Odd $(m_1, m_2, \ldots, m_r)$-cycle systems of K_n

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Abstract

Let $m_1, m_2, \ldots, m_r$ be positive integers with $m_i \geq 3$ for all i. An $(m_1, m_2, \ldots, m_r)$-cycle is defined as the edge-disjoint union of r cycles of lengths $m_1, m_2, \ldots, m_r$. An $(m_1, m_2, \ldots, m_r)$-cycle system of the complete graph K_n is a decomposition of K_n into $(m_1, m_2, \ldots, m_r)$-cycles.

In this paper, the necessary and sufficient conditions for the existence of an $(m_1, m_2, \ldots, m_r)$-cycle system of K_n are given, where m_i $(1 \leq i \leq r)$ are odd integers with $3 \leq m_i \leq n$ and $\sum_{i=1}^{r} m_i = 2^k$ for $k \geq 3$. Moreover, the complete graph with 1-factor removed $K_n - F$ has a similar result.

1. Introduction

In 1981, B. Alspach [3] posed the following conjecture.

Conjecture. *If* $m_1, m_2, \ldots, m_r$ *are integers with* $3 \leq m_i \leq n$ *and* $\sum_{i=1}^{r} m_i = n(n-1)/2$ (n odd) *or* $n(n-2)/2$ (n even), *then the complete graph* K_n (the complete graph with 1-factor removed $K_n - F$) *can be decomposed into cycles of lengths* $m_1, m_2, \ldots, m_r$.

A number of special cases of this conjecture have been done. More recently, Alspach and Gavlas [4] and Šajna [8] have proved that this conjecture is true when all cycle lengths are the same. For various cases when there are combinations of two or three distinct special cycle lengths, refer to [1, 2, 5, 6]. Moreover, the conjecture has been verified for all $n \leq 10$ by Rosa [7]. However, the conjecture is still far from solved.

The *m-cycle* $(v_0, v_1, \ldots, v_{m-1})$, denoted by C_m, is the graph induced by the edges $\{(v_i, v_{i+1}), (v_0, v_{m-1}) \mid i \in Z_{m-1}\}$, and the $(m_1, m_2, \ldots, m_r)$*-cycle*, denoted by $C_{m_1, m_2, \ldots, m_r}$, is the union of edge-disjoint m_i-cycles for $1 \leq i \leq r$. An $(m_1, m_2, \ldots, m_r)$*-cycle system* of a graph G is an ordered pair $(V(G), C)$, where C is a set of $(m_1,$

$m_2, \ldots, m_r$)-cycles whose edges decompose the edge set of G. The $(m_1, m_2, \ldots, m_r)$-cycle is called *odd* if each m_i $(1 \le i \le r)$ is odd; an odd $(m_1, m_2, \ldots, m_r)$-cycle system is defined similarly.

In this paper, it is shown that if m_i $(1 \le i \le r)$ are odd integers with $3 \le m_i \le n$ and $\sum_{i=1}^{r} m_i = 2^k$ for $k \ge 3$, then there exists a decomposition of K_n into cycles with lengths $m_1, m_2, \ldots, m_r$ if and only if $\sum_{i=1}^{r} m_i$ divides $|E(K_n)|$ and n is odd. The complete graph with 1-factor removed $K_n - F$ has also a similar result. It should be mentioned that the analogous consequence (i.e., all m_i are positive even integers) has been proved in [10].

We will use difference methods; in aid of these, we first need some notation.

A *labeling* of an m-cycle C_m is an injection f: $V(C_m) \to \{0, 1, \ldots, n_1\}$ such that the corresponding induced edge labeling f^*: $E(C_m) \to \{1, 2, \ldots, n_2\}$ of the edges of C_m given by

$$f^*(e) = |f(u) - f(v)| \text{ where } e = (u, v),$$

is also an injection, where n_1, n_2 are positive integers and $n_1, n_2 \ge |E(C_m)|$. Let $C_{m_1, m_2, \cdots, m_r}$ be an odd $(m_1, m_2, \ldots, m_r)$-cycle with $\sum_{i=1}^{r} m_i = 2^k$, $k \ge 3$. A labeling of an m_i-cycle $(1 \le i \le r)$ in $C_{m_1, m_2, \cdots, m_r}$ is said to be *proper* if $n_1 = 2|E(C_{m_1, m_2, \ldots, m_r})| + 1$ and $n_2 = |E(C_{m_1, m_2, \ldots, m_r})|$. If each m_i-cycle $(1 \le i \le r)$ in $C_{m_1, m_2, \cdots, m_r}$ has a proper labeling f_i and $\bigcup_{i=1}^{r} f_i^*(E(C_{m_i})) = \{1, 2, \ldots, 2^k\}$, then $C_{m_1, m_2, \cdots, m_r}$ will be called *strongly proper*.

2. The result

To prove the main theorem, we need some preliminary results.

Lemma 2.1. *Let a, b, c, and p be positive integers satisfying that $a + b \pm 1 = c$ and $c < p$. Then, for each positive integer m, the $(4m + 1)$-cycle C_{4m+1} has a labeling f such that the induced edge labeling f^* satisfies $f^*(E(C_{4m+1})) = \{a, b, c, p, p + 1, \ldots, p + 4m - 3\}$.*

Proof. The graph C_{4m+1} is shown in Figure 1. We split the proof into two cases depending on whether $c = a + b + 1$ or $a + b - 1$.

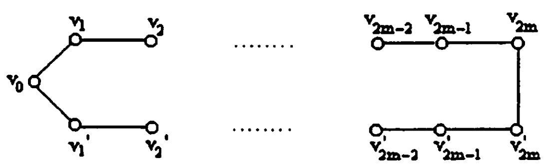

Figure. 1.

Case 1: $c = a + b + 1$.

Consider the following three subcases.

Subcase 1: Suppose that a is odd and b is even.

If $m = 1$, then define 5-cycle as

$$C_5 = (0, a, a+b, p+c, c).$$

If $m \geq 2$, let f be a labeling of C_{4m+1} defined as

$$f(u) = \begin{cases} 0, if\ u = v_0, \\ p+4m+2j-4, if\ u = v_{2j+1}, 0 \leq j \leq m-1, \\ 2p+8m-2j-8, if\ u = v_{2j}, 1 \leq j \leq m-1, \\ a+b+p+6m-5, if\ u = v_{2m}, \\ p+4m-3, if\ u = v_1', \\ b+p+4m+2j-5, if\ u = v_{2j}', 1 \leq j \leq m, \\ b+4m-2j-2, if\ u = v_{2j+1}', 1 \leq j \leq m-1, \end{cases}$$

where each vertex $u \in V(C_{4m+1})$.

Since a is odd and b is even, a routine verification shows that all vertex labels in C_{4m+1} are pairwise distinct and the edge label set $f^*(E(C_{4m+1})) = \{a, b, c, p, p+1, \ldots, p+4m-3\}$ for $m \geq 1$.

Example. The 13-cycle C_{13} with the edge label set $\{1, 2, 4, 6, 7, \ldots, 15\}$ is depicted in Figure 2, where $a = 1$, $b = 2$, and $p = 6$.

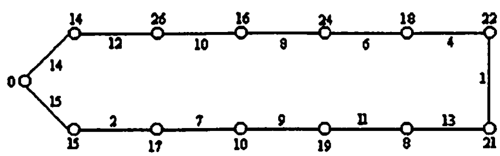

Figure 2.

Subcase 2: Suppose that both a and b are even.

Similar to Subcase 1 and omitted.

Subcase 3: Suppose that both a and b are odd.

Similar to Subcase 1 and omitted.

Case 2: $c = a + b - 1$.

Subcase 1: Suppose that a is odd and b is even.
If $m = 1$, then define 5-cycle as

$$C_5 = (0, a, a+b, p+c+1, c).$$

If $m \geq 2$, then let us introduce a labeling f of C_{4m+1} given by

$$f(u)=\begin{cases}0, if\ u=v_0,\\ p+4m+2j-4, if\ u=v_{2j+1}, 0\le j\le m-1,\\ 2p+8m-2j-8, if\ u=v_{2j}, 1\le j\le m-1,\\ b+p+6m-6, if\ u=v_{2m},\\ p+4m-3, if\ u=v_1',\\ a+b+p+4m+2j-6, if\ u=v_{2j}', 1\le j\le m,\\ a+b+4m-2j-3, if\ u=v_{2j+1}', 1\le j\le m-1,\end{cases}$$

where each vertex $u \in V(C_{4m+1})$.

Similarly, it can be verified that all vertex labels in C_{4m+1} are pairwise distinct and the edge label set $f^*(E(C_{4m+1})) = \{a, b, c, p, p+1, \ldots, p+4m-3\}$ for $m \ge 1$.

Subcase 2: Suppose that both a and b are even.

Similar to Subcase 1 and omitted.

Subcase 3: Suppose that both a and b are odd.

Similar to Subcase 1 and omitted. □

Lemma 2.2. *Let a, b, c, and p be positive integers satisfying that $a+b=c$ and $c<p$. Then, for each positive integer m, the $(4m+3)$-cycle C_{4m+3} has a labeling f such that the induced edge labeling f^* satisfies $f^*(E(C_{4m+3})) = \{a, b, c, p, p+1, \ldots, p+4m-1\}$.*

Proof. The proof is analogous to that of Lemma 2.1.

Case 1: Suppose that a is even and b is odd.

If $m = 0$, then define 3-cycle as

$$C_3 = (0, a, a+b).$$

If $m \ge 1$, let f be a labeling of C_{4m+3} given as

$$f(u)=\begin{cases}0, if\ u=v_0,\\ b, if\ u=v_1,\\ a+b+2j-2, if\ u=v_{2j}, 1\le j\le m,\\ a+b+p+4m-2j-1, if\ u=v_{2j+1}, 1\le j\le m,\\ p+4m+2j-1, if\ u=v_{2j+1}', 0\le j\le m-1,\\ 2p+8m-2j-3, if\ u=v_{2j}', 1\le j\le m,\\ a+b+2p+6m-3, if\ u=v_{2m+1}',\end{cases}$$

where all vertices $u \in V(C_{4m+3})$.

By the same argument, we have that the vertex labels in C_{4m+1} are all pairwise distinct and the edge label set $f^*(E(C_{4m+1})) = \{a, b, c, p, p+1, \ldots, p+4m-1\}$.

Example. The 15-cycle C_{15} with the edge label set $\{1, 2, 3, 5, 6, \ldots, 16\}$ is shown in Figure 3, where $a = 2$, $b = 1$, and $p = 5$.

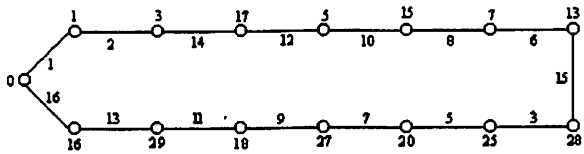

Figure 3.

Case 2: Suppose that both a and b are odd.

Similar to Case 1 and omitted.

Case 3: Suppose that both a and $b$are even.

Similar to Case 1 and omitted. □

In what follows, we will assume that $m_1, m_2, \ldots, m_r$ are odd positive integers with $m_i \geq 3$ for all i and $\sum_{i=1}^{r} m_i \equiv 0 \pmod 4$. Let N_j ($j = 1, 3$) denote the number of m_i's with the property that $m_i \equiv j \pmod 4$. Note that the value r is even since each m_i ($1 \leq i \leq r$) is odd and $\sum_{i=1}^{r} m_i \equiv 0 \pmod 4$.

Lemma 2.3. *If $r \equiv 0 \pmod 4$, then N_1, N_3 are even.*

Proof. Suppose, on the contrary, that N_1, N_3 are odd, say $N_3 = 2t + 1$ and $N_1 = r - 2t - 1$, $0 \leq t \leq r/2 - 1$. For convenience, set $m_i \equiv 3 \pmod 4$, $1 \leq i \leq 2t + 1$, and set $m_j \equiv 1 \pmod 4$, $2t + 2 \leq j \leq r$. Then

$$\sum_{i=1}^{2t+1} m_i \equiv 3(2t+1) \pmod 4 \equiv 2t + 3 \pmod 4$$

and

$$\sum_{j=2t+2}^{r} m_j \equiv r - 2t - 1 \pmod 4.$$

Thus

$$\sum_{i=1}^{r} m_i \equiv r + 2 \pmod 4 \equiv 2 \pmod 4$$

since $r \equiv 0 \pmod 4$. This contradicts the fact that $\sum_{i=1}^{r} m_i \equiv 0 \pmod 4$. □

Using similar techniques as in Lemma 2.3, the following is given.

Lemma 2.4. *If $r \equiv 2 \pmod 4$, then N_1, N_3 are odd.*

The following lemma is vital for the proof of the main result (Theorem 2.8). We need the crucial help of Skolem sequences.

A Skolem sequence of order n is a set of ordered pairs (s_i, t_i), $1 \le i \le n$, such that $t_i - s_i = i$ and $\bigcup_{i=1}^{n}(s_i,t_i) = \{1, 2, ..., 2n\}$. A hooked Skolem sequence of order n is also a set of ordered pairs (s_i, t_i), $1 \le i \le n$, such that $t_i - s_i = i$ and $\bigcup_{i=1}^{n}(s_i,t_i) = \{1, 2, ..., 2n-1, 2n+1\}$.

Note that if $\{(s_i, t_i) \mid 1 \le i \le r\}$ is a (hooked) Skolem sequence of order r, then the set $\{1, 2, ..., 3r\}$($\{1, 2, ..., 3r-1, 3r+1\}$) can be arranged into r triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$, where $a_i = i$, $b_i = r + s_i$ and $c_i = r + t_i$.

Theorem 2.5. [9]

(1) *A Skolem sequence of order n exists if and only if* $n \equiv 0$ *or* 1 (*mod* 4).

(2) *A hooked Skolem sequence of order n exists if and only if* $n \equiv 2$ *or* 3 (*mod* 4).

Lemma 2.6. *Let r be an even positive integer with* $N_1 + N_3 = r$. *Then the set* $\{1, 2, ..., 3r\}$ *can be partitioned into r triples* $\{a_i, b_i, c_i\}$ *such that* $a_i + b_i = c_i$ *for* $1 \le i \le N_3$ *and* $a_i + b_i \pm 1 = c_i$ *for* $N_3 + 1 \le i \le r$.

Proof. Suppose first that $r \equiv 0 \pmod 4$. Then, by Theorem 2.5-(1), a Skolem sequence of order r exists, i.e., we can arrange the set $\{1, 2, ..., 3r\}$ into r triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$. Since $r \equiv 0 \pmod 4$, by Lemma 2.3, N_1 and N_3 are both even. So, for $1 \le i \le N_3$, use the triple $\{a_i, b_i, c_i\}$ and for the remaining triples, of which there are an even number, use $\{a_{j+1}, b_j, c_j\}$ and $\{a_j, b_{j+1}, c_{j+1}\}$, where $j = N_3 + 1, N_3 + 3, ..., r - 1$.

Now suppose that $r \equiv 2 \pmod 4$. Then, by Theorem 2.5-(2), a hooked Skolem sequence of order r exists, i.e., we can arrange the set $\{1, 2, ..., 3r-1, 3r+1\}$ into r triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$. Since $r \equiv 2 \pmod 4$, by Lemma 2.4, N_1 and N_3 are both odd, say $N_1 = 2t + 1$ for $0 \le t \le (r-2)/2$. Subtracting 1 from c_i in the triple with $c_i = 3r + 1$ gives one of the triples with $a_i + b_i \pm 1 = c_i$. There are $r/2 - 1$ pairs of triples remaining with consecutive values for the a_i's. Hence, interchanging the values of the a_i's in t of these pairs will give $N_1 - 1$ more triples with $a_i + b_i \pm 1 = c_i$ while the remaining triples all satisfy $a_i + b_i = c_i$. □

Let $V(K_n) = Z_n$ and let (a, b) be any edge of K_n. We need the following straightforward lemma.

Lemma 2.7. *Let c and d be distinct elements of* Z_n. *If* $|a-b| = i$ *or* $n - i$ *for* $1 \le i \le \lfloor n/2 \rfloor$, *then* $(a + c, b + c) \ne (a + d, b + d)$ *and* $|(a+c)-(b+c)| = i$ *or* $n - i$, *where all addition is taken modulo n.*

Let $m_1, m_2, ..., m_r$ be positive integers with $3 \le m_i \le n$. By Lemma 2.7, it is clear that if there exists a strongly proper $(m_1, m_2, ..., m_r)$-cycle, then there exists a $(m_1, m_2, ..., m_r)$-cycle system of K_n, where $n = 2(\sum_{i=1}^{r} m_i) + 1$.

Now we have all the necessary tools to prove our main result.

Theorem 2.8. *Let* m_i $(1 \le i \le r)$ *be odd integers with* $3 \le m_i \le n$ and $\sum_{i=1}^{r} m_i = 2^k$ *for* $k \ge 3$. *Then there exists an odd* $(m_1, m_2, ..., m_r)$*-cycle system of* K_n *if and only if* $\sum_{i=1}^{r} m_i$ *divides* $|E(K_n)|$ *and n is odd.*

Proof. The necessity follows since each $(m_1, m_2, ..., m_r)$-cycles contains $\sum_{i=1}^{r} m_i$ edges, and each vertex in the m_i-cycle ($1 \le i \le r$) has even degree.

We begin with proving the sufficiency. Let the vertex set of K_n be Z_n, and all addition of vertex labels is done mod n. Obviously, $|E(K_n)| = n(n-1)/2$. Since n is odd, and 2^k divides $n(n-1)/2$, thus $n = s2^{k+1} + 1$, $s \ge 1$ and $k \ge 3$.

Now an odd $(sm_1, sm_2, \ldots, sm_r)$-cycle is an odd $(m_1, \ldots, m_1, m_2, \ldots, m_2, \ldots, m_r, \ldots, m_r)$-cycle with each m_i appearing s times. Note that $\sum_{i=1}^{r} m_i = 2^k$. By Lemma 2.6, the set $\{1, 2, \ldots, 3rs\}$ can be partitioned into rs triples $\{a_i, b_i, c_i\}$ such that N_3 of theses triples satisfy $a_i + b_i = c_i$ and the rest satisfy $a_i + b_i \pm 1 = c_i$, where $N_1 + N_3 = rs$. Next, using the consecutive positive integers $\{3rs + 1, 3rs + 2, \ldots, s2^k\}$ and the rs triples $\{a_i, b_i, c_i\}$ and repeatedly utilizing Lemmas 2.1 and 2.2, we find a proper labeling f_i of each cycle in the odd $(sm_1, sm_2, \ldots, sm_r)$-cycle such that $\bigcup_{i=1}^{r} f_i^*(E(C_{m_i})) = \{1, 2, \ldots, s2^k\}$. Hence the odd $(sm_1, sm_2, \ldots, sm_r)$-cycle is strongly proper and so K_n has an odd $(sm_1, sm_2, \ldots, sm_r)$-cycle system. Furthermore, since the odd $(sm_1, sm_2, \ldots, sm_r)$-cycle is the union of s edge-disjoint odd $(m_1, m_2, \ldots, m_r)$-cycles, it implies that K_n also has an odd $(m_1, m_2, \ldots, m_r)$-cycle system, and the proof is complete. □

Theorem 2.9. *Let* $\sum_{i=1}^{r} m_i = 2^k$ and assume 2^{k+1} *divides* $(n-2)$, *where* $k \ge 3$. *Then there exists an odd* $(m_1, m_2, \ldots, m_r)$*-cycle system of* $K_n - F$.

Proof. Let $F = \{(0, p2^k + 1), (1, p2^k + 2), \ldots, (p2^k, p2^{k+1} + 1)\}$. The remainder of the proof is analogous to that in Theorem 2.8 and so omitted. □

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