On d-antimagic labelings of prisms

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Abstract

We refer to a labeling of a plane graph as a d-antimagic labeling if the vertices, edges and faces of the graph are labeled in such a way that the label of a face and the labels of vertices and edges surrounding that face add up to a weight of the face and the weights of faces constitute an arithmetical progression of difference d. In this paper we deal with d-antimagic labeling of prisms.

1 Introduction

In this paper we consider finite undirected plane graphs without loops and multiple edges. A graph G consists of a vertex set $V(G)$, an edge set $E(G)$ and a face set $F(G)$ with cardinalities v, e and f, respectively. A general reference for graph theoretic notions is [13].

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A *labeling of type* (a, b, c) assigns labels from the set $\{1, 2, 3, ..., av+be+cf\}$ to the vertices, edges and faces of G in such a way that each vertex receives a labels, each edge receives b labels, and each face receives c labels and each number is used exactly once as a label.

The *weight* of a face under a labeling is the sum of labels of the face itself together with labels of vertices and edges surrounding that face.

A labeling of type (a, b, c) is said to be *magic* if for every number s, all s-sided faces have the same weight. We allow different weights for different values of s.

A connected plane graph $G=(V, E, F)$ is said to be *(a, d)-face antimagic* if there exist positive integers a, d and a bijection from the set $\{1, 2, ..., e\}$ onto the edges of G so that the induced mapping $\mu : F(G) \to W$ is also a bijection, where $W=\{a, a+d, ..., a+(f-1)d\}$ is the set of weights of all faces of G.

The (a, d)-face antimagic graph was defined in [1]. Further results can be found in [2]. Other types of antimagic labelings were considered by Hartsfield and Ringel [7], Bodendiek and Walther [5] and Simanjuntak, Miller and Bertault [10].

A labeling of type (a, b, c) of plane graph G is called *d-antimagic* if for every number s, the set of the s-sided face weights is $W_s=\{a_s, a_s+d, a_s+2d, ..., a_s+(f_s-1)d\}$ for some integers a_s and d, where f_s is the number of s-sided faces. We allow different sets W_s for different values of s.

We usually restrict a, b and c to be no greater than one. Labelings of types $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are also called *vertex*, *edge* and *face* labelings, respectively. Furthermore, d-antimagic labelings of types $(1, 1, 0)$ and $(1, 1, 1)$) were defined in [3].

In this paper we deal with d-antimagic labeling of type $(1, 1, 1)$ for prisms and describe these labelings for some values of d.

The prism D_n, $n \geq 3$, is a cubic graph which can be defined as the cartesian product $P_2 \times C_n$ of a path on two vertices with a cycle on n vertices. *Prism* D_n, $n \geq 3$, consists of an outer n-cycle $y_1\ y_2\ ...\ y_n$, an inner n-cycle $x_1\ x_2\ ...\ x_n$, and a set of n spokes x_iy_i, $i = 1, 2, ..., n$. $|V(D_n)| = 2n$, $|E(D_n)| = 3n$, $|F(D_n)| = n+2$. We denote by $z_{4,i}$ the 4-sided face bounded by the edges x_iy_i, x_ix_{i+1}, $x_{i+1}y_{i+1}$ and y_iy_{i+1}; and we denote the inner face and the outer face by $z_{n,1}$ and $z_{n,2}$, respectively (see Figure 1).

It was proved [3] that for $n \geq 3$, the prism D_n is 1-antimagic of type $(1, 1, 1)$ and for $n \equiv 3 \pmod 4$ and $d = 2, 3, 4, 6$ there exist d-antimagic labelings

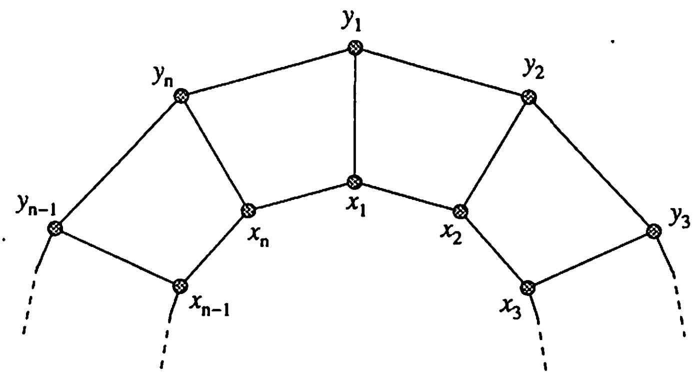

Figure 1: The prism

of type $(1,1,1)$. Subsequently, Bača, Miller and Ryan [4] proved that the prism D_n has a 3-antimagic labeling of type $(1,1,1)$ for $n \geq 3$, $n \neq 4$.

In this paper we shall prove that for $n \geq 3$, the prism D_n is d-antimagic of type $(1,1,1)$ for $d = 2,4,5,6$.

2 Preliminary Results

The basic idea we use to label the prism is fairly simple. We decompose the prism into three parts. The first part is the outer cycle, the second part is the inner cycle and the third part consists of the faces and the spokes. The face weight of any face of the prism is the sum of the corresponding edge weight of the outer cycle with the edge weight of the inner cycle and with the face labeling and the labelings of two spokes. Here we notice that we could label the third part by using the known labeling for cycles if we treat the face labeling as the edge labeling of a cycle and the labelings of spokes as the labelings of vertices; then the sum of the face weight and the labelings of the two spokes equals to the edge weight of the cycle. Now we have tranformed the problem to a labeling problem of three cycles.

We know that a cycle has an $(a,0)$-edge-antimagic labeling [12], an $(a,1)$-edge-antimagic labeling, and also an $(a,2)$-edge-antimagic labeling, so if we use these known results to label the three parts, we can obtain new labelings from their combination.

For example, for $d = 2$, we could first label the inner n-cycle and the outer n-cycle by using the integers $\{2n+1, 2n+2, ..4n\} \cup \{4n+1, 4n+2, ..., 6n\}$

respectively in such a way that both the n-cycles have been assigned an $(a, 1)$-edge-antimagic labeling, and then we could label the third part by using the known result of $(a, 0)$ edge magic labeling with integers $1, 2, \ldots, 2n$. Because we are using magic labeling of cycles, we could start anywhere we want. However, we make sure to assign the label $2n$ to the face which has edge label $2n + 1$. It is easy to see that now the weights of all the 4-sided faces constitute an arithmetical progression of difference 2.

Of course, here the inner n-sided face has a different weight from the outer n-sided face. We swap the edge labelings of the outer cycle and the inner cycle. Now both n-sided faces have the same weight. Then we swap the labels of $2n + 1$ and $2n$. This operation will not affect the weights of any of the 4-sided faces, but the weight of the inner n-sided face is now 1 more than the weight of the outer n-sided face. Now, if we put $6n + 2$ and $6n + 1$ in the inter face and the outer face, the inner n-sided face will have a weight which differs by 2 from the weight of the outer n-sided face. Thus we obtain a 2-antimagic labeling of a prism.

To make this paper self contained, we list here all the known results that will be used in this paper.

For $(a, 1)$-edge-antimagic labeling of cycle C_n, we label the cycle as

$$\begin{aligned} \lambda_1(v_i) &= n + i \quad \text{for } i = 1, 2, ..., n \\ \lambda_1(v_i v_{i+1}) &= \begin{cases} n - i & \text{for } i = 1, 2, ..., n - 1 \\ n & \text{for } i = n \end{cases} \end{aligned}$$

For $(a, 2)$-edge-antimagic labeling of cycle C_n, we label the cycle as

$$\begin{aligned} \lambda_2(v_i) &= 2i \quad \text{for } i = 1, 2, ..., n \\ \lambda_2(v_i v_{i+1}) &= \begin{cases} 2n - 2i - 1 & \text{for } i = 1, 2, ..., n - 1 \\ 2n - 1 & \text{for } i = n \end{cases} \end{aligned}$$

For $(a, 0)$-edge-antimagic labeling of cycle C_n, the labeling is quite complex. When n is odd

$$\lambda(v_i) = \begin{cases} \frac{i+1}{2} & \text{for } i = 1, 3, ..., n \\ \frac{n+i+1}{2} & \text{for } i = 2, 4, ..., n - 1 \end{cases}$$

$$\lambda(v_iv_{i+1}) = \begin{cases} 2n-i & \text{for } i=1,2,...,n-1 \\ 2n & \text{for } i=n \end{cases}$$

It is easy to see that if we put edge labeling in clockwise, we get a $(a,2)$-antimagic labelling.

For $(a,0)$-edge-antimagic labeling of cycle C_n when $n \equiv 0 \bmod 4$

$$\lambda(v_i) = \begin{cases} \frac{i+1}{2} & \text{for } i=1,3,...,n/2+1 \\ \frac{3n}{2} & \text{for } i=2 \\ \frac{n+i}{2} & \text{for } i=4,6,...,n/2 \\ \frac{i+2}{2} & \text{for } i=n/2+2,,n/2+4,...,n \\ \frac{n+i-1}{2} & \text{for } i=n/2+3,n/2+5,...,n-1 \end{cases}$$

If $n \equiv 2 \bmod 4$, then we label the vertices of C_n by

$$\lambda(v_i) = \begin{cases} \frac{i+1}{2} & \text{for } i=1,3,...,n/2 \\ \frac{3n}{2} & \text{for } i=2 \\ \frac{n+i+2}{2} & \text{for } i=4,6,...,n/2-1 \\ \frac{n+6}{4} & \text{for } i=n/2+1 \\ \frac{i+3}{2} & \text{for } i=n/2+2,,n/2+4,...,n-1 \\ \frac{n+i}{2} & \text{for } i=n/2+3,n/2+5,...,n-2 \\ \frac{n+4}{2} & \text{for } i=n \end{cases}$$

We label the edges of C_n in both cases by values $\{n, n+1,..., 3n/2-1\}$ and $\{3n/2+1, 3n/2+2, ..., 2n\}$ in such a way that the common weight for all the edges will be $5n/2+2$.

3 New Results

In this paper we shall prove that for $n \geq 4$, the prism D_n is d-antimagic of type $(1,1,1)$ for $d = 2,4,5,6$.

For $n = 4$, the prism has a special d-antimagic labeling. We first construct the vertex and the edge labelings of D_4 and then we will formulate lemmas.

$$\mu_1(x_i) = \begin{cases} i & \text{for } i = 1,3 \\ 4+i & \text{for } i = 2,4 \end{cases}$$

$$\mu_1(y_i) = \begin{cases} 8-i & \text{for } i = 1,3 \\ 6-i & \text{for } i = 2,4 \end{cases}$$

$$\mu_2(x_i) = \begin{cases} 2i-1 & \text{for } i = 1,2,3 \\ 13 & \text{for } i = 4 \end{cases}$$

$$\mu_2(y_i) = \begin{cases} 8i-1 & \text{for } i = 1,2 \\ 17-2i & \text{for } i = 3,4 \end{cases}$$

$$\xi_1(x_ix_{i+1}) = \begin{cases} 2i-1 & \text{for } i = 1,2 \\ 13-2i & \text{for } i = 3,4 \end{cases}$$

$$\xi_1(y_iy_{i+1}) = \begin{cases} 5+i & \text{for } i = 1,3 \\ i & \text{for } i = 2,4 \end{cases}$$

$$\xi_1(x_iy_i) = \begin{cases} 12 & \text{for } i = 1 \\ 7+i & \text{for } i = 2,3,4 \end{cases}$$

$$\xi_2(x_ix_{i+1}) = \begin{cases} 1+4i & \text{for } i = 1,2 \\ 41-10i & \text{for } i = 3,4 \end{cases}$$

$$\xi_2(y_iy_{i+1}) = \begin{cases} 2 & \text{for } i = 1 \\ 2+2i & \text{for } i = 2,3,4 \end{cases}$$

$$\xi_2(x_iy_i) = \begin{cases} 17-5i & \text{for } i = 1,2 \\ 7-i & \text{for } i = 3,4 \end{cases}$$

$$\xi_3(x_ix_{i+1}) = \begin{cases} 2+5i & \text{for } i = 1,2 \\ 54-13i & \text{for } i = 3,4 \end{cases}$$

$$\xi_3(y_iy_{i+1}) = \begin{cases} 3 & \text{for } i = 1 \\ 2+3i & \text{for } i = 2,3,4 \end{cases}$$

$$\xi_3(x_iy_i) = \begin{cases} 22-6i & \text{for } i = 1,2 \\ 12-2i & \text{for } i = 3,4 \end{cases}$$

Lemma 1 *For $d = 2,4,6$, the prism D_4 has a d-antimagic labeling of type* $(1,1,1)$.

Proof. Let us distinglish three cases.

Case 1. $d = 2$

Label the vertices by $2\mu_1 - 1$, the edges by $2\xi_2$, let the outer 4-sided face and the inner 4-sided face receive the value 25 and 26, respectively, and label the remaining 4-sided faces $z_{4,i}$ with $15 + 2i$. The weights of the 4-

sided faces constitute the set $\{101, 103, 105, 107, 109, 110\}$. If we swap the edge label 18 with the face label 19, the face weight 110 will be changed to 111, and the other face weights will remain the same. Hence we obtain a 2-antimagic labeling of type $(1, 1, 1)$.

Case 2. $d = 4$

The vertices of D_4 are labelled by μ_1 and the edges are labelled by $\xi_3 + 8$. The outer 4-sided face is labelled by 25 and the inner 4-sided face is labeled by 21. Label the other 4-sided faces $z_{4,i}$ by $5 + 4i$ if $1 \le i \le 3$, and label $Z_{4,4}$ by 26. Now we swap the edge value 22 with the face value 25. It is easy to verify that all face weights constitute an arithmetical progression of difference 4.

Case 3. $d = 6$

Label the outer 4-sided face by 26, and the inner 4-sided face by 17 and for $1 \le i \le 2$, label the 4-sided faces $z_{4,i}$ by $15 + 4i$ and for $3 \le i \le 4$, label the faces by $37 - 4i$. If we label the vertices of D_4 by μ_2, the edges by $2\xi_2$, and swap the edge label 20 with the face label 21, then we get a 6-antimagic labeling of type $(1, 1, 1)$. □

Lemma 2 *For $d = 3, 5$, the prism D_4 has a d-antimagic labeling of type $(1, 1, 1)$.*

Proof. Label the vertices of D_4 by μ_1, the edges by $\xi_1 + 8$. Then we have a labeling of type $(1, 1, 0)$ where the weights of the 4-sided faces constitute an arithmetical progression of difference 4. If we use consecutive integers to label the faces, it is easy to see that the resulting labeling is either 3-antimagic or 5-antimagic. □

Next we will prove that for $n \ge 3$ the prism has d-antimagic labeling for $d = 2, 4, 5, 6$. In the following proof, for case $n = 4$, please refer to the above lemmas.

Theorem 1 *For $n \ge 3$, the prism D_n has a 2-antimagic labeling of type $(1, 1, 1)$.*

Proof.

Label the vertices and edges of the prism D_n in the following way.

$$\beta_2(x_i) = \lambda_1(v_i) + 2n \qquad \text{for } i = 1, 2, ..., n$$

$$\beta_2(y_i) = \lambda_1(v_i) + 4n \quad \text{for } i = 1,2,3,...,n$$
$$\beta_2(x_i x_{i+1}) = \lambda_1(v_i v_{i+1}) + 4n \quad \text{for } i = 1,2,...,n$$
$$\beta_2(y_i y_{i+1}) = \lambda_1(v_i v_{i+1}) + 2n \quad \text{for } i = 1,2,...,n$$

Label the 4-sided faces $z_{4,i}$, for $i = 1,2,...,n$, and the spokes, by using one of the (a,0)-edge-antimagic labelings described earlier, depending on n, with integers $1,2,...,2n$. Here we make sure that the maximum label $2n$ is the label of the face which has the label $2n+1$ on one of its edges. Now it is easy to see that the outer n-sided face has the same weight as the inner n-sided face, and the weights of all the 4-sided faces constitute an arithmetical progression of difference 2. Next we swap the labels $2n$ and $2n+1$. Then the outer n-sided face has a weight 1 less than the weight of the inner n-sided face. Then we label the n-sided faces $z_{n,i}$, for $i = 1,2$, of the prism D_n as follows.

$$\beta_2(z_{n,i}) = 6n + 3 - i \quad \text{for } i = 1,2$$

It is easy to see that β_2 is a 2-antimagic labeling of type $(1,1,1)$. □

Theorem 2 *For $n \geq 3$, the prism D_n has a 4-antimagic labeling of type $(1,1,1)$.*

Proof.

Label vertices and edges of the prism D_n as follows.

$$\beta_4(x_i) = \lambda_2(v_i) + 2n \quad \text{for } i = 1,2,...,n$$
$$\beta_4(y_i) = \lambda_2(v_i) + 4n \quad \text{for } i = 1,2,...,n$$
$$\beta_4(x_i x_{i+1}) = \lambda_2(v_i v_{i+1}) + 4n \quad \text{for } i = 1,2,...,n$$
$$\beta_4(y_i y_{i+1}) = \lambda_2(v_i v_{i+1}) + 2n \quad \text{for } i = 1,2,...,n$$

Label the 4-sided faces $z_{4,i}$, for $i = 1,2,...,n$, and the spokes by using one of the (a,0)-edge-antimagic labelings described earlier, depending on n, with integers $1,2,...,2n$. Here we make sure the maximum label $2n$ is the label of the face which has the label $2n+3$ on one of its edges. Now it is easy to see that the outer n-sided face has the same weight as the inner n-sided face, and all the weights of all the 4-sided faces constitute an arithmetical progression of difference 4. If we swap the label of $2n$ and $2n+3$, then the outer n-sided face has weight 3 less than the weight of the inner n-sided face. If the outer n-sided face recives the value $6n+1$ and the inner n-sided face recives the value $6n+2$, then we obtain a 4-antimagic labeling of type $(1,1,1)$. □

Theorem 3 *For $n \geq 3$, the prism D_n has a 5-antimagic labeling of type $(1,1,1)$.*

Proof.

Label vertices and edges of the prism D_n in the following way.

$$\begin{aligned}
\beta_5(x_i) &= \lambda_2(v_i) && \text{for } i = 1,2,...,n \\
\beta_5(y_i) &= \lambda_2(v_i) - 1 && \text{for } i = 1,2,...,n \\
\beta_5(x_i x_{i+1}) &= \lambda_2(v_i v_{i+1}) + 2n && \text{for } i = 1,2,...,n \\
\beta_5(y_i y_{i+1}) &= \lambda_2(v_i v_{i+1}) + 2n + 1 && \text{for } i = 1,2,...,n
\end{aligned}$$

Label the 4-sided faces $z_{4,i}$, for $i = 1,2,...,n$, and the spokes $x_i y_i$, by using a $(a,1)$-edge-antimagic total labeling described before, where

$$\begin{aligned}
\beta_5(z_{4,i}) &= \lambda_1(v_i v_{i+1}) + 4n && \text{for } i = 1,2,...,n \\
\beta_5(x_i y_i) &= \lambda_1(v_i) + 4n && \text{for } i = 1,2,...,n
\end{aligned}$$

Now it is easy to see that the outer n-sided face and the inner n-sided face have the same weight, and the weights of all the 4-sided faces constitute an arithmetical progression of difference 5. If we swap the vertex label 1 with the vertex label 2 and label 3 with label 4, then the inner n-sided face has weight 4 less than the weight of the outer n-sided face. Now we label the n-sided faces $z_{n,i}$, for $i = 1,2$, of the prism D_n as follows.

$$\beta_5(z_{n,i}) = 6n + i \quad \text{for } i = 1,2$$

It is easy to see that λ_5 is a 5-antimagic labeling of type $(1,1,1)$. □

Theorem 4 *For $n \geq 3$, the prism D_n has a 6-antimagic labeling of type* $(1,1,1)$.

Proof.

Label vertices and edges of the prism D_n in the following way.

$$\begin{aligned}
\beta_6(x_i) &= \lambda_2(v_i) && \text{for } i = 1,2,...,n \\
\beta_6(y_i) &= \lambda_2(v_i) - 1 && \text{for } i = 1,2,...,n \\
\beta_6(x_i x_{i+1}) &= \lambda_2(v_i v_{i+1}) + 2n && \text{for } i = 1,2,...,n \\
\beta_6(y_i y_{i+1}) &= \lambda_2(v_i v_{i+1}) + 2n + 1 && \text{for } i = 1,2,...,n
\end{aligned}$$

Label the 4-sided faces $z_{4,i}$, for $i = 1,2,...,n$, and the spokes $x_i y_i$ by

$$\beta_6(z_{4,i}) = \begin{cases} 4n+2 & \text{for } i = 1 \\ 6n+4-2i & \text{for } i = 2,3,...,n. \end{cases}$$

$$\beta_6(x_i y_i) = \begin{cases} 6n-1 & \text{for } i = 1 \\ 4n+2i-3 & \text{for } i = 2,3,...,n \end{cases}$$

Now it is easy to see that the outer n-sided face has the same weight as the inner n-sided face, and the weights of all the 4-sided faces constitute an arithmetical progression of difference 6. If we swap the labels $4n-3$ and $4n+2$, then the outer n-sided face has weight 5 less than the weight of the inner n-sided face. Let external (internal) n-sided face receive the value $6n+1$ $(6n+2)$. Then we have a 6-antimagic labeling of type $(1,1,1)$.
□

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