Trees with Unique Minimum Paired-Dominating Sets

Mustapha Chellali
Department of Mathematics
University of Blida
B.P. 270, Ouled Yaich, Blida, Algeria
e-mail: mchellali@hotmail.com

Teresa W. Haynes
Department of Mathematics
East Tennessee State University
Johnson City, TN 37614 USA
e-mail: haynes@mail.etsu.edu

Abstract

A paired-dominating set of a graph G is a dominating set of vertices whose induced subgraph has a perfect matching. We characterize the trees having unique minimum paired-dominating sets.
Keywords: Domination, total domination, paired-domination.
2000 Mathematics Subject Classification: 05C69.

1 Introduction

Paired-domination was introduced by Haynes and Slater in [9, 10] as a model for assigning backups to guards for security purposes. (See also [1, 3, 4, 11]). For example, in a graph model where each vertex represents a location for either a prisoner or a guard, we assume that a guard placed at a vertex can protect/guard that vertex and all its adjacent vertices. An assignment of guards so that every vertex is protected corresponds to selecting a dominating set for the graph. If we require each site be protected by a guard not on that site, or equivalently, that each site be protected and each guard be adjacent to another guard, we are seeking a total dominating set. If, in addition, guards must be assigned in pairs, that is, each guard must have a designated partner, then we are looking for a paired-dominating set.

Formally, for a graph $G = (V, E)$, a set S is a *dominating set* (respectively, *total dominating set*) if every vertex in $V - S$ (respectively, V) has a neighbor in S. The *domination number* $\gamma(G)$ (respectively, *total domination number* $\gamma_t(G)$) is the minimum cardinality of a dominating (respectively, total dominating) set of G. We call a dominating set of cardinality $\gamma(G)$ a $\gamma(G)$*-set* and use similar notation for other parameters. A set S is a *paired-dominating set* if it dominates V and the induced subgraph $\langle S \rangle$ contains at least one perfect matching. A *paired-dominating set* S *with matching* M is a dominating set $S = \{v_1, v_2, \ldots, v_{2t-1}, v_{2t}\}$ with independent edge set $M = \{e_1, e_2, \ldots, e_t\}$, where each edge e_i joins two elements of S, that is, M is a perfect matching (not necessarily induced) in the induced subgraph $\langle S \rangle$. If $v_j v_k = e_i \in M$, we say that v_j and v_k are *paired* in S. The *paired-domination number* $\gamma_{pr}(G)$ is the minimum cardinality of a paired-dominating set of G. Note that for any graph G without isolated vertices, a paired-dominating set is a total dominating set, and so $\gamma(G) \leq \gamma_t(G) \leq \gamma_{pr}(G)$. Each of these inequalities can be strict. Consider for example the subdivided star $S(K_{1,k})$ for $k \geq 2$, where $\gamma(S(K_{1,k})) = k < \gamma_t(S(K_{1,k})) = k + 1 < \gamma_{pr}(S(K_{1,k})) = 2k$. For more through treatment of domination and its variations, see [6, 7].

Gunther, Hartnell, Markus, and Rall [5] studied graphs with unique minimum dominating sets, and Haynes and Henning [8] studied graphs, called *UTD-graphs*, with unique minimum total dominating sets. In this paper, we consider the same problem for paired-domination. A graph G will be called a *unique paired-domination graph*, or just a *UPD-graph*, if it has a unique $\gamma_{pr}(G)$-set. For example, the graph mK_2 and the paths P_n with $n \equiv 0 \,(mod\, 4)$ are UPD-graphs. Note that each of these graphs has a unique matching associated with their unique minimum paired-dominating set, but for UPD-graphs, in general, the matching is not necessarily unique. (A *corona* $G \circ K_1$ is the graph formed from G by adding a new vertex v' for each $v \in V(G)$ and the edge vv'.) For instance, the corona $C_4 \circ K_1$ has a unique minimum paired-dominating set S, namely, the vertices of the C_4, but $\langle S \rangle$ has more than one possible matching. However, in Section 2, we show that for a tree with a unique minimum paired-dominating set S, $\langle S \rangle$ has exactly one perfect matching. Also, in Section 2 we observe properties of UPD-graphs and give a sufficient condition for a tree to be a UPD-tree. Then, in Section 3, we give a constructive characterization of UPD-trees. First we state some more definitions.

For notation and graph theory terminology, we in general follow [2, 6].

Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. For any vertex $v \in V$, the *open neighborhood of* v is the set $N(v) \doteq \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. The *private neighborhood* $\text{pn}(v, S)$ of $v \in S$ is defined by $\text{pn}(v, S) = N(v) - N(S - \{v\})$. For a vertex v in a rooted tree T, we let T_v denote the maximal subtree of T at rooted at v, i.e., T_v is the subtree induced by v and its descendants in T. A vertex of degree one is called an *endvertex* or a *leaf* and its neighbor is called a *support* vertex. A tree T is a *double star* if it contains exactly two vertices that are not leaves.

2 UPD-Trees

We begin with some straightforward observations.

Observation 1 *If v is a support vertex of a graph G, then v is in every paired-dominating set and in some minimum dominating set of G.*

Observation 2 *If T is a tree with $\gamma(T) = \gamma_{pr}(T)$, then no support vertex is paired with its leaf.*

Observation 3 *If T is a UPD-tree of order $n \geqslant 4$, then no support vertex is paired with a leaf in any matching of the unique $\gamma_{pr}(T)$-set.*

Proof. Let T be a tree of order $n \geqslant 4$ with a unique $\gamma_{pr}(T)$-set S and v a support vertex of T. By Observation 1, $v \in S$. Suppose that v is paired with its leaf u in a matching M associated with S. Clearly, u is the unique leaf adjacent to v. Let x be a vertex adjacent to v. Then $x \in S$, for otherwise, $(S - \{u\}) \cup \{x\}$ with matching $(M - \{uv\}) \cup \{vx\}$ is a second $\gamma_{pr}(T)$-set, contradicting the uniqueness of S. Therefore assume that x is paired with y in S. Then y has a private neighbor, say z, for otherwise, $S - \{u, y\}$ with matching $(M - \{uv, xy\}) \cup \{vx\}$ is a paired-dominating set of T with cardinality less than $|S| = \gamma_{pr}(T)$, a contradiction. But then $(S - \{u\}) \cup \{z\}$ with matching $(M - \{uv, xy\}) \cup \{vx, yz\}$ is a $\gamma_{pr}(T)$-set, again contradicting the uniqueness of S. ∎

As noted in the introduction, the unique minimum paired-dominating set of a UPD-graph may have more than one perfect matching. However, our next result shows that this is not the case for UPD-trees.

Proposition 4 *For any tree T, if S is a unique $\gamma_{pr}(T)$-set with a matching M, then M is the only perfect matching in $\langle S \rangle$.*

Proof. Let T be a tree with a unique $\gamma_{pr}(T)$-set S. Assume there are two perfect matching in $< S >$, namely, M and M'. Consider the spanning forest $G_S = (S, (M \cup M') - (M \cap M'))$. Then $\deg(v) \leqslant 2$ for any vertex v of G_S. Thus each component of G_S is an isolated vertex or a nontrivial path since T is a tree. But a leaf in G_S is unsaturated by either M or M'. Hence, G_S is a set of isolates implying that $M = M'$ and we are finished. ■

Note that the converse of Proposition 4 in not true; for example, consider the corona of the path P_3.

We have seen that for any graph G without isolated vertices, $\gamma(G) \leq \gamma_t(G) \leq \gamma_{pr}(G)$. Next we show that equality of the domination number and paired-domination number is a sufficient condition for a tree to be a UPD-tree. Note that this is not true in general, however. For example, a cycle C_4 has $\gamma(C_4) = \gamma_{pr}(C_4) = 2$ and C_4 is not a UPD-graph.

Theorem 5 *If T is a tree with $\gamma(T) = \gamma_{pr}(T)$, then T is a UPD-tree.*

Proof. We proceed by induction on the order n of T. Since there is no tree T of order 2 or 3 with $\gamma(T)=\gamma_{pr}(T)$, let $n \geq 4$. If $n \in \{4, 5\}$, it is a simple exercise to check that $\gamma(T) = \gamma_{pr}(T)$ if and only if T is a doublestar and hence, T is a UPD-tree, establishing the base case.

Let $n \geq 6$ and suppose that any tree T of order n', for $4 \leq n' < n$, such that $\gamma(T) = \gamma_{pr}(T)$ is a UPD-tree. Let T be a tree of order n such that $\gamma(T) = \gamma_{pr}(T)$ and S a $\gamma_{pr}(T)$-set.

Root T at a vertex r and let u be a support vertex at maximum distance from r. From Observation 1, we have $u \in S$. By our choice of u, every child of u is a leaf. Therefore, Observation 2 implies that u is paired in S with its parent, say w. Observation 2 also implies that every child of w besides u is a leaf. Since $\gamma(T) = \gamma_{pr}(T)$, T is not a star and so $deg(w) \geq 2$. If $w = r$, then T is a doublestar and hence is a UPD-tree.

Thus, assume that $w \neq r$ and let v be the parent of w in T. If $deg(v) = 1$, then again T is a double star and the result holds. Hence assume that $deg(v) \geqslant 2$. We consider two cases.

Case 1. $v \notin pn(w, S)$. Let $a \neq w$ be a vertex in $N[v] \cap S$ and let b be paired with a in S. (Note that a can be v.) If $x \in \{a, b, w\}$ has no private neighbor in $V - S$, then $S - \{x\}$ is a dominating set of T with cardinality less than $\gamma(T)$, a contradiction. Thus, w is a support vertex and each of a and b has a private neighbor in $V - S$. Let $T' = T - T_w$. Then T' has cardinality at least four. Since $S \cap V(T')$ is a (paired-) dominating set of T', it follows that $\gamma(T') \leq \gamma(T) - 2$ and $\gamma_{pr}(T') \leq \gamma_{pr}(T) - 2$. On the other hand, for any $\gamma_{pr}(T')$-set (respectively, $\gamma(T')$-set) S', $S' \cup \{u, w\}$ is a paired-dominating set (respectively, dominating set) of T. Thus, $\gamma(T) \leq \gamma(T') + 2$ and $\gamma_{pr}(T) \leq \gamma_{pr}(T') + 2$, and we have $\gamma(T') = \gamma(T) - 2 = \gamma_{pr}(T) - 2 = \gamma_{pr}(T')$. Applying our inductive hypothesis to T', T' is a UPD-tree. Note that $S \cap V(T')$ is the unique $\gamma_{pr}(T')$-set. Since u and w must be in every $\gamma_{pr}(T)$-set, we deduce that T is a UPD-tree.

Case 2. $v \in pn(w, S)$. Thus, v is not in S and hence, v is not a support vertex. By our choice of u and Observation 2, it must be the case that every child of v is in S. Hence, w is the only child of v, i.e., $deg(v) = 2$. Let z be the parent of v in T. Since neither z nor v is in S, z must be dominated by a vertex, say a, in S. Let b be the vertex paired with a in S. Then b must have a private neighbor in $V - S$, for otherwise, $S - \{b\}$ is a dominating set of T with cardinality less that $\gamma(T)$, a contradiction.

Let $T' = T - T_v$. It follows that T' has order at least four. Obviously, $S \cap V(T')$ (pair-) dominates T', so $\gamma(T') \leq \gamma(T) - 2$ and $\gamma_{pr}(T') \leq \gamma_{pr}(T) - 2$. If $S' \neq S - \{w, u\}$ (pair-) dominates T', then $S' \cup \{w, u\}$ (pair-) dominates T. Hence, $\gamma(T) \leq \gamma(T') + 2$ and $\gamma_{pr}(T) \leq \gamma_{pr}(T') + 2$. Therefore, $\gamma(T') = \gamma_{pr}(T')$. Applying our inductive hypothesis to T', T' is a UPD-tree. Note that $S \cap V(T')$ is the unique $\gamma_{pr}(T')$-set. Suppose that T has a $\gamma_{pr}(T)$-set $D \neq S$. Then $u \in D$ and $|D \cap V(T_w)| \geq 2$. Moreover, if $w \notin D$ or if $D \cap V(T') \neq S \cap V(T')$, then $|D \cap V(T - T_w)| > \gamma_{pr}(T') = \gamma_{pr}(T) - 2$, a contradiction. Hence, S is the unique $\gamma_{pr}(T)$-set and T is a UPD-tree. ∎

The converse of Theorem 5 is false as can be seen by considering the path P_8 which has a unique minimum paired-dominating set, but $\gamma_{pr}(P_8) = 4 > \gamma(P_8) = 3$. Also, we observe that for a tree T, while $\gamma(T) = \gamma_t(T) = \gamma_{pr}(T)$ implies that T is a UPD-tree, $\gamma_t(T) = \gamma_{pr}(T)$ is not sufficient to imply that T is a UPD-tree. For example, consider the tree T shown in Figure 1, where $\gamma_t(T) = \gamma_{pr}(T)$, but T is not a UPD-tree.

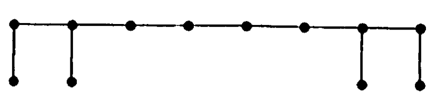

Figure 1: Tree T with $\gamma_t(T) = \gamma_{pr}(T) = 6$.

In the next section we give a characterization of UPD-trees. The UTD-trees were characterized in [8]. Although our examples of UPD-trees thus far have been UTD-trees, we note that not all UPD-trees are UTD-trees and vice versa. For example, the subdivided star of order $n \geq 5$ is a UTD-tree, but not a UPD-tree, since in any $\gamma_{pr}(T)$-set, a support vertex can be paired with a leaf. To see that not all UPD-trees are UTD-trees, consider the tree T in Figure 2, where $S = \{u_1, u_2, u_3, u_4, u_7, u_8\}$ is the unique $\gamma_{pr}(T)$-set, while both S and $\{u_1, u_2, u_3, u_6, u_7, u_8\}$ are $\gamma_t(T)$-sets.

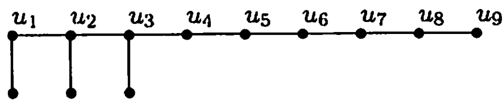

Figure 2: Tree T is a UPD-tree, but not a UTD-tree.

3 Characterization

In this section, we present a constructive characterization for UPD-trees. For this purpose, we define a family of trees as follows.

Let $\mathcal{T}$ the collection of trees T that can be obtained from a sequence T_1, T_2, ..., T_k ($k \geq 1$) of trees, where T_1 is the path P_4 with support vertices x and y, $T = T_k$, and, if $k \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the following operations. Let $A(T_1) = \{x, y\}$ and $B(T_1) = \emptyset$. Let H be the path P_4 with support vertices u and w.

- **Type-1 operation:** Attach a leaf to any vertex in $A(T_i)$. Let $A(T_{i+1}) = A(T_i)$ and $B(T_{i+1}) = B(T_i)$.
- **Type-2 operation:** Attach a copy of H by adding edge uv, where $v \in A(T_i) \cup B(T_i)$. Let $A(T_{i+1}) = A(T_i) \cup \{u, w\}$ and $B(T_{i+1}) = B(T_i)$.
- **Type-3 operation:** Attach a copy of H by adding an edge from a leaf of H to a vertex in $V(T_i) - A(T_i)$. Let $A(T_{i+1}) = A(T_i) \cup \{u, w\}$ and $B(T_{i+1}) = B(T_i)$.
- **Type-4 operation:** Attach a copy of H by adding a new vertex z and edges uz and zv, where $v \in V(T_i)$. Let $A(T_{i+1}) = A(T_i) \cup \{u, w\}$ and $B(T_{i+1}) = B(T_i) \cup \{z\}$.

We begin with a lemma.

Lemma 6 *If $T \in \mathcal{T}$, then*

(a) $A(T) \cap B(T) = \emptyset$,

(b) $A(T)$ *is the unique* $\gamma_{pr}(T)$*-set, and*

(c) *if* $v \in B(T)$, *then* v *is not the only private neighbor of any vertex in* $A(T)$.

Proof. Parts (a) and (c) follow directly from the way a tree $T \in \mathcal{T}$ is constructed. To prove part (b), let $T \in \mathcal{T}$. Then T can be obtained from a sequence $T_1, T_2, \ldots, T_k$ ($k \geq 1$) of trees, where T_1 is the path P_4, $T = T_k$, and, if $k \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the four operations defined. We use the terminology of the construction for sets $A(T)$, $B(T)$, and the graph H with support vertices u and w. If $k = 1$, then $T = P_4$ and clearly, $A(T)$ is the unique $\gamma_{pr}(T)$-set. This establishes our basis case.

Assume that the result holds for all trees $T \in \mathcal{T}$ that can be constructed from a sequence of length at most $k-1$, and let $T' = T_{k-1}$. Applying our inductive hypothesis to $T' \in \mathcal{T}$ shows that $A(T')$ is the unique $\gamma_{pr}(T')$-set. Clearly, if T is obtained from T' using a Type-1 operation, then $\gamma_{pr}(T) = \gamma_{pr}(T')$ and $A(T') = A(T)$ is the unique $\gamma_{pr}(T)$-set.

Note that if T is obtained from T' using a Type-2, 3, or 4 operation, then $A(T') \cup \{u, w\}$ is a paired-dominating set of T. Thus, $\gamma_{pr}(T) \leq \gamma_{pr}(T') + 2$.

Suppose that T is obtained from T' using a Type-2 or Type-4 operation. Since u and w are support vertices in T, by Observation 1, u and w are in every $\gamma_{pr}(T)$-set. Let v be the vertex of T' adjacent to u. If $\gamma_{pr}(T) < \gamma_{pr}(T') + 2$, then $\gamma_{pr}(T) = \gamma_{pr}(T')$ implying that there is a $\gamma_{pr}(T' - v)$-set S' of cardinality $\gamma_{pr}(T') - 2$. No neighbor of v is in S' for otherwise S' pair-dominates T', a contradiction. Hence, $S' \cup \{v, x\}$ where $x \in N(v)$ is a $\gamma_{pr}(T')$-set, and since $A(T')$ is the unique $\gamma_{pr}(T')$-set, $A(T') = S' \cup \{v, x\}$. But from the construction, each vertex in $A(T')$ has a private neighbor in $V(T') - A(T')$ with respect to $A(T')$, contradicting our assumption that S' pair-dominates $T' - v$. Hence, $\gamma_{pr}(T') = \gamma_{pr}(T) - 2$ and it follows that $A(T)$ is the unique $\gamma_{pr}(T)$-set.

If T is obtained from T' using a Type-3 operation, then w is a support vertex in T and by Observation 1 must be in any $\gamma_{pr}(T)$-set S. Since v is not in $A(T')$, it follows that the common neighbor of u and v is not dominated by a $\gamma_{pr}(T')$-set and so w must be paired with u in any $\gamma_{pr}(T)$-set. A similar argument as before shows that $A(T)$ is the unique $\gamma_{pr}(T)$-set.
■

Theorem 7 *A tree T is a UPD-tree if and only if $T = P_2$ or $T \in \mathcal{T}$.*

Proof. If $T = P_2$, then T is a UPD-tree. Lemma 6 states that any tree $T \in \mathcal{T}$ is a UPD-tree.

To prove the converse, we proceed by induction on the order n of a UPD-tree T. If $n = 2$, then $T = P_2$. Since there is no tree T of order 3 with a unique $\gamma_{pr}(T)$-set, let $n = 4$. Then Observation 3 implies that $T = P_4$ and hence, $T \in \mathcal{T}$. If $n = 5$, then again Observation 3 implies that T is a double star. Thus, T can be obtained from a P_4 using a Type-1 operation, and so $T \in \mathcal{T}$.

Let $n \geqslant 6$, and assume that any UPD-tree T' of order $n' < n$ is in $\mathcal{T}$. Let T be a UPD-tree of order n with the unique $\gamma_{pr}(T)$-set S and an associated matching M.

If any support vertex, say v, of T is adjacent to two or more leaves, then let T' be the tree obtained from T by removing a leaf adjacent to v. Then v is still a support vertex in T', and Observation 1 implies that $v \in S$ and v is in every $\gamma_{pr}(T')$-set. It follows that $S \cap V(T')$ is the unique $\gamma_{pr}(T')$-set and hence, T' is a UPD-tree of order at least five. Applying our inductive hypotheses to T', we have that $T' \in \mathcal{T}$. Since T can be formed from T' by a Type-1 operation, $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is adjacent to exactly one leaf.

Root T at a vertex r and let u be a support vertex at maximum distance from r and u' be the leaf adjacent to u. Then every child of u must be a leaf and so $deg(u) = 2$. Observations 1 and 3 imply that $u \in S$ and is paired with its parent, say w, and that $deg(w) \geq 2$. Moreover, Observation 3 implies that every child of w except u is a leaf. Thus, $2 \leq deg(w) \leq 3$. Since $n \geq 6$ and we have assumed that every support vertex is adjacent to exactly one leaf, $w \neq r$. Thus, let v be the parent of w in the rooted tree T. Again since every support vertex is adjacent to exactly one leaf, v is not a leaf.

We consider two cases.

Case 1. $v \in S$. Then v is paired with a vertex, say y, in S. Now w must be a support vertex, for otherwise, $(S - \{w\}) \cup \{u'\}$ with matching $(M - \{uw\}) \cup \{uu'\}$ is a $\gamma_{pr}(T)$-set, contradicting the uniqueness of S.

Let $T' = T - T_w$. Observation 3 implies that y has degree at least two in T, and so T' has order at least three. Obviously, $S - \{u, w\}$ with matching $M - \{uw\}$ is a paired-dominating set of T', and so $\gamma_{pr}(T') \leq \gamma_{pr}(T) - 2$. If T' has $\gamma_{pr}(T')$-set $S' \neq S \cap V(T')$ with an associated matching M', then

$S' \cup \{w, u\}$ with matching $M' \cup \{wu\}$ is either a paired-dominating set of T with cardinality less than $\gamma_{pr}(T)$ or a $\gamma_{pr}(T)$-set not equal to S. In both cases, we reach a contradiction. Thus, $S \cap V(T')$ is the unique $\gamma_{pr}(T')$-set. Applying our inductive hypothesis, we have that $T' \in \mathcal{T}$. From Lemma 6, $v \in A(T')$. Thus, T can be obtained from T' using a Type-2 operation, and so $T \in \mathcal{T}$.

Case 2. $v \notin S$. By Observation 1, v is not a support vertex. From our choice of u and Observation 3, it follows that every child of v is in S.

First assume that $v \in pn(w, S)$, that is, w is the only neighbor of v in S. Then w is the only child of v, i.e., $deg(v) = 2$. Let y be the parent of v in T. Since $y \notin S$, there is a vertex, say a, in $N(y) \cap S$ to dominate y. Let b be paired with a in S. Now b must have a private neighbor in $V - S$, for otherwise, $(S - \{b\}) \cup \{y\}$ with matching $(M - \{ab\}) \cup \{ay\}$ is a $\gamma_{pr}(T)$-set different from S, a contradiction. Let $T' = T - T_v$. The tree T' has order at least four and it is straightforward to see that $S \cap V(T')$ is the unique $\gamma_{pr}(T')$-set. Applying our inductive hypothesis to T', we have $T' \in \mathcal{T}$. If $deg(w) = 2$, then T can be obtained from T' using a Type-3 operation. If $deg(w) = 3$, then w is a support vertex and T can be obtained from T' using a Type-4 operation. In either case, $T \in \mathcal{T}$.

Next assume that $v \notin pn(w, S)$ and let a be another neighbor of v in S, where b is the vertex in S paired with a. Observation 3 implies that $deg(b) \geq 2$. Now w must be a support vertex, for otherwise, $(S - \{w\}) \cup \{u'\}$ with matching $(M - \{wu\}) \cup \{uu'\}$ is a second $\gamma_{pr}(T)$-set, a contradiction. A similar argument shows that each of a and b must also have a private neighbor in $(V - S) - \{v\}$. Note also that for any child y of v, y is in S and hence, T_y is isomorphic to T_w.

If $deg(v) = 2$, then let $T' = T - T_v$. Then T' has order at least four. If $S' \neq S - \{u, w\}$ is a $\gamma_{pr}(T')$-set, then $S' \cup \{u, w\}$ is either a second $\gamma_{pr}(T)$-set or a paired-dominating set of T with cardinality less than $\gamma_{pr}(T)$, a contradiction in either case. Hence, T' is a UPD-tree. Applying our inductive hypothesis to T', we have $T' \in \mathcal{T}$. Since T can be obtained from T' using a Type-4 operation, $T \in \mathcal{T}$.

If $deg(v) \geq 3$, then let $T' = T - T_w$. It is straightforward to see that T' has order at least four and, as before, $S \cap V(T')$ is the unique $\gamma_{pr}(T')$-set. Applying our inductive hypothesis to T', we have $T' \in \mathcal{T}$. From the way T' is constructed, it follows that $v \in B(T')$. Hence, T can be obtained from T' using a Type-2 operation implying that $T \in \mathcal{T}$. ■

References

[1] M. Blidia, M. Chellali, and T. W. Haynes, Characterizations of trees with equal paired and double domination numbers, submitted for publication.

[2] G. Chartrand and L. Lesniak, *Graphs & Digraphs: Third Edition*, Chapman & Hall, London, 1996.

[3] M. Chellali and T. W. Haynes, Paired and double domination in graphs, to appear in *Utilitas Math.*

[4] S. Fitzpatrick and B. Hartnell, Paired-domination. *Discuss. Math.-Graph Theory* 18 (1998) 63–72.

[5] G. Gunther, B. Hartnell, L. R. Markus, and D. Rall, Graphs with unique minimum dominating sets. *Congr. Numer.* 101 (1994), 55–63.

[6] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[7] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[8] T. W. Haynes and M. A. Henning, Trees with unique minimum total dominating sets, *Discuss. Math.-Graph Theory* 22 (2002) 233–246.

[9] T. W. Haynes and P. J. Slater, Paired-domination in graphs. *Networks* 32 (1998) 199–206.

[10] T. W. Haynes and P. J. Slater, Paired-domination and the paired-domatic number. *Congr. Numer.* 109 (1995) 65 72.

[11] K. E. Proffitt, T. W. Haynes, and P. J. Slater, Paired-domination in grid graphs. *Congr. Numer.* 150 (2001) 161 172.