Exponential lower bounds for the numbers of Skolem-type sequences

G. K. Bennett, M. J. Grannell, T. S. Griggs
Department of Pure Mathematics
The Open University
Walton Hall
Milton Keynes MK7 6AA
UNITED KINGDOM

Abstract

It was shown by Abrham that the number of pure Skolem sequences of order n, $n \equiv 0$ or $1 \pmod 4$, and the number of extended Skolem sequences of order n, are both bounded below by $2^{\lfloor n/3 \rfloor}$. These results are extended to give similar lower bounds for the numbers of hooked Skolem sequences, split Skolem sequences and split-hooked Skolem sequences.

1 Introduction

A *pure Skolem sequence*, sometimes simply called a *Skolem sequence*, of order n is a sequence $\{s_1, s_2, \ldots, s_{2n}\}$ of $2n$ integers satisfying the following conditions.

C1. For each $k \in \{1, 2, \ldots, n\}$ there are precisely two elements of the sequence, say s_i and s_j, such that $s_i = s_j = k$.

C2. If $s_i = s_j = k$ and $i < j$ then $j - i = k$.

For example, $\{4, 1, 1, 5, 4, 2, 3, 2, 5, 3\}$ is a pure Skolem sequence of order 5. It is well known that a pure Skolem sequence of order n exists if and only if $n \equiv 0$ or $1 \pmod 4$. For this and other existence results mentioned below see, for example, [2, 3].

An *extended Skolem sequence* of order n is a sequence $\{s_1, s_2, \ldots, s_{2n+1}\}$ of $2n + 1$ integers satisfying C1 and C2 above and such that precisely one element of the sequence is zero. An extended Skolem sequence of order n exists for every positive integer n. If the zero element of an extended

Skolem sequence of order n appears in the $2n$-th position, i.e. $s_{2n} = 0$, then the sequence is called a *hooked Skolem sequence.* A hooked Skolem sequence of order n exists if and only if $n \equiv 2$ or 3 (mod 4). If the zero element of an extended Skolem sequence of order n appears in the $(n+1)$-th position, i.e. $s_{n+1} = 0$, then the sequence is called a *split Skolem sequence.* A split Skolem sequence of order n exists if and only if $n \equiv 0$ or 3 (mod 4).

A *split-hooked Skolem sequence* of order n is a sequence $\{s_1, s_2, \ldots, s_{2n+2}\}$ of $2n+2$ integers satisfying C1 and C2 above and such that $s_{n+1} = s_{2n+1} = 0$. A split-hooked Skolem sequence of order n exists if and only if $n \equiv 1$ or 2 (mod 4) and $n \neq 1$.

The various types of Skolem sequence described above may be used to construct solutions to Heffter's first and second difference problems. These, in turn, may be used to construct cyclic Steiner triple systems (STSs).

Heffter's first difference problem is concerned with partitioning $\{1, 2, \ldots, 3n\}$ into n triples (a_i, b_i, c_i), $1 \le i \le n$, such that $a_i + b_i = c_i$ or $a_i + b_i + c_i \equiv 0 \pmod{6n+1}$. From a solution to this problem, a cyclic STS$(6n+1)$ may be formed by taking the set of all triples $\{0, a_i, a_i + b_i\}$ as a set of starters. A solution to Heffter's first difference problem may be obtained from either a pure Skolem sequence of order n, or a hooked Skolem sequence of order n, according as $n \equiv 0$ or 1 (mod 4), or $n \equiv 2$ or 3 (mod 4). Take the (pure or hooked) Skolem sequence $\{s_1, s_2, \ldots\}$ and for each $k \in \{1, 2, \ldots, n\}$, denote by a_k and b_k the suffices such that $s_{a_k} = s_{b_k} = k$ and $b_k - a_k = k$. Then the set of triples $\{(k, a_k + n, b_k + n) : 1 \le k \le n\}$ forms a solution to Heffter's first difference problem.

Heffter's second difference problem is concerned with partitioning $\{1, 2, \ldots, 3n+1\} \setminus \{2n+1\}$ into n triples (a_i, b_i, c_i), $1 \le i \le n$, such that $a_i + b_i = c_i$ or $a_i + b_i + c_i \equiv 0 \pmod{6n+3}$. From a solution to this problem, a cyclic STS$(6n+3)$ may be formed by taking the set of all triples $\{0, a_i, a_i+b_i\}$ together with $\{0, 2n+1, 4n+2\}$ as a set of starters. A solution to Heffter's second difference problem may be obtained from either a split Skolem sequence of order n, or a split-hooked Skolem sequence of order n, according as $n \equiv 0$ or 3 (mod 4), or $n \equiv 1$ or 2 (mod 4). Take the (split or split-hooked) Skolem sequence $\{s_1, s_2, \ldots\}$ and for each $k \in \{1, 2, \ldots, n\}$, denote by a_k and b_k the suffices such that $s_{a_k} = s_{b_k} = k$ and $b_k - a_k = k$. Then the set of triples $\{(k, a_k + n, b_k + n) : 1 \le k \le n\}$ forms a solution to Heffter's second difference problem.

In [1] Abrham obtained the exponential lower bound $2^{\lfloor n/3 \rfloor}$ for the numbers of pure and extended Skolem sequences of order n (with $n \equiv 0$ or 1 (mod 4) in the pure case). The bound for pure Skolem sequences provides a lower bound for the number of solutions to Heffter's first difference problem in the cases $n \equiv 0$ or 1 (mod 4), and hence for the number of cyclic STS$(6n+1)$s in these cases. The extended Skolem sequences of order n constructed by Abrham in the cases $n \equiv 0$ or 3 (mod 4) are in fact split

Skolem sequences, and so it follows directly from [1] that the number of split Skolem sequences of order n is at least $2^{\lfloor n/3 \rfloor}$ for $n \equiv 0$ or 3 (mod 4). This bound provides a lower bound for the number of solutions to Heffter's second difference problem in the cases $n \equiv 0$ or 3 (mod 4) and hence for the number of cyclic STS($6n+3$)s in these cases. It should, however, be noted that any one solution to Heffter's first (respectively second) difference problem produces 2^n distinct cyclic STS($6n$+1)s (respectively STS($6n$+3)s) obtained by replacing each starter $\{0, a_i, a_i+b_i\}$ by $\{0, b_i, a_i+b_i\}$, and that the resulting systems may, or may not, be isomorphic. Nevertheless, the establishment of lower bounds for the numbers of hooked and split-hooked Skolem sequences would also be of some interest as these would provide lower bounds for the numbers of cyclic STS($6n+1$)s for $n \equiv 2$ or 3 (mod 4), and STS($6n+3$)s for $n \equiv 1$ or 2 (mod 4).

Our method of tackling these problems is similar to Abrham's. We denote by h_n the number of hooked Skolem sequences of order n, and by $\bar{h}_n$ the number of split-hooked Skolem sequences of order n.

2 Results

Our methods are constructive and rely on so-called $(m, 3, c)$-systems. A set $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$, where each D_i is a triple of positive integers (a_i, b_i, a_i+b_i) with $a_i < b_i$ and $\bigcup_{i=1}^{m} D_i = \{c, c+1, \ldots, c+3m-1\}$ is called an $(m, 3, c)$-system. As remarked in [1], such a system exists if and only if

(i) $m \geq 2c-1$, and

(ii) $m \equiv 0$ or 1 (mod 4) if c is odd, or $m \equiv 0$ or 3 (mod 4) if c is even.

Given an $(m, 3, c)$-system $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$, where $D_i = (a_i, b_i, a_i+b_i)$, and putting $r = c+3m-1$, a sequence $\{x_{-r}, x_{-r+1}, \ldots, x_{r-1}, x_r\}$ may be constructed by putting $x_{-(a_i+b_i)} = a_i$, $x_{-b_i} = a_i$, $x_{-a_i} = a_i + b_i$, $x_{a_i} = b_i$, $x_{b_i} = a_i + b_i$, $x_{a_i+b_i} = b_i$ for $i = 1, 2, \ldots, m$, and $x_j = 0$ for $-c < j < c$. For example, if $c = 2$ and $m = 3$, and if $\mathcal{D} = \{D_1, D_2, D_3\}$ where $D_1 = (2, 6, 8)$, $D_2 = (3, 7, 10)$, $D_3 = (4, 5, 9)$ then the constructed sequence is

$$\{3, 4, 2, 3, 2, 4, 9, 10, 8, 0, 0, 0, 6, 7, 5, 9, 8, 10, 6, 5, 7\}.$$

Observe that for each $k \in \{c, c+1, \ldots, r\}$ the two positions where k appears in such a sequence are precisely k apart. Further observe that, independently for each $i \in \{1, 2, \ldots, m\}$, we may replace x_j for $j \in \{-a_i - b_i, -b_i, -a_i, a_i, b_i, a_i+b_i\}$ by x'_j where $x'_{-(a_i+b_i)} = b_i$, $x'_{-b_i} = a_i+b_i$, $x'_{-a_i} = b_i$, $x'_{a_i} = a_i + b_i$, $x'_{b_i} = a_i$, $x'_{a_i+b_i} = a_i$. Thus we may obtain 2^m distinct sequences of length $2r+1$ each of which has the property that for each

$k \in \{c, c+1, \ldots, r\}$, the two positions where k appears are precisely k apart. Each such sequence has zeros in the central $2c-1$ positions. We will denote any one such sequence by $x(\mathcal{D})$.

A hooked Skolem sequence of order $n = 3m + c$ may be constructed by taking an $(m, 3, c)$-system, $\mathcal{D}$, forming a sequence $x(\mathcal{D})$ as above, replacing the central $2c - 1$ zeros by an extended Skolem sequence of order $c - 1$ with its zero in its $(c+1)$-th position, replacing this zero by n and, finally, adjoining the two new entries 0 and n on the right-hand side. For example, taking the $(3,3,2)$-system given earlier with $x(\mathcal{D}) = \{3,4,2,3,2,4,9,10,8,0,0,0,6,7,5,9,8,10,6,5,7\}$, replacing the central 0, 0, 0 by 1, 1, 11, and adjoining 0 and 11 on the right-hand side gives the sequence

$$\{3,4,2,3,2,4,9,10,8,1,1,11,6,7,5,9,8,10,6,5,7,0,11\}$$

which is a hooked Skolem sequence of order 11.

In fact, with $c = 2$ and $m \equiv 0$ or 3 (mod 4), $m \geq 3$, a hooked Skolem sequence of order n may be constructed in this way for every $n \equiv 2$ or 11 (mod 12) with $n \geq 11$. Since there are 2^m distinct sequences $x(\mathcal{D})$ which may be employed in this construction, the number of hooked Skolem sequences of order n ($n \equiv 2$ or 11 (mod 12), $n \geq 11$) is at least $2^m = 2^{(n-2)/3} = 2^{\lfloor n/3 \rfloor}$. Thus $h_n \geq 2^{\lfloor n/3 \rfloor}$ for $n \equiv 2$ or 11 (mod 12), $n \geq 11$.

A similar argument with $c = 3$ and $m \equiv 0$ or 1 (mod 4), $m \geq 5$, gives $h_n \geq 2^{(n-3)/3}$ for $n \equiv 3$ or 6 (mod 12), $n \geq 18$. A slightly better bound is achieved in these cases by taking $c = 6$ and $m \equiv 0$ or 3 (mod 4), $m \geq 11$; the number of choices for $x(\mathcal{D})$ here is $2^{(n-6)/3}$ but there are six choices for the extended Skolem sequence of order 5 having its zero in its seventh position. These are

$$\begin{array}{lcl} (1) & : & \{4,2,5,2,4,3,0,5,3,1,1\} \\ (2) & : & \{2,4,2,5,3,4,0,3,5,1,1\} \\ (3) & : & \{1,1,3,4,5,3,0,4,2,5,2\} \\ (4) & : & \{1,1,2,5,2,4,0,3,5,4,3\} \\ (5) & : & \{2,3,2,4,3,5,0,4,1,1,5\} \\ (6) & : & \{3,1,1,3,4,5,0,2,4,2,5\} \end{array}$$

Consequently $h_n \geq 6 \cdot 2^{(n-6)/3} > 2^{\lfloor n/3 \rfloor}$ for $n \equiv 3$ or 6 (mod 12), $n \geq 39$.

With $c = 7$ and $m \equiv 0$ or 1 (mod 4), $m \geq 13$, the construction gives hooked Skolem sequences of orders $n \equiv 7$ or 10 (mod 12), $n \geq 46$. The number of choices for $x(\mathcal{D})$ is $2^{(n-7)/3}$ but a computer search shows that there are 18 choices for the extended Skolem sequence of order 6 having its zero in its eighth position. Consequently $h_n \geq 18 \cdot 2^{(n-7)/3} > 2^{\lfloor n/3 \rfloor}$ for $n \equiv 7$ or 10 (mod 12), $n \geq 46$.

Combining the above results gives the following theorem.

Theorem 1 *The number h_n of hooked Skolem sequences of order n satisfies $h_n \geq 2^{\lfloor n/3 \rfloor}$ for $n \equiv 2$ or 3 (mod 4) and $n \geq 46$.*

A split-hooked Skolem sequence of order $n = 3m+c+1$ with $c \geq 3$ may be constructed by taking an $(m, 3, c)$-system, $\mathcal{D}$, forming a sequence $x(\mathcal{D})$, and replacing the central $2c-1$ zeros by an extended Skolem sequence of order $c-1$ with its zero in its $(c+2)$-th position and its entries "2" in its positions $(c-1)$ and $(c+1)$. These two "2" entries are replaced by n and $n-1$ respectively and, finally, five new entries $n, n-1, 2, 0, 2$ are adjoined on the right-hand side. For example, if $c = 4$, then using the extended Skolem sequence $\{1, 1, 2, 3, 2, 0, 3\}$, the operations are as follows.

$$
\begin{array}{c}
x(\mathcal{D}) = \ldots, 0, 0, 0, 0, 0, 0, 0, \ldots \\
\downarrow \\
\ldots, 1, 1, 2, 3, 2, 0, 3, \ldots \\
\downarrow \\
\ldots, 1, 1, n, 3, n-1, 0, 3 \ldots \\
\downarrow \\
\ldots 1, 1, n, 3, n-1, 0, 3, \ldots\ldots, n, n-1, 2, 0, 2
\end{array}
$$

The final result is a split-hooked Skolem sequence of order $n = 3m+c+1$.

With $c = 4$ and $m \equiv 0$ or 3 (mod 4), $m \geq 7$, a split-hooked Skolem sequence of order n may be constructed for every $n \equiv 2$ or 5 (mod 12) with $n \geq 26$. Since there are 2^m distinct sequences $x(\mathcal{D})$, the number of split-hooked Skolem sequences of order n ($n \equiv 2$ or 5 (mod 12), $n \geq 26$) is at least $2^m = 2^{(n-5)/3}$. A slightly better bound is achieved in these cases by taking $c = 13$ and $m \equiv 0$ or 1 (mod 4), $m \geq 25$. The number of choices for $x(\mathcal{D})$ here is $2^{(n-14)/3}$, but computer search shows that there are 2308 choices for the extended Skolem sequence of order 12 having its 0 and 2s in the correct positions. Consequently, $\bar{h}_n \geq 2308 \cdot 2^{(n-14)/3} > 2^{\lfloor n/3 \rfloor}$ for $n \equiv 2$ or 5 (mod 12) and $n \geq 89$.

With $c = 9$ and $m \equiv 0$ or 1 (mod 4), a bound can be established for $n \equiv 1$ or 10 (mod 12). However, a better bound is found by taking $c = 12$ and $m \equiv 0$ or 3 (mod 4), $m \geq 23$. The number of choices for $x(\mathcal{D})$ here is $2^{(n-13)/3}$, but computer search shows that there are 396 choices for the extended Skolem sequence of order 11 having its 0 and 2s in the correct positions. Consequently, $\bar{h}_n \geq 396 \cdot 2^{(n-13)/3} > 2^{\lfloor n/3 \rfloor}$ for $n \equiv 1$ or 10 (mod 12) and $n \geq 82$.

Finally, with $c = 17$ and $m \equiv 0$ or 1 (mod 4), $m \geq 33$, a bound can be established for $n \equiv 6$ or 9 (mod 12), $n \geq 117$. The number of choices for $x(\mathcal{D})$ here is $2^{(n-18)/3}$, but computer search shows that there are many thousands of choices for the extended Skolem sequence of order 16 having its

0 and 2s in the correct positions. Consequently, $\bar{h}_n > 64 \cdot 2^{(n-18)/3} = 2^{\lfloor n/3 \rfloor}$ for $n \equiv 6$ or 9 (mod 12) and $n \geq 117$.

Combining the above results gives the following theorem.

Theorem 2 *The number $\bar{h}_n$ of split-hooked Skolem sequences of order n satisfies $\bar{h}_n \geq 2^{\lfloor n/3 \rfloor}$ for $n \equiv 1$ or 2 (mod 4) and $n \geq 117$.*

3 Concluding Remarks

It seems highly likely that the numbers of the various types of Skolem sequence of order n greatly exceed $2^{\lfloor n/3 \rfloor}$ for large values of n lying in the appropriate residue classes. Taking the numbers N of distinct extended Skolem sequences (which exist for all n) as tabulated in [2] for $n = 1, 2, \ldots, 14$, and fitting a model of the form $N = ab^n$ to the data for $n = 7, 8, \ldots, 14$ by a least-squares method gives $a = 0.0015$ and $b = 6.190$ with a good fit. The establishment of improved lower bounds for the various types of Skolem sequence mentioned in this paper is an interesting open problem.

References

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