Some Identities Involving the Generalized Fibonacci and Lucas Numbers

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Abstract

In this paper, by using the generating function method, we obtain a series of identities involving the generalized Fibonacci and Lucas numbers.

1. Introduction

For the generalized Fibonacci and Lucas numbers

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n, \tag{1}$$

where $\alpha = (p + \sqrt{D})/2$, $\beta = (p - \sqrt{D})/2$, $D = p^2 - 4q$, p and q are real numbers, and $pq \neq 0$, many authors investigated various properties of $\{U_n\}$ and $\{V_n\}$. Various identities involving the generalized Fibonacci and Lucas numbers were established. In [1], Neville Robbins showed that:

$$\sum_{a+b=n} U_a U_b = \frac{(n-1)V_n - 2qU_{n-1}}{p^2 - 4q}, \quad n \geq 1,$$

$$\sum_{a+b=n} V_a V_b = (n+1)V_n + 2U_{n+1}, \quad n \geq 1.$$

In [2], Zhang calculated the summation of the form

$$\sum_{a_1+a_2+\cdots+a_m=n} U_{a_1} U_{a_2} \cdots U_{a_m}.$$

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For instance, Zhang proved that

$$\begin{aligned}\sum_{a+b+c=n} U_aU_bU_c &= \frac{u_1^2}{2(p^2-4q)^2}\{[(p^3-4pq)n^2-(3p^3-6pq)n \\ &+(2p^3+4pq)]U_{n-1}+[(4q^2-p^2q)n^2+3p^2qn \\ &-2(p^2q+4q^2)]U_{n-2}\}, \quad n\geq 2.\end{aligned}$$

Zhang's results were generalized (see [3]).

In this paper, we are interested in computing the summation of the forms

$$\sum_{a_1+a_2+\cdots+a_m=n} \binom{a_1+k}{k}\binom{a_2+k}{k}\cdots\binom{a_m+k}{k} U_{a_1}U_{a_2}\cdots U_{a_m}$$

and

$$\sum_{a_1+a_2+\cdots+a_m=n} \binom{a_1+k}{k}\binom{a_2+k}{k}\cdots\binom{a_m+k}{k} V_{a_1}V_{a_2}\cdots V_{a_m},$$

where k is a nonnegative integer. These problems are interesting because they can help us to find some new convolution properties. In the next section, we will work out a series of identities involving the generalized Fibonacci and Lucas numbers.

2. Main Results

In this section, we state the main results of this paper.

Theorem 1. Let $\{U_n\}$ and $\{V_n\}$ be the generalized Fibonacci and Lucas sequences, respectively. Then

$$\sum_{a+b=n}(a+1)(b+1)U_aU_b = \frac{1}{D^2}[\binom{n+3}{3}DV_n+4qU_{n+1}-2(n+1)V_{n+2}], \quad (2)$$

$$\sum_{a+b+c=n}(a+1)(b+1)(c+1)U_aU_bU_c = \frac{U_n}{D}\binom{n+5}{5} - \frac{3U_{n+2}}{D^2}\binom{n+3}{3}$$

$$+\frac{3p(n+1)V_{n+2}}{D^3}-\frac{12pqU_{n+1}}{D^3}+\frac{6qV_{n+1}}{D^3}\binom{n+2}{2}-\frac{6q^2(n+1)U_n}{D^3}, \tag{3}$$

$$\sum_{a+b=n}\binom{a+2}{2}\binom{b+2}{2}U_aU_b = \frac{1}{D}[\binom{n+5}{5}V_n-\frac{2U_{n+3}}{D}\binom{n+2}{2} +\frac{6q(n+1)V_{n+2}}{D^2}-\frac{12q^2U_{n+1}}{D^2}], \tag{4}$$

$$\sum_{a+b=n}(a+1)(b+1)V_aV_b=\binom{n+3}{3}V_n+\frac{2(n+1)V_{n+2}}{D}-\frac{4qU_{n+1}}{D}, \tag{5}$$

$$\sum_{a+b+c=n}(a+1)(b+1)(c+1)V_aV_bV_c=\frac{3}{D}[V_{n+2}\binom{n+3}{3}-2qU_{n+1}\binom{n+2}{2}] +\binom{n+5}{5}V_n+\frac{3(n+1)(3q^2V_n+V_{n+4})}{D^2}-\frac{12q(U_{n+3}+q^2U_{n-1})}{D^2}, \tag{6}$$

$$\sum_{a+b=n}\binom{a+2}{2}\binom{b+2}{2}V_aV_b = \binom{n+5}{5}V_n+\frac{2U_{n+3}}{D}\binom{n+2}{2} -\frac{6(n+1)qV_{n+2}}{D^2}+\frac{12q^2U_{n+1}}{D^2}. \tag{7}$$

Proof. We only show that identities (2–3) hold. The proofs of (4–7) follow the same pattern.

Consider the generating function of the sequence $\{\binom{n+k}{k}U_n\}$:

$$U_k(x)=\sum_{n=0}^{\infty}\binom{n+k}{k}U_nx^n.$$

By using (1) and the formula

$$\sum_{n=0}^{\infty}\binom{n+k}{k}x^n=\frac{1}{(1-x)^{k+1}},\quad (|x|<1), \tag{8}$$

we have

$$U_k(x)=\frac{1}{\alpha-\beta}[\frac{1}{(1-\alpha x)^{k+1}}-\frac{1}{(1-\beta x)^{k+1}}],\quad |x|<\min(\frac{1}{|k\alpha|},\frac{1}{|k\beta|}).$$

Hence

$$U_k^m(x) = \frac{1}{(\alpha-\beta)^m}\sum_{i=0}^{m}\binom{m}{i}\frac{(-1)^{m-i}}{(1-\alpha x)^{ki+i}(1-\beta x)^{(k+1)(m-i)}}. \qquad (9)$$

Comparing the coefficients of x^n on both sides of (9), one can compute the summation

$$\sum_{a_1+a_2+\cdots+a_m=n}\binom{a_1+k}{k}\binom{a_2+k}{k}\cdots\binom{a_m+k}{k}U_{a_1}U_{a_2}\cdots U_{a_m}.$$

Since $\frac{1}{(1-\alpha x)(1-\beta x)} = \frac{1}{\alpha-\beta}(\frac{\alpha}{1-\alpha x} - \frac{\beta}{1-\beta x}),\quad \alpha\beta = q,$

$$U_1^2(x) = \frac{1}{(\alpha-\beta)^2}[\frac{1}{(1-\alpha x)^4} + \frac{1}{1-\beta x)^4} - \frac{2}{(1-\alpha x)^2(1-\beta x)^2}],$$

$$\begin{aligned} U_1^3(x) &= \frac{1}{(\alpha-\beta)^3}[\frac{1}{(1-\alpha x)^6} - \frac{1}{(1-\beta x)^6} - \frac{3}{(1-\alpha x)^4(1-\beta x)^2} \\ &\quad + \frac{3}{(1-\alpha x)^2(1-\beta x)^4}], \end{aligned}$$

after some calculus, one can verify that

$$\begin{aligned} U_1^2(x) = \frac{1}{(\alpha-\beta)^2}\{\frac{1}{(1-\alpha x)^4} + \frac{1}{(1-\beta x)^4} - \frac{2}{(\alpha-\beta)^2}[\frac{\alpha^2}{(1-\alpha x)^2} + \frac{\beta^2}{(1-\beta x)^2}] \\ + \frac{4q}{(\alpha-\beta)^3}(\frac{\alpha}{1-\alpha x} - \frac{\beta}{1-\beta x})\}, \qquad (10) \end{aligned}$$

$$\begin{aligned} U_1^3(x) &= \frac{1}{(\alpha-\beta)^3}\{\frac{1}{(1-\alpha x)^6} - \frac{1}{(1-\beta x)^6} - \frac{3\alpha^2}{(\alpha-\beta)^2(1-\alpha x)^4} \\ &\quad + \frac{3\beta^2}{(\alpha-\beta)^2(1-\beta x)^4} + \frac{3(\alpha^2-\beta^2)}{(\alpha-\beta)^4}[\frac{\alpha^2}{(1-\alpha x)^2} + \frac{\beta^2}{(1-\beta x)^2}] \\ &\quad - \frac{12q(\alpha^2-\beta^2)}{(\alpha-\beta)^5}(\frac{\alpha}{1-\alpha x} - \frac{\beta}{1-\beta x}) + \frac{6q\alpha}{(\alpha-\beta)^3(1-\alpha x)^3} \\ &\quad + \frac{6q\beta}{(\alpha-\beta)^3(1-\beta x)^3} - \frac{6q^2}{(\alpha-\beta)^4}[\frac{1}{(1-\alpha x)^2} - \frac{1}{(1-\beta x)^2}]\}. \end{aligned}$$

(11)

Comparing the coefficients of x^n on both sides of (10–11) and noting the formula (8), the geometric formula and the definition (1), we can obtain identities (2) and (3). This completes the proof.

Corollary 1.

$$\sum_{a+b=n}(a+1)(b+1)U_{ak}U_{bk} = \frac{1}{D^2}\left[\binom{n+3}{3}DV_{nk}+\frac{4q^kU_{nk+k}}{U_k^3}-\frac{2(n+1)V_{nk+2k}}{U_k^2}\right],$$

$$\sum_{a+b+c=n}(a+1)(b+1)(c+1)U_{ak}U_{bk}U_{ck} = \frac{3(n+1)[U_{2k}V_{2nk+2k}-2q^{2k}U_{nk}]}{D^3U_k^4} + \frac{U_{nk}}{D}\binom{n+5}{5}-\frac{12V_kq^kU_{nk+k}}{D^3U_k^4}-\frac{3U_{nk+2k}}{D^2U_k^2}\binom{n+3}{3}+\frac{6q^kV_{nk+k}}{D^3U_k^3}\binom{n+2}{2},$$

$$\sum_{a+b=n}\binom{a+2}{2}\binom{b+2}{2}U_{ak}U_{bk} = \frac{1}{D}\left[\binom{n+5}{5}V_{nk}-\frac{2U_{nk+3k}}{DU_k^3}\binom{n+2}{2}+\frac{6(n+1)q^kV_{nk+2k}}{D^2U_k^4}-\frac{12q^{2k}U_{nk+k}}{D^2U_k^5}\right],$$

$$\sum_{a+b=n}(a+1)(b+1)V_{ak}V_{bk} = \binom{n+3}{3}V_{nk}+\frac{2(n+1)V_{nk+2k}}{DU_k^2}+\frac{4q^kU_{nk+k}}{DU_k^3},$$

$$\sum_{a+b+c=n}(a+1)(b+1)(c+1)V_{ak}V_{bk}V_{ck} = \binom{n+5}{5}V_{nk}+\frac{3V_{nk+2k}}{DU_k^2}\binom{n+3}{3} + \frac{3(n+1)(3q^{2k}V_{nk}+V_{nk+4k})}{D^2U_k^4} - \frac{12q^k(U_{nk+3k}+q^{2k}U_{nk-k})}{D^2U_k^5} - \frac{6q^kU_{nk+k}}{DU_k^3}\binom{n+2}{2},$$

$$\sum_{a+b=n}\binom{a+2}{2}\binom{b+2}{2}V_{ak}V_{bk} = \binom{n+5}{5}V_{nk}+\frac{2U_{nk+3k}}{DU_k^3}\binom{n+2}{2} - \frac{6(n+1)q^kV_{nk+2k}}{D^2U_k^4}+\frac{12q^{2k}U_{nk+k}}{D^2U_k^5},$$

where k is a positive integer.

Proof. It is well known that $\{U_n\}$ and $\{V_n\}$ satisfy the linear recurrence relation

$$W_n = pW_{n-1} - qW_{n-2}, \quad n \geq 2.$$

Suppose that

$$U'_n = \frac{(\alpha^k)^n - (\beta^k)^n}{\alpha^k - \beta^k} = \frac{U_{nk}}{U_k}, \quad V'_n = \alpha^{nk} + \beta^{nk} = V_{nk}. \tag{12}$$

The definition (12) implies that $\{U'_n\}$ and $\{V'_n\}$ satisfy the linear recurrence relation

$$W_n = V_k W_{n-1} - q^k W_{n-2}, \quad n \geq 2.$$

Applying Theorem 1 to the sequences $\{U'\}$ and $\{V'_n\}$, and noticing that $V_k^2 - 4q^k = DU_k^2$ and $U_k V_k = U_{2k}$, we can prove that Corollary 1 holds. This completes the proof.

Theorem 2. Let $\{U_n\}$ and $\{V_n\}$ be the generalized Fibonacci and Lucas sequences, respectively. Then

$$\sum_{a+b=n} (a+1)(b+1)U_a^2 U_b^2 = \frac{1}{D^2}\{\frac{2}{Dp^2}[(n+1)V_{2n+4} - \frac{2q^2 U_{2n+2}}{U_2}] + \binom{n+3}{3}(V_{2n} + 4q^n) - \frac{4(n+1)V_n V_{n+2}}{D} + \frac{8qV_n U_{n+1}}{D}\},$$

$$\sum_{a+b=n} (a+1)(b+1)V_a^2 V_b^2 = \binom{n+3}{3}(V_{2n} + 4q^n) + \frac{2}{Dp^2}[(n+1)V_{2n+4} - \frac{2q^2 U_{2n+2}}{U_2}] + \frac{4(n+1)(V_{2n+2} + V_2 q^n)}{D} - \frac{8qU_{n+1}V_n}{D}.$$

Corollary 2.

$$\sum_{a+b=n} (a+1)(b+1)U_{ak}^2 U_{bk}^2 = \frac{1}{D^2}\{\frac{2}{DU_{2k}^2}[(n+1)V_{2nk+4k} - \frac{2q^{2k}U_{2nk+2k}}{U_{2k}}] + \binom{n+3}{3}(V_{2nk} + 4q^{nk}) - \frac{4(n+1)V_{nk}V_{nk+2k}}{DU_k^2} + \frac{8q^k V_{nk} U_{nk+k}}{DU_k^3}\},$$

$$\sum_{a+b=n}(a+1)(b+1)V_{ak}^2V_{bk}^2 = \frac{2}{DU_{2k}^2}[(n+1)V_{2nk+4k} - \frac{2q^{2k}U_{2nk+2k}}{U_{2k}}]$$

$$+\binom{n+3}{3}(V_{2nk}+4q^{nk}) + \frac{4(n+1)(V_{2nk+2k}+V_{2k}q^{nk})}{DU_k^2} - \frac{8q^kU_{nk+k}V_{nk}}{DU_k^3},$$

where k is a positive integer.

The proofs of Theorem 2 and Corollary 2 are similar to that of Theorem1 and Corollary 2, respectively, and therefore are omitted here.

From Theorems 1–2 we can establish some congruences. For example, if $p = -q = 1$ in (4) and (6), we have

$$\sum_{a+b=n}\binom{a+2}{2}\binom{b+2}{2}F_aF_b = \frac{1}{5}[\binom{n+5}{5}L_n - \frac{2F_{n+3}}{5}\binom{n+2}{2} - \frac{6(n+1)L_{n+2}}{25} - \frac{12F_{n+1}}{25}]$$

and

$$\sum_{a+b+c=n}(a+1)(b+1)(c+1)L_aL_bL_c = \frac{3}{5}[\binom{n+3}{3}L_{n+2} + 2\binom{n+2}{2}F_{n+1}]$$

$$+\binom{n+5}{5}L_n + \frac{3(n+1)(3L_n+L_{n+4})}{25} + \frac{12(F_{n+3}+F_{n-1})}{25},$$

where $F_n(L_n)$ denotes the n^{th} term of the Fibonacci (Lucas) sequences ($U_n = F_n$ and $V_n = L_n$ when $p = -q = 1$). Therefore, we get the congruences:

$$25\binom{n+5}{5}L_n - 10\binom{n+2}{2}F_{n+3} - 6(n+1)L_{n+2} - 12F_{n+1} \equiv 0 \pmod{125},$$

$$25\binom{n+5}{5}L_n + 15[\binom{n+3}{3}L_{n+2} + 2\binom{n+2}{2}F_{n+1}]$$
$$+3(n+1)(3L_n+L_{n+4}) + 12(F_{n+3}+F_{n-1}) \equiv 0 \pmod{25}.$$

Using various identities involving Fibonacci and Lucas numbers, the last identity can be reduced to the following simplified form when $n \geq 1$:

$$5[\binom{n+2}{3} + n]L_{n+1} - 2nL_n + [10\binom{n+1}{2} + 12F_n] \equiv 0 \pmod{25}.$$

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