Counting rooted spanning forests in complete multipartite graphs

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Abstract

Jin and Liu discovered an elegant formula for the number of rooted spanning forests in the complete bipartite graph K_{a_1,a_2}, with b_1 roots in the first vertex class and b_2 roots in the second vertex class. We give a simple proof to their formula, and a generalization for complete m-partite graphs, using the multivariate Lagrange inverse.

Y. Jin and C. Liu [4] give a formula for $f(m,l;n,k)$, the number of spanning forests of the labelled complete bipartite graph $K_{n,m}$, where in the forest every tree is rooted, there are k roots in the first vertex class (among the n vertices) and l roots in the second vertex class (among the m vertices), and the trees in the forest are not ordered. They discovered the elegant formula

$$f(m,l;n,k) = \binom{m}{l}\binom{n}{k} n^{m-l-1} m^{n-k-1}(km + ln - lk). \qquad (1)$$

The goal of the present note is generalization of (1) from complete bipartite to complete multipartite graphs, through a simple proof using the multivariate Lagrange inverse.

Let $f(a_1,b_1;...;a_m,b_m)$ denote the number of spanning forests of the labelled complete multipartite graph $K_{a_1,a_2,...,a_m}$, where in the forest every tree is rooted, there are b_i roots in the i^{th} vertex class for $i = 1,2,...,m$, and the trees in the forest are not ordered. Let $w_i(t_1,...,t_m)$ denote the multivariate exponential generating function (EGF) of the numbers

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$f(a_1, 0; ...; a_i, 1; ...; a_m, 0)$ (the number of rooted spanning trees of the complete multipartite graph $K_{a_1, a_2, ..., a_m}$, if the root has to be in the i^{th} class), i.e.

$$w_i(t_1, ..., t_m) = \sum_{a_1=0}^{\infty} \cdots \sum_{a_i=1}^{\infty} \cdots \sum_{a_m=0}^{\infty} f(a_1, 0; ...; a_i, 1; ...; a_m, 0) \prod_{k=1}^{m} \frac{t_k^{a_k}}{a_k!}. \quad (2)$$

The key identity for our argument is

$$t_i e^{(w_1+w_2+...+w_m)-w_i} = w_i \quad \text{for } i = 1, 2, ..., m. \quad (3)$$

The proof of formula (3) is based on the following combinatorial decomposition. Given a rooted spanning tree of the complete multipartite graph $K_{a_1, a_2, ..., a_m}$, where the root is in the i^{th} class, remove the root vertex from the tree to obtain a spanning forest of $K_{a_1, a_2, ..., a_i - 1, ..., a_m}$, and mark the former neighbors of the eliminated root vertex as roots in the forest. This decomposition establishes a bijection between the following two sets:
the set of rooted spanning trees of the complete multipartite graph $K_{a_1, a_2, ..., a_m}$, where the root is in the i^{th} vertex class,
and
the set of some ordered pairs, where the first entry of the ordered pair is one of the vertices of the i^{th} vertex class, the second element of the ordered pair is a rooted spanning forest of $K_{a_1, a_2, ..., a_i - 1, ..., a_m}$, where the vertex from the first entry is removed from the i^{th} vertex class, and the trees of the forest are not ordered.
Now $t_i e^{(w_1+w_2+...+w_m)-w_i}$ is the EGF of the set of ordered pairs in question, according to the Exponential Formula; and w_i is the same EGF by the bijection. Set $\Phi_i(w_1, w_2, ..., w_m) = e^{(w_1+w_2+...+w_m)-w_i}$.

According to the multiplication rule of EGF's, $\prod_{k=1}^{m} w_k^{b_k}$ is the multivariate exponential generating function of the number of rooted spanning forests of complete m-partite graphs, with b_k roots in the k^{th} vertex class, where the trees rooted in the same part *are ordered*; hence

$$f(a_1, b_1; ...; a_m, b_m) = \frac{a_1! a_2! \cdots a_m!}{b_1! b_2! \cdots b_m!} [t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m}] \prod_{k=1}^{m} w_k^{b_k}. \quad (4)$$

According to Part 1 of Theorem 1.2.9 (Multivariate Lagrange Formula) from [3], (3) implies

$$[t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m}] \prod_{k=1}^{m} w_k^{b_k} = [\lambda_1^{a_1} \cdots \lambda_m^{a_m}] \left\{ \det \left| \delta_{ij} - \frac{\lambda_j}{\Phi_i} \cdot \frac{\partial \Phi_i}{\partial \lambda_j} \right| \right. \quad (5)$$

$$\left. \times \; \lambda_1^{b_1} \cdots \lambda_m^{b_m} \prod_{k=1}^{m} e^{a_i(w_1+...+w_m)-a_i w_i} \right\}, \quad (6)$$

where Φ_i is a short-hand notation for $\Phi_i(\lambda_1, ..., \lambda_m)$. Observe that $\frac{\lambda_j}{\Phi_i} \cdot \frac{\partial \Phi_i}{\partial \lambda_j} = (1 - \delta_{ij})\lambda_j$, and the for the determinant in (5) we have the well-known evaluation

$$\det\left|\delta_{ij} - (1-\delta_{ij})\lambda_j\right| = (\lambda_1+1)\cdots(\lambda_m+1)\left(1 - \frac{\lambda_1}{\lambda_1+1} - \ldots - \frac{\lambda_m}{\lambda_m+1}\right) \quad (7)$$

(see for example Exercise 225 in [2]). Now (7) is easily rewritten as

$$1 - \sum_{j=2}^{m}(j-1) \sum_{1 \le i_1 < i_2 < \cdots < i_j \le m} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_j}, \quad (8)$$

and (8) is rewritten as

$$m\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_j} \cdot \sum_{l_1=0}^{1}\sum_{l_2=0}^{1}\cdots\sum_{l_m=0}^{1}(1 - l_1 - l_2 - \ldots - l_m)(\lambda_1)_{l_1}(\lambda_2)_{l_2}\cdots(\lambda_m)_{l_m}, \quad (9)$$

where $(x)_t$ stands for the falling factorial, $(x)_0 = 1$ and $(x)_1 = x$. Introducing the notation $A = a_1 + a_2 + \ldots + a_m$ and using (9), we find that (5) and (6) are equal to

$$[\lambda_1^{a_1-b_1}\lambda_2^{a_2-b_2}\cdots\lambda_m^{a_m-b_m}] \sum_{l_1=0}^{1}\sum_{l_2=0}^{1}\cdots\sum_{l_m=0}^{1}(1 - l_1 - l_2 - \ldots - l_m)$$

$$\times(\lambda_1)_{l_1}(\lambda_2)_{l_2}\cdots(\lambda_m)_{l_m} e^{(A-a_1)\lambda_1} e^{(A-a_2)\lambda_2}\cdots e^{(A-a_m)\lambda_m}$$

$$= \left(\prod_{k=1}^{m}\frac{(A-a_k)^{a_k-b_k-1}}{(a_k-b_k)!}\right)\sum_{l_1=0}^{1}\sum_{l_2=0}^{1}\cdots\sum_{l_m=0}^{1}(1 - l_1 - l_2 - \ldots - l_m) \quad (10)$$

$$\times\left(\prod_{j=1}^{m}(A-a_j)^{1-l_j}(a_j-b_j)_{l_j}\right). \quad (11)$$

Combining (10), (11), and (4), we obtain the main result:

Theorem 1

$$f(a_1, b_1; ...; a_m, b_m) = \left(\prod_{k=1}^{m}\binom{a_k}{b_k}(A-a_k)^{a_k-b_k-1}\right) \quad (12)$$

$$\times\sum_{l_1=0}^{1}\sum_{l_2=0}^{1}\cdots\sum_{l_m=0}^{1}(1 - l_1 - l_2 - \ldots - l_m)\left(\prod_{j=1}^{m}(A-a_j)^{1-l_j}(a_j-b_j)_{l_j}\right). \quad (13)$$

For the case $m = 2$, formula (12), (13) specializes to the formula of Jin and Liu (1), and formula (12), (13) yields a closed formula for every fixed m. Note that for the case $m = 2$ we do not even have to evaluate the determinant in general, since for $m = 2$ simply

$$\det\left|\delta_{ij} - (1 - \delta_{ij})\lambda_j\right| = 1 - \lambda_1\lambda_2.$$

It is worth noting that the papers [1, 5] count ordered forests by *inventory*. Summing up their counting formula for certain restricted types of inventories should provide an alternative way of obtaining Theorem 1. It is not clear for me, however, if an obvious way of summation would give Theorem 1 from their counting formula. However, their counting formula involves determinants pretty similar to (7).

References

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