Graph Orientations and Signings That Are Saturated with Alternating Cycles

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Abstract. The edges of a graph can be either directed or signed (2-colored) so as to make some of the even-length cycles of the underlying graph into alternating cycles. If a graph has a signing in which every even-length cycle is alternating, then it also has an orientation in which every even-length cycle is alternating, but not conversely. The existence of such an orientation or signing is closely related to the existence of an orientation in which every even-length cycle is a directed cycle.

1 Motivation

Suppose G is any finite, simple graph. Choosing between two possible directions for each edge of G produces an *orientation* G^d of G, while choosing between two possible signs (positive and negative) for each edge of G produces a *signing* G^s of G. A *cycle* of either G^d or G^s means a cycle in G, and an *even cycle* [or *odd cycle*] is a cycle of even [respectively, odd] length. (Notice that a cycle of G^d need not be a directed cycle; in other words, 'cycle' refers to what is often called a semicycle.)

Call a cycle an *alternating cycle* in G^d or G^s if consecutive edges are oppositely directed or signed. (In oriented graphs, these are often called antidirected cycles, or anticycles.) Clearly, every alternating cycle must be an even cycle.

There are two simple senses in which a directed or signed graph can be 'saturated' with alternating cycles:

- A strong sense in which every cycle is alternating.
- A weak sense in which every even cycle is alternating.

If G^s or G^d is saturated with alternating cycles in the strong sense, then every cycle is even, so the underlying graph G must be bipartite. The weak sense can be viewed as saying that every cycle that *can* be alternating actually *is* alternating.

If a graph G has a signing in which every cycle is alternating, then G must have an orientation in which every cycle is alternating (just direct all the edges of the underlying bipartite graph out of one color class and into the other). But the converse fails; $K_{2,3}$ has no signing in which every cycle is alternating.

Indeed, it is easy to see that a graph has an orientation in which every cycle is alternating if and only if it is bipartite, while a graph has a signing in which every cycle is alternating if and only if it is a bipartite *cactus* (where 'cactus' means that no edge is in more than one cycle). It is also easy to see that a graph has an orientation in which every cycle is directed if and only if it is a cactus. Therefore, *a graph has a signing in which every cycle is alternating if and only if it has both an orientation in which every cycle is alternating and an orientation in which every cycle is directed.*

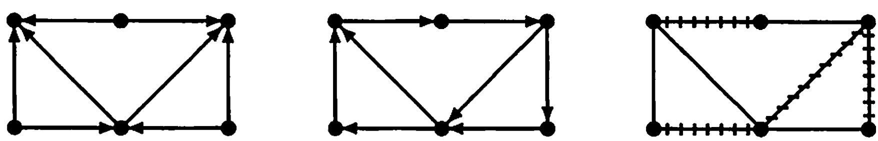

Figure 1: A graph with an orientation (on the left) with every even cycle alternating, an orientation (in the middle) with every even cycle directed, and a signing (the negative edges are hatched) with every even cycle alternating.

Section 2 will characterize the graphs that have signings or orientations that are saturated with alternating cycles in the weak sense, and to show that the same relationship holds between signings and orientations that are saturated with alternating cycles in the weak sense as holds in the strong sense.

A third sense of saturation of alternating cycles in orientations and signings was studied by Hoàng [2]: in his sense, every *hole* (meaning every induced cycle of length at least four) is alternating. This is indeed a different notion than the weak sense of saturation above, since K_4 has an orientation (and a signing) in which every hole is alternating, but no orientation (or signing) in which every even cycle is alternating, while C_5 has an orientation (and a signing) in which every even cycle is alternating, but no orientation (or signing) in which every hole is alternating. Graphs that have orientations or signings in which every hole is alternating are shown in [2] to be *perfect graphs* (meaning that every induced subgraph has chromatic number equal to the order of a largest clique). If only perfect graphs are considered, then Hoàng's sense of saturation reduces to every induced even cycle being alternating, and so becomes strictly weaker than the weak sense considered in the present paper. But, with or without the restriction to perfect graphs, the class of graphs having an orientation in which all holes are alternating is incomparable to the class of graphs having a signing in which all holes are alternating: $K_{2,3}$ has such an orientation but no such signing, while the complement of C_6 has such a signing but no such orientation.

Bang-Jensen and Gutin [1] also study alternating cycles and interactions of edge orientations and edge signings, but not in terms of saturation of alternating cycles.

2 Results

Call a graph G *nearly bipartite* if $G - v$ is bipartite for some $v \in V(G)$. (Thus, bipartite graphs are also nearly bipartite.)

Theorem 1 *A graph has an orientation in which every even cycle is alternating if and only if each block is nearly bipartite.*

Proof. First suppose G is a block and $G - v$ is bipartite with the vertices of $G - v$ colored black and white to show the bipartition. Form G^d by directing the edges of $G - v$ from white vertices and toward black vertices, and then directing the edges incident to v away from white vertices or toward black vertices. No cycle through v can be an even cycle, and so every even cycle in G^d will be alternating.

Conversely, suppose G has a block that is not nearly bipartite, say containing two odd cycles C_1 and C_2. Suppose for the moment that C_1 and C_2 are vertex disjoint. Then G must contain a subgraph as shown in Figure 2, where the vertex-disjoint paths P and P' can have either odd or even positive lengths. There are

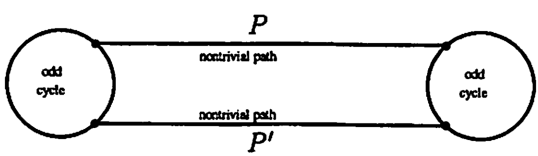

Figure 2: Six paths, involving two disjoint odd cycles within a common block.

three cases for the subgraph in Figure 2, depending on the parities of the lengths of P and P'. Since the endpoints of P and P' within either odd cycle will break that cycle into an odd path and an even path, each of the three cases will contain two even cycles; Figure 3 shows a smallest example of each case with the edges of one of the two even cycles oriented so as to make an alternating cycle. It is easy to check that, in each case, the second even cycle cannot also be oriented so as to be alternating.

So we can assume that C_1 and C_2 intersect in one or more paths (each of length ≥ 0). If there is more than one path, then the parts of C_1 and C_2 between those paths will combine to form additional cycles; if all those additional cycles are even, then forming the symmetric difference of some of them with C_1 will form

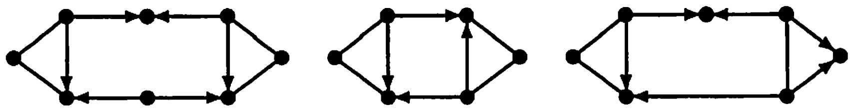

Figure 3: The three smallest examples for the subgraph from Figure 2 with, in each, one of the two even cycles oriented so as to be alternating.

another odd cycle that intersects C_2 in exactly one path. Thus, we can assume that C_1 and C_2 intersect in a single path (possibly of length 0).

Since G is not nearly bipartite, there must be a third odd cycle C_3, and since every two odd cycles intersect, C_3 must intersect both C_1 and C_2. Again, as above, we can assume that C_1 and C_3 intersect in a path, C_2 and C_3 intersect in a path, and $C_1 \cup C_2 \cup C_3$ form a subgraph G' of G as in Figure 4.

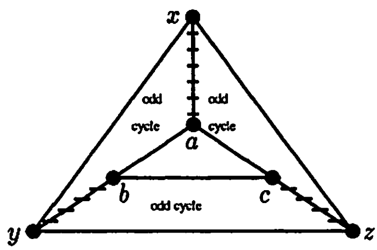

Figure 4: Nine paths, forming three odd cycles that pairwise intersect in disjoint paths (each possibly of length 0) indicated by hatching.

Suppose G (and so G') is oriented so that every cycle is alternating, and also suppose, for the moment, that the cycle C_{abc} composed of paths P_{ab} ($= P_{ba}$, the a-to-b path that contains none of the other labeled vertices), P_{bc}, and P_{ca} is an odd cycle. Then the cycle $C_{aczyb} = P_{ac} \cup P_{cz} \cup P_{zy} \cup P_{yb} \cup P_{ba}$ will have even length, and so vertex a will have in-degree 0 or 2 in the orientation of C_{aczyb}, and so in C_{abc}. Similarly, each of the vertices a, b, and c will have in-degree 0 or 2 in C_{abc}. If all three have in-degree 2 or all three have 0, then each of P_{ab}, P_{bc}, and P_{ca} will have even length; if two of the three are 2 and the other is 0, or vice versa, then two of the paths P_{ab}, P_{bc}, and P_{ca} will have odd length and the third will have even length. But in none of these possible cases is C_{abc} an odd cycle.

Therefore C_{abc} must be an even cycle, as similarly must C_{xyz}. But then, in the subgraph G', the even cycle C_{xyz} would have to be the symmetric difference of the even cycle C_{abc} and the three odd cycles C_1, C_2, and C_3, which is impossible. Therefore, there could not have been an orientation of G' (and so of G) in which every even cycle is alternating. □

A *theta graph* is a graph that is a subdivision (or 'homeomorph') of $K_{1,1,2}$ (= K_4 minus an edge). A cactus is thus a connected graph that has no theta subgraph. An *even theta graph* is a theta graph in which each of the three cycles has even length.

Theorem 2 *A graph has an orientation in which every even cycle is directed if and only if there are no even theta subgraphs.*

Proof. First suppose G is 2-connected and contains no even theta subgraph. If G has no even cycle, then the edges can be directed arbitrarily. Otherwise, suppose C is any even cycle and direct its edges cyclically. By [4, Theorem 4.2.7], G can be built up from C by repeatedly adding paths of new edges and new internal vertices whose endpoints are previously-existing vertices. Suppose, along the way in this process, a path P is to be added so as to connect vertices x and y in a 2-connected graph G^- where enough edges of G^- have been directed so as to make all the even cycles of G^- be directed cycles, but only edges in even cycles have been directed.

If x and y are in an even cycle of G^- that consists of two odd x, y-paths P_1 and P_2, then P must have even length (to avoid $P \cup P_1 \cup P_2$ forming an even theta subgraph). If x and y are already connected by an even path P' in G^-, then none of the edges of P' could have been in an even cycle of G^-, because $P \cup P'$ together with that even cycle would form an even theta subgraph in G. So P' would be the only even path connecting x and y in G^- and none of the edges of P' would have been directed yet. Then the edges of $P \cup P'$ could be directed so that all the even cycles in $G^- \cup P$ are directed cycles.

If x and y are in an even cycle that consists of two even x, y-paths P_1 and P_2, then P must have odd length (to avoid $P \cup P_1 \cup P_2$ forming an even theta subgraph). If x and y are already connected by an odd path P' in G^-, then none of the edges of P' could have been in an even cycle of G^-, because $P \cup P'$ together with that even cycle would form an even theta subgraph in G. So P' would be the only odd path connecting x and y in G^- and none of the edges of $P \cup P'$ would have been directed yet. Then the edges of $P \cup P'$ could be directed so that all the even cycles in $G^- \cup P$ are directed cycles.

In the remaining case, suppose x and y are not in an even cycle, and so are joined by a unique odd path P_1 and a unique even path P_2. Let P' be the one of P_1 and P_2 whose length has the same parity as the length of P. Then none of the edges of P' could have been in an even cycle of G^-, because $P \cup P'$ together with that even cycle would form an even theta subgraph in G. So none of the edges of $P \cup P'$ would have been directed yet. Then the edges of $P \cup P'$ could be directed so that all the even cycles in $G^- \cup P$ are directed cycles.

Continuing in this way—repeating the process within each block if G is not 2-connected—will eventually direct enough edges so that all the even cycles in G are directed cycles; any edges that remain undirected can be directed arbitrarily without disturbing this.

The converse follows from an even theta subgraph having no orientation in which all three cycles are directed. □

Theorem 3 *A graph has a signing in which every even cycle is alternating if and only if each block is nearly bipartite and there are no even theta subgraphs.*

Proof. First suppose G is a block with no even theta subgraph and $G - v$ is bipartite. Thus $G - v$ will be a bipartite cactus, so one can sign all the edges of $G - v$ that are in cycles so as to make every cycle alternating. There can be at most one even cycle of G that contains v (because G is a block and contains no even theta subgraph) and that even cycle must be edge-disjoint from every cycle of $G - v$ (since G contains no even theta subgraph and $G - v$ contains no odd cycles). Any even cycle C of G through v will have none of its edges signed yet, and so those edges can be signed so as to make C—and so every even cycle of G—alternating. Finally, any edges of G that remain unsigned can be signed arbitrarily without disturbing this.

Conversely, suppose G^s is a signed graph in which every even cycle is alternating. So G contains no even theta subgraph. Suppose G is 2-connected but not nearly bipartite (arguing toward a contradiction). Then G must contain two vertex-disjoint odd cycles within a common block. Hence, G must contain a subgraph as shown in Figure 2, and there are three cases, depending on the parities of the lengths of P and P'; a smallest example for each case is shown by the underlying graphs from Figure 3. Signing each edge directed 'clockwise' positive and each each edge directed 'counterclockwise' negative would make one of the two even cycles alternating. It is easy to check that, in each case, the second even cycle cannot also be signed so as to be alternating. □

Corollary 4 *A graph has a signing in which every even cycle is alternating if and only if it has both an orientation in which every even cycle is alternating and an orientation in which every even cycle is directed.*

Proof. Suppose a 2-connected graph G has an orientation D_1^d in which the even cycles are precisely the alternating cycles, and an orientation D_2^d in which the even cycles are precisely the directed cycles. Let G^s be the signing obtained by making an edge of G positive if and only if it is directed in the same way in both D_1^d and D_2^d. Then in G^s, the even cycles will be precisely the alternating cycles.

The converse follows directly from Theorems 1, 2, and 3. □

Figure 5 illustrates the independence of these two sorts of orientations. Note that if G has a signing G^s in which every even cycle is alternating and also has an orientation G^d in which every even cycle is alternating, then the orientation obtained by reversing the direction of precisely the positive edges of G^d has every even cycle a directed cycle. Moreover, 'alternating' and 'directed' can be interchanged for G^d in the preceding sentence.

Figure 5: On the left, a graph with an orientation in which every even cycle is alternating, but none in which every even cycle is directed; on the right, a graph with an orientation in which every even cycle is directed, but none in which every even cycle is alternating; neither has a signing in which every even cycle is alternating.

References

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