Embeddings of Steiner Quadruple Systems using Extensions of Linear Spaces

Chester W.J. Liu
Department of International Business, Chang Jung University, 396 Sec.1 Chang Jung Road, Kway Jen, Tainan, TAIWAN 711
cwjliu@mail.cju.edu.tw
and
Peter R. Wild
Department of Mathematics, Royal Holloway, University of London, Egham Hill, Egham, Surrey, TW20 0EX, UK
p.wild@rhbnc.ac.uk

Abstract

Using a linear space on v points with all block sizes $|B| \equiv 0$ or 1 (mod 3), Doyen and Wilson construct a Steiner triple system on $2v+1$ points that embeds a Steiner triple system on $2|B|+1$ points for each block B. We generalise this result to show that if the linear space on v points is extendable in a suitable way, there is a Steiner quadruple system on $2v+2$ points that embeds a Steiner quadruple system on $2(|B|+1)$ points for each block B.

Key words: Steiner Quadruple System, linear space, embedding, substructure.

1 Introduction

A *Steiner system* $S(t,k;v)$ is a pair $(P,\mathcal{A})$ where P is a v-set and $\mathcal{A}$ is a collection of k-subsets of P such that every t-subset of P is contained in exactly one member of $\mathcal{A}$. The elements of P are called points and the elements of $\mathcal{A}$ are called blocks. The cases $t=2$, $k=3$ and $t=3$, $k=4$ are called *Steiner triple system* (STS) and *Steiner quadruple system* (SQS) respectively. An STS with $|P|=v$ is said to be of *order* v and is referred to as an STS(v). Similarly an SQS with $|P| = v$ is referred to as an SQS(v). Kirkman [4] proved that there exists an STS(v) if and only if $v \equiv 1$ or 3 (mod 6). Hanani [2] proved that there exixts an SQS(v) if and only if $v \equiv 2$ or 4 (mod 6). If K is a set of integers with $k \geq 2$ for all $k \in K$, an $S(t, K; v)$ is a pair $(P, \mathcal{A})$ where P is a point-set with $|P| = v$ and $\mathcal{A}$ is a block-set with block sizes $k \in K$ such that every t distinct points of P is contained in exactly one block of $\mathcal{A}$. The case $t = 2$ is called a *linear space.*

If $(P, \mathcal{A})$ and $(Q, \mathcal{B})$ are two Steiner systems such that $P \subseteq Q$ and $\mathcal{A} \subseteq \mathcal{B}$, then we say $(P, \mathcal{A})$ is embedded in $(Q, \mathcal{B})$ and that $(Q, \mathcal{B})$ contains $(P, \mathcal{A})$ as a substructure. If an STS(u) is embedded in an STS(v) then $v \geq 2u + 1$ [1]. It is immediate from this that if an SQS($u + 1$) is embedded in an SQS(v) then $v \geq 2u + 2$. Doyen and Wilson [1] have shown that given a linear space on u points with each block B of size $|B| \equiv 0$ or 1 (mod 3) there exists an STS($2u + 1$) embedding an STS($2|B| + 1$) for each block B. In this paper, we generalize Doyen and Wilson's construction under the assumption that there is a linear space with parameters as described above that is extendable to an $S(3, K; u + 1)$ by adjoining a point p such that the blocks of $S(3, K; u + 1)$ not containing p have size $l \equiv 2$ or 4 (mod 6). We construct an SQS($2u + 2$) that contains a sub-SQS($2|B| + 2$) for each block B of the linear space. The simplest example of such a linear space is an STS(u) that is extendable to an SQS($u + 1$) (see Liu and Wild [3]).

2 Preliminaries

We begin by establishing some properties of the substructures that are ingredients of our construction.

A *groop divisible design* on v points with groop size g and block size k is called a *t-GD*$[k, g; v]$ if every subset of t distinct points that contains no two points from the same groop is contained in exactly one block. A *transversal quadrangle*, $[x, y, z, w]$, is the 3-$GD[4, 2; 8]$ that has the four groops $\{x, x'\}$, $\{y, y'\}$, $\{z, z'\}$, $\{w, w'\}$ and contains the block $\{x, y, z, w\}$. Similarly $[a_1, \ldots, a_l]$ denotes a 3-$GD[4, 2; 2l]$ which has l groops $\{a_1, a_1'\}, \ldots,$ $\{a_l, a_l'\}$.

The internal structure of an SQS, $(P, \mathcal{A})$, at a point $p \in P$ is an incidence structure with point set $P' = P \backslash \{p\}$ and block set $\mathcal{A}'$ consisting of those triples of points of P' which together with p form a block belonging to $\mathcal{A}$. The structure $(P', \mathcal{A}')$ is an STS. An SQS is an extension of an STS if the STS is isomorphic to the internal structure of the SQS at some point of the SQS. An extension of an STS(u) is an SQS($u + 1$). Similarly, an

$S(3,K;u+1)$ is an extension of a linear space on u points if the linear space on u points is isomorphic to the internal structure of the $S(3,K;u+1)$ at some point of the $S(3,K;u+1)$. If an $S(3,K;u+1)$ is an extension of a linear space $S(2,K';u)$ then $k+1 \in K$ for each $k \in K'$.

Lemma 1 If there exists an SQS(l), then there exists a 3-$GD[4,2;2l]$ which contains the SQS(l) as a substructure.

Proof Let the SQS(l) be $(P,\mathcal{A})$. Let $Q = P \cup P'$ where $P' = \{x' \mid x \in P\}$ is a disjoint copy of P. For every block in SQS(l), say $\{x,y,z,w\}$, we make a copy of the transversal quadrangle $[x,y,z,w]$. The 8 quadruples of $[x,y,z,w]$ are

$$\{x,y,z,w\},\{x',y',z',w'\},\{x',y,z,w'\},\{x,y',z,w'\},$$

$$\{x,y,z',w'\},\{x',y',z,w\},\{x',y,z',w\},\{x,y',z',w\},$$

and $\mathcal{B}$ is the union of all such collections. Since $\{x,y,z,w\}$ is a block of $[x,y,z,w]$, $\mathcal{A} \subseteq \mathcal{B}$ and the SQS(l) is a sub structure of $(Q,\mathcal{B})$. The points of Q are partitioned into $|P|$ groops of size 2, i.e. , $\{x,x'\}$ for $x \in P$. We note that for each $\{x,y,z,w\} \in \mathcal{A}$ no block of $[x,y,z,w]$ contains two points from the same groop. Thus no block in $\mathcal{B}$ contains a groop. Let α, β, γ be three distinct points of Q, no two in a groop. Then α, β, γ correspond to three distinct points of P, say x, y, z determined by their corresponding groops. Now x, y, z belong to a unique block $\{x,y,z,w\}$ of $\mathcal{A}$ and so α, β, γ belong to a unique block of the 3-$GD[4,2;8]$, $[x,y,z,w]$. Hence α, β, γ belong to a block of $\mathcal{B}$. Now $\mathcal{B}$ has

$$8 \cdot |\mathcal{A}| = 8 \cdot \frac{l(l-1)(l-2)}{24} = \frac{l(l-1)(l-2)}{3}$$

blocks and each block has 4 triples. Now there are l groops and so $l(2l-2)$ triples that contain a groop. So there are

$$\binom{2l}{3} - 2(l-1)\cdot l = \frac{2l(2l-1)(2l-2)}{3\cdot 2\cdot 1} - \frac{6(2l-2)l}{3\cdot 2} = \frac{l(l-1)(l-2)}{3}\cdot 4$$

triples α, β, γ of Q, no two in a groop. It follows that each such triple is on a unique block in $\mathcal{B}$ and $(Q,\mathcal{B})$ is a 3-$GD[4,2;2l]$.

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Lemma 2 A 3-$GD[4,2;2l]$ can be uniquely embedded in an SQS($2l$).

Proof We can complete a 3-$GD[4,2;2l]$ to an SQS($2l$) by just adding to the blocks of the 3-$GD[4,2;2l]$, $\binom{l}{2}$ blocks of type $\{x, x', y, y'\}$ for every pair $\{x, x'\}$, $\{y, y'\}$ of groops of the 3-$GD[4,2;2l]$. Indeed every triple of points that contains a groop is of type $\{x, x', y\}$ where $\{x, x'\}$, $\{y, y'\}$ are distinct groops and will appear in exactly one such quadruple. Moreover, any triple contained in one of these $\binom{l}{2}$ blocks contains a groop. Every other triple of points appears in exactly one block of the 3-$GD[4,2;2l]$. The $\frac{l(l-1)(l-2)}{3}$ blocks of the 3-$GD[4,2;2l]$ together with these $\binom{l}{2}$ blocks make up the $\frac{l(l-1)(l-2)}{3} + \binom{l}{2} = \frac{l(l-1)(2l-1)}{6}$ blocks of the SQS($2l$). Further if an SQS($2l$) contains a 3-$GD[4,2;2l]$ then the block of the SQS($2l$) containing x, x', y must be of the form $\{x, x', y, y'\}$ since, for any point $w \notin \{x, x', y, y'\}$, x, y,w belong to a block of the 3-$GD[4,2;2l]$. So the embedding is unique.

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Corollary 1 An SQS($2l$) contains a 3-$GD[4,2;2l]$ if and only if its points can be partitioned into l groops of 2 points such that the $\binom{l}{2}$ unions of pairs of groops are blocks of the SQS($2l$).

Proof That such a partitioning of the points exists if the SQS($2l$) contains a 3-$GD[4,2;2l]$ follows from the Lemma 2. Conversely, if there is such a partitioning then any three points of the SQS($2l$) belonging to distinct groops belong to a unique block which is not one of the unions of groops. It follows that the blocks of SQS($2l$) which are not a union of groops form the blocks of a 3-$GD[4,2;2l]$.

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We remark that if the 3-$GD[4,2;2l]$ comes from an SQS(l) as in Lemma 1, then the SQS($2l$) of Lemma 2 contains a sub SQS(l).

3 Main Result

We now present our construction.

Suppose there exists $(P, \mathcal{A})$ an $S(3, K; u+1)$ with such a point p such that each block not containing p has size $l \equiv 2$ or 4 (mod 6) and each block containg p has size $k+1$ with $k \equiv 0$ or 1 (mod 3), i.e. there exists an SQS(l) for each $l \in K_1$ and there exists an SQS($2(k+1)$) for each $k+1 \in K_2$ where K_1 is the set of block sizes of blocks not containing p and K_2 is the set of block sizes of blocks that do contain p. We construct $(Q, \mathcal{B})$ an SQS($2u+2$), embedding an SQS($2(k+1)$) as a substructure for each $k+1 \in K_2$.

The construction :
Let $S(2,K';u)$ be the internal structure of $S(3,K;u+1)$ at p. Let the $S(2,K';u)$ and the $S(3,K;u+1)$ be $(P_p,\mathcal{A}')$ and $(P,\mathcal{A})$ respectively. Consider the set $Q = P_p \cup P'_p \cup \{\infty\} \cup \{p\}$ of cardinality $2u+2$ where $P'_p = \{x' | x \in P_p\}$ is a disjoint copy of P_p. Let $\mathcal{A} = \mathcal{A}_{K_2} \cup \mathcal{A}_{K_1}$ be such that $\mathcal{A}_{K_2}$ and $\mathcal{A}_{K_1}$ are the sets of blocks of $S(3,K;u+1)$ containing p and not containing p respectively. Thus $\mathcal{A}_{K_2} = \{\{p\} \cup A | A \in \mathcal{A}'\}$. As there exists an SQS$(2(k+1))$ for each $k+1 \in K_2$, then necessarily $2(k+1) \equiv 2$ or $4 \pmod 6$ and the block sizes of the $S(2,K;u)$ have $k \equiv 0$ or $1 \pmod 3$. Now by Doyen and Wilson's construction [1], there exists an STS$(2u+1)$, $(Q_p,\mathcal{B}')$ say, containing, for every $A \in \mathcal{A}'$ of size k, a sub STS$(2k+1)$, where $Q_p = P_p \cup P'_p \cup \{\infty\}$. Each such sub STS contains, by construction, the blocks $\{x, x', \infty\}$ for $x \in A$. Moreover each sub STS may be a Steiner triple system of our choosing for the given parameters. Since there exists an SQS$(2(k+1))$ for each $k+1 \in K_2$, we may choose each such STS$(2k+1)$ to be isomorphic to an internal structure of an SQS$(2(k+1))$. Hence a sub SQS$(2(k+1))$ on the point set $Q_p \cup \{p\}$ can be constructed from every sub STS$(2k+1)$ contained in $(Q_p,\mathcal{B}')$ by forming the following two types of quadruples:

(a) add p to each block of the sub STS$(2k+1)$. Let $\mathcal{B}'_A$ denote the collection of these blocks containing p corresponding to a block $A \in \mathcal{A}'$;

(b) the quadruples of points, not containing p, of the sub STS$(2k+1)$ that are preimages of blocks of the SQS$(2(k+1))$ under the isomorphism between the sub STS$(2k+1)$ and the internal structure of the SQS$(2(k+1))$. Let $\mathcal{B}_A$ denote the collection of these blocks corresponding to block $A \in \mathcal{A}'$.

Then $\mathcal{B}'_A \cup \mathcal{B}_A$ are the blocks of a sub SQS$(2(k+1))$ with point set consisting of the point p adjoined to the points of the sub STS$(2k+1)$ determined by A.

For each block $B = \{a_1, \ldots, a_l\}$ of $S(3,K;u+1)$ not containing p, $l \equiv 2$ or $4 \pmod 6$, as there exists an SQS(l). Construct a 3-$GD[4,2;2l]$, $[a_1, \ldots, a_l]$, corresponding to B by making a copy of $[x,y,z,w]$ for every block $\{x,y,z,w\}$ in SQS(l) as in Lemma 1. Let $\mathcal{B}_B$ denote the collection of blocks of this 3-$GD[4,2;2l]$ for each $B \in \mathcal{A}_{K_1}$. Then we put

$$\mathcal{B} = (\cup_{A\in\mathcal{A}'}\mathcal{B}'_A) \cup (\cup_{A\in\mathcal{A}'}\mathcal{B}_A) \cup (\cup_{B\in\mathcal{A}_{K_1}}\mathcal{B}_B)$$

and show that $(Q,\mathcal{B})$ is an SQS$(2u+2)$ in the following theorem.

Theorem 3.1 If there exists an $S(3, K; u+1)$ with a point p such that there exists an SQS(l) for each $l \in K_1$ and there exists an SQS($2(k+1)$) for each $k+1 \in K_2$ where K_1 is the set of block sizes of blocks not containing p and K_2 is the set of block sizes of blocks that do contain p, then there exists an SQS($2u+2$) embedding a sub SQS($2(k+1)$) for each $k+1 \in K_2$. If further, the SQS($2(k+1)$)s for $k+1 \in K_2$ have the property that there is a partition of the points into $k+1$ groups of size 2 such that each of the $\binom{k+1}{2}$ unions of two groups is a block, then the SQS($2u+2$) embeds a sub SQS($2l$) for each $l \in K_1$.

Proof Let $(Q, \mathcal{B})$ arise from the construction described above. We partition the triples of points in Q into the following classes,

(i) triples that contain p ;

(ii) triples that contain ∞ but not p ;

(iii) triples from $Q \setminus \{\infty, p\}$.

We show that every triple of points in Q belongs to a unique block in $\mathcal{B}$:

(i) a triple containing p is on the unique block in $\cup_{A \in \mathcal{A}'} \mathcal{B}'_A$ corresponding to the unique block of STS($2u+1$) containing the two points other than p in the triple;

(ii) triples containing ∞ of the form $\{x, y, \infty\}$, $\{x', y', \infty\}$, $\{x, y', \infty\}$ where $x, y \in P_p$ are only on the corresponding block of type (b) which does not contain p in $\cup_{A \in \mathcal{A}'} \mathcal{B}_A$; the remaining possible form $\{x, x', \infty\}$ of a triple containing ∞ where $x \in P_p$ is on the unique block $\{x, x', \infty, p\}$ of type (a) in $\cup_{A \in \mathcal{A}'} \mathcal{B}'_A$;

(iii) triples of types $\{x, y, z\}$, $\{x', y', z'\}$, $\{x, y, z'\}$, $\{x, y', z'\}$ where $x,y,z \in P_p$ are of the form $\{\alpha, \beta, \gamma\}$ where α, β, γ are three distinct points of $P_p \cup P'_p$, no two in a groop (i.e. a pair $\{x, x'\}$). Then α, β, γ correspond to three distinct points of P_p, say x, y, z :

- if x, y, z are not on a line in the $S(2, K; u)$, then x, y, z belong to a unique block $\{a_1, \cdots, a_l\}$ in $\mathcal{A}_{K_1}$ and so α, β, γ belong to exactly one block of the 3-$GD[4, 2; 2l]$, $[a_1, \cdots, a_l]$ in $\cup_{B \in \mathcal{A}_{K_1}} \mathcal{B}_B$;
- if x, y, z are on a line $A_0 \in \mathcal{A}'$ in the $S(2, K; u)$, then x, y, z are contained in the unique block $A \cup \{p\}$ in $\mathcal{A}_{K_2}$ and so
  - $\{\alpha, \beta, \gamma\}$ is contained in exactly one quadruple of type (b) in $\cup_{A \in \mathcal{A}'} \mathcal{B}_A$ if $\{\alpha, \beta, \gamma\}$ is not a line in the sub STS determined by A_0 ;

- $\{\alpha, \beta, \gamma\}$ is contained in exactly one quadruple of type (a) in $\cup_{A \in \mathcal{A}'} \mathcal{B}'_A$ if $\{\alpha, \beta, \gamma\}$ is a line in the sub STS determined by A_0 ;

the triples of types $\{x, x', y\}$ and $\{x, x', y'\}$ where $x, y \in P_p$, contain a groop:

- x, x', y are 3 non-collinear points of the sub STS determined by the unique line of the $S(2, K; u)$ through x and y and so $\{x, x', y\}$ is contained in exactly one quadruple of type (b) in $\cup_{A \in \mathcal{A}'} \mathcal{B}_A$;
- x, x', y' are 3 non-collinear points of the sub STS determined by the unique line of the $S(2, K; u)$ through x and y and so $\{x, x', y'\}$ is contained in exactly one quadruple of type (b) in $\cup_{A \in \mathcal{A}'} \mathcal{B}_A$.

It follows that each triple of points of Q is contained in exactly one quadruple of $\mathcal{B}$ and so $(Q, \mathcal{B})$ is an SQS$(2u+2)$. By the construction of quadruples of types (a) and (b) this SQS contains a sub SQS$(2(k+1))$ for each block $A \cup \{p\} \in \mathcal{A}_{K_2}$ of size $k+1$.

Suppose now that each SQS$(2(k+1))$ for $k+1 \in K_2$ has a partition of the points into $k+1$ groops of 2 points such that each of the $\binom{k+1}{2}$ unions of two groops is a block. We may assume that these groops correspond (under the isomorphism referred to above) to $\{\infty, p\}$ and $\{x, x'\}$ for $x \in A$, where A is the block of $\mathcal{A}'$ corresponding to the SQS$(2(k+1))$. Then the quadruples that do not contain ∞ in type (b) include all those of the form $\{x, x', y, y'\}$, i.e. the union of two groops. Let $\mathcal{C}_A$ denote the collection of these blocks determined by line $A \in \mathcal{A}'$. Let the quadruples of the form $\{x, x', \infty, p\}$ from type (a) corresponding to $A \in \mathcal{A}'$ be denoted as $\mathcal{F}_A$. Write p' instead of ∞. Then $\mathcal{C}_A \cup \mathcal{F}_A$ contains $\{x, x', y, y'\}$ for every pair $x, y \in (\{p\} \cup A)$ and each SQS$(2(k+1))$ corresponding to a block $A \cup \{p\}$ in $\mathcal{A}_{K_2}$ of size $k+1$ has the property that there is a partition of the points into $k+1$ groops of 2 such that each of the $\binom{k+1}{2}$ unions of two such groops is a block $\{x, x', y, y'\}$ in $\mathcal{B}$. Consider a block $B = \{a_1, \ldots, a_l\}$ of $S(3, K; u+1)$ not containing p. Now $\mathcal{B}_B$ and the blocks of type $\{x, x', y, y'\}$ from the set $\cup_{A \in \mathcal{A}'} \mathcal{C}_A$ for every pair $x, y \in \{a_1, \ldots, a_l\}$, form a sub SQS$(2l)$ by Lemma 2. This completes the proof of the theorem.

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4 Conclusion

Doyen and Wilson's construction of an STS containing sub STS using a linear space has been used to construct an SQS containing sub SQS, which is under the assumption that the linear space is extendable to an $S(3, K; u+1)$ with suitable properties. In general, an $S(3, K; u+1)$ is necessarily the extension of an $S(2, K; u)$, but the latter need not have the properties required by Doyen and Wilson's construction. A solution to the question of the existence of extensions of linear spaces will help to deal with the embedding problem of SQS.

Our construction has used 3-$GD[4, 2; 2l]$, $[a_1, \ldots, a_l]$, arising from SQS(l) as in Lemma 1. These exist for $l \equiv 2$ or 4 (mod 6) and so we have placed this restriction on block sizes $l \in K_1$ of $S(3, K; u+1)$. This restriction is not essential and the construction works whenever there exists a 3-$GD[4, 2; 2l]$. However, in general, the question remains open whether there exists 3-$GD[4, 2; 2l]$ for $l \not\equiv 2$ or 4 (mod 6).

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