) + d(y) \geq (n+1)/2$ for each pair of vertices $x, y \in V'$ and each pair of vertices $x, y \in V''$, then $\lambda(u, v) = \min\{d(u), d(v)\}$ for all pairs u and v of vertices in G.

Corollary 2.15 (Dankelmann, Volkmann [4] 1995) Let G be a bipartite graph of order n. If $d(x) + d(y) \geq (n+1)/2$ for all nonadjacent vertices x and y in G, then $\lambda(G) = \delta(G)$.

Corollary 2.16 (Volkmann [18] 1988) Let G be a bipartite graph of order n. If $n \leq 4\delta(G) - 1$, then $\lambda(G) = \delta(G)$.

Example 2.17 Let $p \geq 2$ be an integer and let H_1 and H_2 be two copies of the complete bipartite graph $K_{p,p}$ with the bipartitions

$$V(H_1) = \{x_1, x_2, \ldots, x_p\} \cup \{x'_1, x'_2, \ldots, x'_p\}$$

and

$$V(H_2) = \{y_1, y_2, \ldots, y_p\} \cup \{y'_1, y'_2, \ldots, y'_p\}.$$

We define the bipartite graph G as the union of H_1 and H_2 together with the new edges $x_1y_1, x_2y_2, \ldots, x_py_p$. Then, G is of order $n = 4p$, $\delta(G) = p$, and

$$d(x) + d(y) \geq 2p = \lceil (4p+1)/2 \rceil - 1 = \lceil (n+1)/2 \rceil - 1$$

for all pairs x and y of vertices in G. However,

$$\lambda(x_i, y_i) = p < \min\{d(x_i), d(y_i)\} = p + 1$$

for $i = 1, 2, \ldots, p$. Consequently, G is not maximally local-edge-connected.

This example shows that the condition $d(x) + d(y) \geq (n+1)/2$ in Corollary 2.14 as well as in Theorem 2.13 is best possible. The family of graphs in the next example will demonstrate that the condition $\delta \geq 2$ in Theorem 2.13 and Corollary 2.14 are necessary.

Example 2.18 Let $p \geq 2$ be an integer and let H be the complete bipartite graph $K_{p,p-1}$ with the bipartition $V(H) = \{x_1, x_2, \ldots, x_p\} \cup \{y_1, y_2, \ldots, y_{p-1}\}$ and let w be a further vertex. We define the bipartite graph G as the union of H and w together with the new edge wx_1. Then, G is of order $n = 2p$, $\delta(G) = 1$, and

$$d(x) + d(y) \geq p + 1 \geq (n+1)/2$$

for all pairs x and y of vertices which are contained in the same partite set of G. However,

$$\lambda(x_1, y_i) = p - 1 < \min\{d(x_1), d(y_i)\} = p$$

for $i = 1, 2, \ldots, p-1$. Consequently, G is not maximally local-edge-connected.

References

[1] B. Bollobás, On graphs with equal edge-connectivity and minimum degree, *Discrete Math.* **28** (1979), 321-323.

[2] G. Chartrand, A graph-theoretic approach to a communication problem, *SIAM J. Appl. Math.* **14** (1966), 778-781.

[3] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, 3rd Edition, Wadsworth, Belmont, CA, 1996.

[4] P. Dankelmann and L. Volkmann, New sufficient conditions for equality of minimum degree and edge-connectivity, *Ars Combin.* **40** (1995), 270-278.

[5] P. Dankelmann and L. Volkmann, Degree sequence conditions for maximally edge-connected graphs and digraphs, *J. Graph Theory* **26** (1997), 27-34.

[6] P. Dankelmann and L. Volkmann, Degree sequence conditions for maximally edge-connected graphs depending on the clique number, *Discrete Math.* **211** (2000), 217-223.

[7] J. Fàbrega and M.A. Fiol, Maximally connected digraphs, *J. Graph Theory* **13** (1989), 657-668.

[8] J. Fàbrega and M.A. Fiol, Bipartite graphs and digraphs with maximum connectivity. *Discrete Appl. Math.* **69** (1996), 271-279.

[9] G. Fricke, O.R. Oellermann, and H.C. Swart, The edge-connectivity, average edge-connectivity and degree conditions, manuscript (2000).

[10] D.L. Goldsmith and R.C. Entringer, A sufficient condition for equality of edge-connectivity and minimum degree of a graph, *J. Graph Theory* **3** (1979), 251-255.

[11] D.L. Goldsmith and A.T. White, On graphs with equal edge-connectivity and minimum degree, *Discrete Math.* **23** (1978), 31-36.

[12] A. Hellwig and L. Volkmann, Maximally edge-connected digraphs, *Austral. J. Combin.*, to appear.

[13] L. Lesniak, Results on the edge-connectivity of graphs, *Discrete Math.* **8** (1974), 351-354.

[14] J. Plesník, Critical graphs of given diameter, *Acta Fac. Rerum Natur. Univ. Comenian Math.* **30** (1975), 71-93.

[15] J. Plesník and S. Znám, On equality of edge-connectivity and minimum degree of a graph, *Arch. Math.* (Brno) **25** (1989), 19-25.

[16] T. Soneoka, H. Nakada, M. Imase, and C. Peyrat, Sufficient conditions for maximally connected dense graphs, *Discrete Math.* **63** (1978), 53-66.

[17] P. Turán, An extremal problem in graph theory, *Mat. Fiz. Lapok* **48** (1941), 436-452.

[18] L. Volkmann, Bemerkungen zum p-fachen Kantenzusammenhang in Graphen, *An. Univ. Buccuresti Mat.* **37** (1988), 75-79.

[19] L. Volkmann, Edge-connectivity in p-partite graphs, *J. Graph Theory* **13** (1989), 1-6.

[20] L. Volkmann, *Fundamente der Graphentheorie*, Springer-Verlag, Wien New York 1996.

[21] J.-M. Xu, A sufficient condition for equality of arc-connectivity and minimum degree of a digraph, *Discrete Math.* **133** (1994), 315-318.