An Introduction to Maximum Sum Permutations of Graphs

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ABSTRACT: This paper introduces the problem of finding a permutation ϕ on the vertex set $V(G)$ of a graph G such that the sum of the distances from each vertex to its image under ϕ is maximized. We let $\mathcal{S}(G) = \max \sum_{v \in V(G)} d(v, \phi(v))$, where the maximum is taken over all permutations ϕ of $V(G)$. Explicit formulae for several classes of graphs as well as general bounds are presented.

1. Introduction. We assume G is a connected graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. As usual, the distance between two vertices v_i and v_j is denoted by $d(v_i, v_j)$ and equals the minimum number of edges in a v_i-v_j path. Because G is connected, each $d(v_i, v_j)$ is finite. The eccentricity of a vertex v_i is $e(v_i) = \max \{d(v_i, v_j) : 1 \le j \le n\}$, the maximum distance from v_i to another vertex. The neighborhood of a vertex v_i is $N(v_i) = \{v_j : v_iv_j \in E(G)\}$, and the degree of v_i is $\deg(v_i) = |N(v_i)|$ which is the number of vertices at distance one from v_i. The distance of vertex v_i is $d(v_i) = \sum \{d(v_i, v_j) : 1 \le j \le n\}$, the sum of the distances from v to all of the vertices in $V(G)$.

The radius of G is the minimum eccentricity of a vertex, $r(G) = \min \{e(v) : v \in V(G)\}$, and the center of G is the set of vertices of minimum eccentricity, $C(G) = \{v_i \in V(G) : e(v_i) = r(G)\}$. The median of G is $M(G) = \{v_i \in V(G) : d(v_i) \le d(v_j)$ for $1 \le j \le n\}$, the set of vertices of minimum distance.

Here we are interested in permutations $\phi : V(G) \to V(G)$ which maximize the sum of the distances from each vertex to its image under ϕ. Let Π denote the set of all permutations from $V(G)$ onto $V(G)$. We define the maximum sum permutation distance $\mathcal{S}(G)$ to be

$$\mathcal{S}(G) = \max_{\phi \in \Pi} \sum_{i=1}^{n} d(v_i, \phi(v_i)).$$

ARS COMBINATORIA 72(2004), pp. 307-310

We identify permutation $\phi : V(G) \to V(G)$ with the permutation ϕ : { 1, 2, ..., $n\} \to \{1, 2, ..., n\}$ by equating having $\phi(v_i) = v_j$ with $\phi(i) = j$.

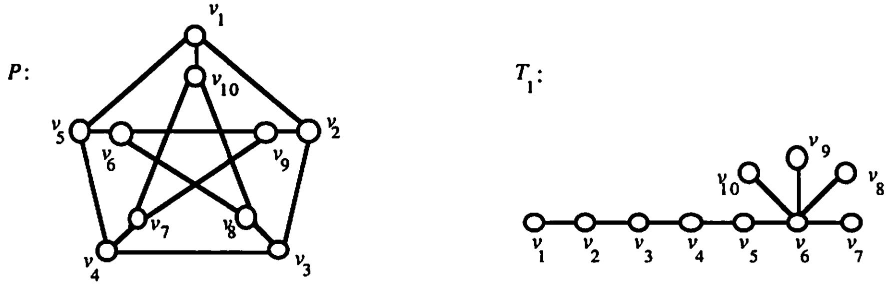

Figure 1. Petersen graph P and a tree T_1.

For example, ϕ = (1, 3, 5, 2, 4) (6, 7, 8, 9, 10) satisfies $\phi(v_1) = v_3$, $\phi(v_2) = v_4$, $\phi(v_3) = v_5$, $\phi(v_4) = v_1$, $\phi(v_5) = v_2$, $\phi(v_6) = v_7$, $\phi(v_7) = v_8$, $\phi(v_8) = v_9$, $\phi(v_9) = v_{10}$, and $\phi(v_{10}) = v_6$. For the Petersen graph P in Figure 1, $\sum_{i=1}^{10} d(v_i, \phi(v_i)) = 10 \cdot 2 = 20 = \mathcal{S}(P)$, while for tree T_1 of Figure 1, $\sum_{i=1}^{10} d(v_i, \phi(v_i))$ = 2 + 2 + 2 + 3 + 3 + 1 + 2 + 2 + 2 + 1 = 20. Note that for Ψ = (1, 6, 3, 7, 5, 9) (2, 10, 4, 8) and tree T_1, we have $\sum_{i=1}^{10} d(v_i, \psi(v_i))$ = 5 + 5 + 4 + 3 + 2 + 3 + 2 + 5 + 6 + 3 = 38. As will be shown, $\mathcal{S}(T_1) = 38$.

2. Examples and bounds. We can precisely bound $\mathcal{S}(G)$ for graphs G of order n.

Theorem 1. For every connected graph G of order $n \geq 2$, we have $n \leq \mathcal{S}(G) \leq \lfloor n^2/2 \rfloor$ and these bounds are sharp.

Proof. Let ϕ be any derangement of $V(G)$, that is, $\phi(v_i) \neq v_i$ for $1 \leq i \leq n$. Then $\mathcal{S}(G) \geq \sum_{i=1}^{n} d(v_i, \phi(v_i)) \geq n \cdot 1 = n$. For a median vertex $v \in M(G)$ we have $d(v) \leq n^2/4$. Let ϕ be a maximum sum permutation of G. Then $\mathcal{S}(G) = \sum_{i=1}^{n} d(v_i, \phi(v_i)) \leq \sum_{i=1}^{n} (d(v_i, v) + d(v, \phi(v_i))) \leq 2 \cdot d(v) \leq 2(n^2/4) = n^2/2$.

Clearly for complete graph K_n we have $\mathcal{S}(K_n) = n$ for $n \geq 2$. For path P_n let ϕ be the permutation with $\phi(v_i) = v_{n-i+1}$ for $1 \leq i \leq n$, and it is easily verified that $\sum_{i=1}^{n} d(v_i, \phi(v_i)) = \lfloor n^2/2 \rfloor$. ❑

We note that for cycles C_n we have $\mathcal{S}(C_n) = n^2/2$ when n is even and $\mathcal{S}(C_n) = n^2/2 - n/2$ if n is odd. Complete graph K_n is the only order n graph G with $\mathcal{S}(G) = n$.

For the eccentricity $e(v_i)$ we clearly have $d(v_i, \phi(v_i)) \leq e(v_i)$.

Theorem 2. $\mathcal{S}(G) \leq \sum_{i=1}^{n} e(v_i)$. ❑

For example, the k-cube Q_k has $V(Q_k)$ equal to the set of all binary k-tuples with two vertices adjacent if and only if they differ in exactly one position. Each vertex has a unique vertex farthest from it, $d((e_1, e_2, \ldots, e_k), (1-e_1, 1-e_2, \ldots, 1-e_k)) = k$. Hence, for Q_k we have $n = 2^k$, and $\mathcal{S}(Q_k) = k \cdot 2^k$.

For each edge e in a tree T, let $w(e)$ be the order of the smaller of the two components of $T - e$.

Theorem 3. For any tree T, we have $\mathcal{S}(T) = \sum_{e \in E(T)} 2 \cdot w(e)$.

Proof. Note that an edge e is on the unique path from v_i to $\phi(v_i)$ for a permutation $\phi: V(T) \to V(T)$ if and only if v_i and $\phi(v_i)$ are in different components of $T - e$. It follows that the maximum number of v_i- $\phi(v_i)$ paths containing e is $2 \cdot w(e)$. Thus $\mathcal{S}(T) = \max_{\phi \in \Pi} \sum_{i=1}^{n} d(v_i, \phi(v_i)) \leq \sum_{e \in E(T)} 2 \cdot w(e)$.

If $M(T) = \{u, v\}$ let T_u and T_v be the components of $T - uv$ containing u and v, respectively. We have $|V(T_u)| = |V(T_v)|$, and so there is a permutation $\phi: V(T) \to V(T)$ with $\phi(v_i) \in V(T_v)$ if and only if $v_i \in V(T_u)$. For such a ϕ every edge e is on $2 \cdot w(e)$ of the v_i- $\phi(v_i)$ paths, and so $\mathcal{S}(T) = \sum_{e \in E(T)} 2 \cdot w(e)$.

If the median consists of one vertex, $M(T) = \{u\}$, then each component of $T - u$ has less than $n/2$ vertices. We can define permutation $\phi: V(T) \to V(T)$ such that every edge is on $2 \cdot w(e)$ of the v_i- $\phi(v_i)$ paths iteratively, as follows. Let C_1 and C_2 be the two largest components of $T - u$. Select $x \in V(C_1)$ and $y \in V(C_2)$ such that x and y are endpoints of T. Let $\phi(x) = y$ and $\phi(y) = x$, replace T by $T - x - y$ and iterate this process until only one or two vertices remain. If it is just u, let $\phi(u) = u$, and if it is u and w then let $\phi(u) = w$ and $\phi(w) = u$. ❑

3. Distance-separating edge partitions. In this concluding section we generalize Theorem 3. An edge set $F \subseteq E(G)$ is called distance-separating if $G - F$ has exactly two components, say with vertex sets V_1 and V_2, and every path from any vertex $x \in V_1$ to any vertex $y \in V_2$ of length $d(x, y)$ contains exactly one edge in F. Note that in trees edge set F is distance-separating if and only if F consist of exactly one edge. We let $w(F)$ be the order of the smaller of the two components of $G - F$. A distance-separating edge partition for G is a collection $\{E_1, E_2, \ldots, E_t\}$ where each E_i is a distance-separating edge set, $\bigcup_{i=1}^{t} E_i = E(G)$, and $1 \le i < j \le t$ implies $E_i \cap E_j = \varnothing$.

Theorem 4. Assume $\{E_1, E_2, \ldots, E_t\}$ is a distance-separating edge partition of graph G. Then $\mathcal{S}(G) \le \sum_{i=1}^{t} 2 \cdot w(E_i)$.

Proof. Let V_i^1 and V_i^2 be the vertex sets of the components of $G - E_i$ with $w(E_i) = |V_i^1| \le |V_i^2|$. For any permutation $\phi: V(G) \to V(G)$ consider a collection $\{P_1, P_2, \ldots, P_n\}$ where P_i is a v_i-$\phi(v_i)$ path of length $d(v_i, \phi(v_i))$. For each $e \in E(G)$, let $f(e)$ be the number of paths P_i that contain e, and let $f(E_i) = \sum_{e \in E_i} f(e)$. We have $\sum_{j=1}^{n} d(v_j, \phi(v_j)) = \sum_{e \in E(G)} f(e) = \sum_{i=1}^{t} f(E_i) \le \sum_{i=1}^{t} 2 \cdot w(E_i)$.

We note that $\mathcal{S}(G) = \sum_{i=1}^{t} 2 \cdot w(E_i)$ if and only if there is a permutation $\phi: V(G) \to V(G)$ such that for $1 \le i \le t$ we have $x \in V_i^1$ implying that $\phi(x) \in V_i^2$. ❑

For example, consider the grid graph $G_{r,s}$ which is the cartesian product $P_r \times P_s$ of paths P_r and P_s. Theorem 4 can be used to see that $\mathcal{S}(G_{r,s}) = s\lfloor r^2/2 \rfloor + r\lfloor s^2/2 \rfloor$.

We have further results for maximum sum permutations of graphs under study.