ANTIMAGIC LABELINGS OF GENERALIZED PETERSEN GRAPHS THAT ARE PLANE

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ABSTRACT. We deal with the problem of labeling the vertices, edges and faces of a plane graph in such a way that the label of a face and the labels of the vertices and edges surrounding that face add up to a weight of that face and the weights of all s-sided faces constitute an arithmetic progression of difference d. In this paper, we describe various antimagic labelings for the generalized Petersen graph $P(n, 2)$. The paper concludes with a conjecture.

1. INTRODUCTION AND DEFINITIONS

We consider finite undirected graphs without loops and multiple edges. Denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively. Let $|V(G)| = v$ and $|E(G)| = e$. Sedláček [14] defined a graph to be *magic* if it had an edge-labeling, with range the real numbers, such that the sum of the labels around any vertex equalled a constant, independent of the choice of the vertex. These labelings have been studied by Stewart [15] who called a labeling *supermagic* if the labels were consecutive integers, starting from 1.

An *antimagic* graph is a graph whose edges can be labeled with the integers $1, 2, ..., e$ so that the sum of the labels at any given vertex is different from the sum of the labels at any other vertex, that is, no two vertices have the same sum. The concept of an antimagic graph was introduced by Hartsfield and Ringel [10]. They conjecture that every tree other than K_2 is antimagic and, more strongly, that every connected graph other than K_2 is antimagic.

Let *weight* $w(x)$ of a vertex $x \in V(G)$ under an edge labeling $g : E(G) \to \{1, 2, ..., e\}$ be the sum of values $g(xy)$ assigned to all edges incident to a given vertex x.

Bodendiek and Walther [6] defined an (a, d)-*antimagic* graph as a special case of an antimagic graph as follows. A connected graph $G = (V, E)$ is said to be (a, d)-*antimagic* if there exist positive integers a, d and a bijection $g : E(G) \to \{1, 2, ..., e\}$ such that the induced mapping $h_g : V(G) \to W$ is also a bijection, where $W = \{w(x) : x \in V(G)\} = \{a, a+d, a+2d, ..., a+(v-1)d\}$ is the set of the weights of vertices.

The problem of deciding whether a given graph is magic or antimagic is very difficult. A beautiful survey on magic graphs can be found in Gallian [9]. Until now only few infinite families of graphs are known to be antimagic.

(a, d)-antimagic labelings for special graphs called parachutes are described in [7] and [8], for prisms and antiprisms in [1],[2] and [13].

Let $F(G)$ be the face set and $|F(G)| = f$ be the number of the faces of a plane graph G.

Assume that $\alpha, \beta, \gamma \in \{0, 1\}$. A labeling of type (α, β, γ) assigns labels from the set $\{1, 2, 3, ..., \alpha v + \beta e + \gamma f\}$ to the vertices, edges and faces of G in such a way that each vertex receives α labels, each edge receives β labels, and each face receives γ labels and each number is used exactly once as a label. Labelings of types $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are also called *vertex*, *edge* and *face* labelings, respectively.

The *weight* of a face under a labeling is the sum of labels (if present) carried by that face and the edges and vertices surrounding it.

A labeling of type (α, β, γ) is said to be *face-magic* if for every number s, all s-sided faces have the same weight. We allow different weights for different s.

The notion of face-magic labeling of plane graphs was defined by Ko-Wei Lih in [12]. Ko-Wei Lih called such a labeling *magic* but this notion of being magic is entirely different from those defined in [11], [14] and [15].

A labeling of type (α, β, γ) of plane graph G is called *d-antimagic* if for every number s the set of s-sided face weights is $W_s = \{a_s, a_s + d, a_s + 2d, ..., a_s + (f_s - 1)d\}$ for some integers a_s and d, where f_s is the number of s-sided faces. We allow different sets W_s for different s. d-antimagic labeling of type $(1, 1, 0)$ (respectively of type $(1, 1, 1)$) was defined in [3].

Two labelings g and g' of a plane graph G are said to be *d-complementary* if, for every number s, the sums of the g-weight and the g'-weight of each s-sided face constitute an arithmetic progression of difference d, $d \geq 0$, depending on s.

Let n, m be integers such that $n \geq 3$, $1 \leq m < n$ and $n \neq 2m$. For such n, m, the generalized Petersen graph $P(n, m)$ is defined by $V(P(n, m)) = \{x_i,\ y_i : 1 \leq i \leq n\}$ and $E(P(n, m)) = \{y_i y_{i+1},\ x_i x_{i+m},\ x_i y_i : 1 \leq i \leq n\}$ (subscripts are to be read modulo n). The standard Petersen graph is the instance $P(5, 2)$. By definition, $P(n, m)$ is a 3-regular graph which has $2n$ vertices and $3n$ edges. Generalized Petersen graphs were first defined by Watkins [16]. Note that $P(n, m_1) \cong P(n, m_2)$ if $m_1 + m_2 = n$ or $m_1 m_2 \equiv \pm 1 \pmod{n}$.

If $m = 1$ and $n \geq 3$ or $m = 2$ and n is even, $n \geq 6$, then the generalized Petersen graph $P(n, m)$ is plane.

In [3] it was proved that for $n \geq 3$, $n \equiv 3 \pmod 4$ and $d \in \{1,2,3,4,6\}$, the generalized Petersen graph $P(n,1)$ has a d-antimagic labeling of type $(1,1,1)$. 3-antimagic labeling of type $(1,1,1)$ for $P(n,1)$ and for all n, $n \geq 3$, can be found in [4].

In this paper, we consider the case where n is even and $m = 2$, and we present several necessary conditions for the graph $P(n,2)$ to be d-antimagic and describe various d-antimagic labelings of type $(1,1,1)$ for $P(n,2)$. Note that the face set $F(P(n,2))$ contains n 5-sided faces, one internal $\frac{n}{2}$-sided face and one external $\frac{n}{2}$-sided face.

2. Necessary conditions

In this section we shall find bounds for a feasible value d for the vertex labeling, the edge labeling and the face labeling of the generalized Petersen graph $P(n,2)$.

First, we will consider the weights of $\frac{n}{2}$-sided faces. Under a vertex [edge] labeling λ, the vertices x_i, $1 \leq i \leq n$ [the edges $x_i x_{i+2}$, $1 \leq i \leq n$], could conceivably receive the smallest labels $1, 2, \ldots, n$ or, at the other extreme, the largest labels $n+1, n+2, \ldots, 2n$ $[2n+1, 2n+2, \ldots, 3n]$, or anything in between. Consequently, we have

$$\sum_{i=1}^{n} i \leq \sum_{i=1}^{n} \lambda(x_i) \leq \sum_{i=1}^{n} (n+i)$$

$$\left[\sum_{i=1}^{n} i \leq \sum_{i=1}^{n} \lambda(x_i x_{i+2}) \leq \sum_{i=1}^{n} (2n+i)\right].$$

The sum of weights for both $\frac{n}{2}$-sided faces is

$$\sum_{i=1}^{n} \lambda(x_i) = a_{\frac{n}{2}} + (a_{\frac{n}{2}} + d)$$

$$\left[\sum_{i=1}^{n} \lambda(x_i x_{i+2}) = a_{\frac{n}{2}} + (a_{\frac{n}{2}} + d)\right].$$

For d-antimagic vertex [edge] labeling of $P(n,2)$, the minimum weight of a $\frac{n}{2}$-sided face is $a_{\frac{n}{2}} \geq 1 + 2 + \ldots + \frac{n}{2}$.

Thus

$$d = \sum_{i=1}^{n} \lambda(x_i) - 2a_{\frac{n}{2}} \leq kn^2$$

$$\left[d = \sum_{i=1}^{n} \lambda(x_i x_{i+2}) - 2a_{\frac{n}{2}} \leq kn^2\right],$$

where $0 < k \leq \frac{9}{4}$.

We can see that the upper bound for parameter d is very large in this case.

Now, we consider the weights of 5-sided faces.

Theorem 1. *For every generalized Petersen graph $P(n,2)$, $n \geq 6$, there is no d-antimagic vertex labeling with $d \geq 10$.*

Proof. Suppose that λ_1 is d-antimagic vertex labeling of $P(n,2)$. If the vertices x_i, $1 \leq i \leq n$, receive labels $1, 2, ..., n$ or, at the other extreme, labels $n+1, n+2, ..., 2n$, or anything in between, then

$$(1) \qquad (9n+5)\frac{n}{2} \leq 2\sum_{i=1}^{n} \lambda_1(x_i) + 3\sum_{i=1}^{n} \lambda_1(y_i) \leq (11n+5)\frac{n}{2}.$$

The sum of all the weights of 5-sided faces in $P(n,2)$ is

$$(2) \qquad a_5 + (a_5 + d) + ... + (a_5 + (n-1)d) = na_5 + d\binom{n}{2},$$

where the minimum value of weight which can be assigned to a 5-sided face is $a_5 \geq 1+2+3+4+5$. Thus, from (1) and (2) we get the following Diophantine equation

$$(3) \qquad 2\sum_{i=1}^{n} \lambda_1(x_i) + 3\sum_{i=1}^{n} \lambda_1(y_i) = na_5 + d\binom{n}{2}.$$

From (3) it follows that $d < 11$.

On the other hand, the maximum weight of a 5-sided face under a d-antimagic vertex labeling is no more than

$$\sum_{i=1}^{5}(|V(P(n,2))| + 1 - i) = 10n - 10$$

and then $a_5 + (n-1)d \leq 10n - 10$.

However, a_5 is always greater than 15, so $d < 10$, which completes the proof.

□

Theorem 2. *For every graph $P(n,2)$, $n \geq 6$, there is no d-antimagic edge labeling with $d \geq 15$.*

Proof. Let λ_2 be a d-antimagic edge labeling of $P(n,2)$. Under the edge labeling λ_2, the edges $x_i x_{i+2}$, $1 \leq i \leq n$, can receive the smallest labels $1, 2, ..., n$ or, at the other extreme, the largest labels $2n+1, 2n+2, ..., 3n$, or anything in between.

The sum of all the edge labels used to calculate the weights of 5-sided faces is equal to

$$(4) \qquad \frac{(13n+5)n}{2} \leq \sum_{i=1}^{n} \lambda_2(x_i x_{i+2}) + 2\sum_{i=1}^{n} \lambda_2(x_i y_i) + 2\sum_{i=1}^{n} \lambda_2(y_i y_{i+1}) \leq \frac{(17n+5)n}{2} .$$

Combining (2) and (4), we get

$$(5) \quad \sum_{i=1}^{n} \lambda_2(x_i x_{i+2}) + 2\sum_{i=1}^{n} \lambda_2(x_i y_i) + 2\sum_{i=1}^{n} \lambda_2(y_i y_{i+1}) = n a_5 + d\binom{n}{2} ,$$

which will give the range of feasible values for d.

By direct computation from (5) we obtain that $d < 17$.

However, the maximum 5-sided face weight (under a d-antimagic edge labeling of $P(n,2)$) is no more than

$$\sum_{i=1}^{5} (|E(P(n,2))| + 1 - i) = 15n - 10 .$$

Hence $a_5 + (n-1)d \leq 15n - 10$. Since $a_5 \geq 15$, we get that $d < 15$ and we arrive at the desired result.

□

In light of Theorem 1, Theorem 2 and the fact that under d-antimagic face labeling $F(P(n,2)) \to \{1, 2, ..., n\}$ the parameter d is no more than 1, we obtain the following result.

Theorem 3. *Let $P(n,2)$, $n \geq 6$, be a generalized Petersen graph which admits d_1-antimagic vertex labeling λ_1, d_2-antimagic edge labeling λ_2 and 1-antimagic face labeling λ_3, $d_1 \geq 0$, $d_2 \geq 0$. If the labelings λ_1, $v + \lambda_2$ and $v + e + \lambda_3$ combine to a d-antimagic labeling of type $(1,1,1)$ then the parameter $d \leq 24$.*

3. 1-ANTIMAGIC LABELING

Theorem 4. *If n is even, $n \geq 6$, then the generalized Petersen graph $P(n,2)$ has an 1-antimagic labeling of type $(1,1,1)$.*

Proof. It was proved [5] that the generalized Petersen graph $P(n,2)$, n even, $n \geq 6$, $n \neq 10$, is face-magic of type $(1,1,0)$. For $n = 10$, the $P(n,2)$ is the graph of a *dodecahedron*, which has face-magic labeling of type $(1,1,0)$ (see [12], Theorem 10). If we define a face labeling g_1 with values in the set $\{v+e+1, v+e+2, ..., v+e+f-2\} \cup \{v+e+f-1, v+e+f\}$ such that the 5-sided faces are labeled with $v+e+1, ..., v+e+f-2$ and the $\frac{n}{2}$-sided faces with $v+e+f-1$ and $v+e+f$, we obtain the 1-antimagic labeling of type $(1,1,1)$.

□

4. CASE $n \equiv 2 \pmod 4$

Let $I = \{1, 2, ..., n\}$ be an index set. In the sequel we shall use the following functions

$$\psi(x) = \begin{cases} 1 & \text{if} \quad x \equiv 1 \pmod 2 \\ 0 & \text{if} \quad x \equiv 0 \pmod 2 \end{cases}$$

$$\lambda(x,y) = \begin{cases} 1 & \text{if} \quad x \leq y \\ 0 & \text{otherwise} \end{cases}$$

$$\varrho(x) = \begin{cases} 1 & \text{if} \quad x \equiv 0 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$$

to simplify later notations.

Now, if $n \equiv 2 \pmod 4$, $n \geq 6$, $n \neq 10$, construct the function $h : V(P(n,2)) \cup E(P(n,2)) \to \{1, 2, ..., v+e\}$ as follows:

$h(x_i) = (2n - \frac{i-1}{2})\psi(i) + (n + \frac{i}{2})\psi(i+1),$

$h(y_i) = (\frac{i+1}{2})\psi(i) + (n + 1 - \frac{i}{2})\psi(i+1),$

$h(y_iy_{i+1})= (n+\frac{i+1}{2})\psi(i)+\frac{3n+i}{2}\psi(i+1),$
$h(x_iy_i)= \frac{12n+4-i}{4}\varrho(i)+\frac{10n+3-i}{4}\varrho(i+1)+\frac{11n+4-i}{4}\varrho(i+2)+\frac{9n+3-i}{4}\varrho(i+3),$
$h(x_ix_{i+2})= \frac{n+1-i}{2}\psi(i)+\left[(n-\frac{i}{2})\lambda(i,n-2)+n\lambda(n,i)\right]\psi(i+1)$
for $i\in I$ with indices taken modulo n.

Theorem 5. *If $n\equiv 2 \pmod 4$, $n\geq 6$, $n\neq 10$, then the generalized Petersen graph $P(n,2)$ has a face-magic labeling of type $(1,1,1)$.*

Proof. We define the vertex and the edge labeling g_1 of the generalized Petersen graph $P(n,2)$ in the following way:

$g_1(x_i)=h(x_i),\quad g_1(y_i)=h(y_i),\quad g_1(x_iy_i)=h(x_iy_i)$ and

$g_1(y_iy_{i+1})=[2n+h(y_iy_{i+1})]\psi(i)+[\frac{5n}{2}+h(y_iy_{i+1})]\psi(i+1)$ for $i\in I$.

$$g_1(x_ix_{i+2})=\begin{cases}5n+h(x_ix_{i+2}) & \text{if } i\in I-\{n-1\}\\ 5n & \text{if } i=n-1.\end{cases}$$

It is easy to verify that the labeling g_1 uses each integer from the set $g_1(V(P(n,2)))\cup g_1(E(P(n,2)))=\{1,2,3,...,\frac{7n}{2}-1,\frac{7n}{2}\}\cup\{4n+1,4n+2,...,\frac{9n}{2}-1,\frac{9n}{2}\}\cup\{5n\}\cup\{5n+2,5n+3,...,6n-1,6n\}$ exactly once.

The weights of the 5-sided faces and $\frac{n}{2}$-sided faces, under the labeling g_1, constitute the sets $W_5(g_1)=\{\frac{87n+18}{4}\}\cup\{\frac{87n+18}{4}+i:i=2,3,...,\frac{n}{2}\}\cup\{\frac{91n+18}{4}+i:i=1,2,...,\frac{n}{2}\}$, $W_{\frac{n}{2}}(g_1)=\{\frac{n}{4}(14n+2)-1,\frac{n}{4}(14n+2)\}$.

If g_2 is the face labeling 0-complementary to the labeling g_1 with values in the set $g_2(F(P(n,2)))=\{\frac{7n}{2}+1,\frac{7n}{2}+2,...,4n-1,4n\}\cup\{\frac{9n}{2}+1,\frac{9n}{2}+2,...,5n-1\}\cup\{5n+1\}\cup\{6n+1,6n+2\}$, where the $\frac{n}{2}$-sided faces are labeled by $6n+1$ and $6n+2$, then the labelings g_1 and g_2 combine to a labeling of type $(1,1,1)$ which has the common weight for all 5-sided faces equal to $\frac{107n+22}{4}$, and the common weight for both $\frac{n}{2}$-sided faces equal to $\frac{7n^2+13n+2}{2}$. This proves that $P(n,2)$ has the face-magic (0-antimagic) labeling of type $(1,1,1)$.

□

Theorem 6. *The graph of the dodecahedron has a 2-antimagic labeling of type $(1,1,1)$.*

Proof. The graph of the dodecahedron is the generalized Petersen graph $P(10,2)$. Ko-Wei Lih [12] proved that the graph of the dodecahedron has a 1-antimagic labeling of type $(1,0,0)$ and a 1-antimagic labeling of type $(0,1,0)$ which are 0-complementary. It means that there exist a 1-antimagic vertex labeling L_1 and a 1-antimagic edge labeling L_2 such that

$L_1(V(P(10,2))) = \{1,2,3,...,2n\}$, $L_2(E(P(10,2))) = \{1,2,3,...,3n\}$ and the labelings L_1 and $2n + L_2$ combine to a face-magic labeling of type $(1,1,0)$.

Define the vertex labeling g_3 and the edge labeling g_4 of $P(10,2)$ as follows:

$$g_3(x) = 2L_1(x) - 1, \quad x \in V(P(10,2))$$

$$g_4(x) = 2L_2(x), \quad x \in E(P(10,2)).$$

The labelings g_3 and g_4 are 2-antimagic and 0-complementary, that is, the labelings g_3 and g_4 combine to a labeling where all 5-sided faces have the same weight, say z.

Let g_5 be the face labeling with values in the set $\{4n + 2i - 1 : i = 1,2,...,n\} \cup \{6n+1, 6n+2\}$.

Label the faces of $P(10,2)$ by g_5 so that the values $6n+1$ and $6n+2$ will be given to faces which have the common edge labeled by $6n$. The labelings g_3, g_4 and g_5 combine to a labeling of type $(1,1,1)$ and the weights of 5-sided faces constitute the set $\{z+4n+2i-1 : i \in I\} \cup \{z+6n+1, z+6n+2\}$.

If we swap the edge value $6n$ with the face value $6n + 1$ then the face weight $z+6n+1$ will not be change but the face weight $z+6n+2$ will changed to $z + 6n + 3$. So, the resulting labeling of type $(1,1,1)$ is 2-antimagic.

□

Theorem 7. *If $n \equiv 2 \pmod 4$, $n \geq 6$, $n \neq 10$ and $2 \leq d \leq 3$, then the generalized Petersen graph $P(n,2)$ has a d-antimagic labeling of type $(1,1,1)$.*

Proof. Let us distinguish two cases.

Case 1. $d = 2$.

Construct the function $g_6 : V(P(n,2)) \cup E(P(n,2)) \to \{1,2,3,...,3n-2, 3n-1\} \cup \{3n+1\} \cup \{\frac{7n}{2}+1, \frac{7n}{2}+2,...,5n-1,5n\} \cup \{\frac{11n}{2}+1, \frac{11n}{2}+2,...,6n-1,6n\}$ as follows:

$$g_6(x_i) = h(x_i), \quad g_6(y_i) = h(y_i), \quad g_6(x_iy_i) = \tfrac{3n}{2} + h(x_iy_i),$$

$$g_6(y_iy_{i+1}) = [\tfrac{7n}{2} + h(y_iy_{i+1})]\psi(i) + [4n + h(y_iy_{i+1})]\psi(i+1),$$

$$g_6(x_ix_{i+2}) = [2n + h(x_ix_{i+2})]\lambda(i, n-1) + (3n+1)\lambda(n,i)$$

for $i \in I$.

It is not difficult to check that $W_5(g_6) = \{\frac{99n+18}{4} + i : i = 1,2,...,\frac{n}{2}\} \cup \{\frac{103n+18}{4} + i : i = 2,3,...,\frac{n}{2}\} \cup \{\frac{103n+26}{4}\}$ and $W_{\frac{n}{2}}(g_6) = \{\frac{n}{4}(8n+2), \frac{n}{4}(8n+2)+1\}$ are the sets of weights of 5-sided faces and $\frac{n}{2}$-sided faces.

Let g_7 be the 2-complementary face labeling to the labeling g_6 with labels from the set $\{3n\} \cup \{3n+2, 3n+3, ..., \frac{7n}{2}-1, \frac{7n}{2}\} \cup \{5n+1, 5n+$

$2, \dots, \frac{11n}{2}-1, \frac{11n}{2}\} \cup \{6n+1, 6n+2\}$.

Label the vertices, edges and faces of $P(n,2)$ by g_6 and g_7, respectively. We obtain the resulting labeling of type $(1,1,1)$ in which the weights of 5-sided faces ($\frac{n}{2}$-sided faces) constitute the arithmetical progression $\{\frac{115n+18}{2}+2i : i \in I\}$ ($\{2n^2+\frac{13n}{2}+1, 2n^2+\frac{13n}{2}+3\}$).

Case 2. $d=3$.

Consider the following function $g_8 : V(P(n,2)) \cup E(P(n,2)) \to \{1,2,3, \dots, 3n-1, 3n\} \cup \{3n+3i-2 : i \in I\} \cup \{3n+3i-1 : i \in I\}$ where

$g_8(x_i)= \frac{6n-3i+3}{2}\psi(i) + \frac{3i}{2}\psi(i+1)$,

$g_8(y_i)= \frac{3n+3i+1}{2}\psi(i) + \frac{3n-3i+4}{2}\psi(i+1)$,

$g_8(y_iy_{i+1})= \frac{3n-3i-1}{2}\psi(i) + [\frac{6n-3i-4}{2}\lambda(i,n-2) + (3n-2)\lambda(n,i)]\psi(i+1)$,

$g_8(x_iy_i)= \frac{6n+3i-1}{2}\psi(i) + \frac{9n+3i-4}{2}\psi(i+1)$,

$g_8(x_ix_{i+2})= \frac{6n+3i+1}{2}\psi(i) + \frac{12n-3i+4}{2}\psi(i+1)$

for $i \in I$ with indices taken modulo n.

The reader may verify that, under the labeling g_8, the common weight for all 5-sided faces is $24n+3$ and the common weight for both $\frac{n}{2}$-sided faces is $3n^2+n$.

Define the face labeling $g_9 : F(P(n,2)) \to \{3n+3i : i \in I\} \cup \{6n+1, 6n+2\}$ so that the 5-sided faces ($\frac{n}{2}$-sided faces) will be labeled by $3n+3$, $3n+6$, $\dots$, $6n$ ($6n+1$, $6n+2$) and so that the values $6n+1$ and $3n+3$ will be given to faces which have the common edge labeled by $3n+5$. The labelings g_8 and g_9 combine to a labeling of type $(1,1,1)$ and the weights of 5-sided faces constitute an arithmetic progression of difference 3; $27n+6$, $27n+9, \dots$, $30n+3$, while the weights of $\frac{n}{2}$-sided faces are $3n^2+7n+1$, $3n^2+7n+2$.

If we swap the edge value $3n+5$ with the face value $3n+3$ then the weight $3n^2+7n+1$ of $\frac{n}{2}$-sided face will change to $3n^2+7n-1$ and the weights of the 5-sided faces will not be changed. Hence, we have 3-antimagic labeling of type $(1,1,1)$. □

5. Case $n \equiv 0 \pmod 4$

In the sequel we shall investigate labelings of $P(n,2)$ if $n \equiv 0 \pmod 4$. Define the labelings g and g^*, for $i \in I$, as follows:

$g(y_i)= \frac{n+1-i}{2}\psi(i) + \frac{n+i}{2}\psi(i+1)$,

$$g(x_i) = \begin{cases} n+2 & \text{if } i=2 \\ \frac{3n+6-i}{2} & \text{if } i \equiv 2 \pmod 4, i>2 \\ \frac{3n+2-i}{2} & \text{if } i \equiv 0 \pmod 4 \\ \frac{3n+3+i}{2} & \text{if } i \equiv 1 \pmod 4 \\ \frac{3n-1+i}{2} & \text{if } i \equiv 3 \pmod 4 \end{cases}$$

$$g(y_i y_{i+1}) = (n - \tfrac{i-1}{2})\psi(i) + [\tfrac{i+2}{2}\lambda(i, n-2) + \lambda(n, i)]\psi(i+1),$$

$$g(x_i y_i) = \begin{cases} \frac{3n}{2} & \text{if } i=1 \\ n+\frac{i-1}{2} & \text{if } i \equiv 1 \pmod 4, i>1 \\ 2n & \text{if } i=2 \\ \frac{3n+i-2}{2} & \text{if } i \equiv 2 \pmod 4, i>2 \\ 2n-\frac{i-1}{2} & \text{if } i \equiv 3 \pmod 4 \\ n+\frac{i-2}{2} & \text{if } i \equiv 0 \pmod 4 \end{cases}$$

$$g(x_i x_{i+2}) = \begin{cases} 2n+1 & \text{if } i=1 \\ \frac{5n+3-i}{2} & \text{if } i \equiv 1 \pmod 4, i>1 \\ 3n-\frac{i-2}{2} & \text{if } i \equiv 2 \pmod 4 \\ \frac{5n-i-1}{2} & \text{if } i \equiv 3 \pmod 4, i<n-1 \\ \frac{5n}{2} & \text{if } i=n-1 \\ 3n+1-\frac{i}{2} & \text{if } i \equiv 0 \pmod 4 \end{cases}$$

$$g^*(y_i) = [\tfrac{2n+i-3}{2}\lambda(i,3) + \tfrac{n+i-3}{2}\lambda(5,i)]\psi(i) + [\tfrac{6-i}{2}\lambda(i,4) + \tfrac{n-i+6}{2}\lambda(6,i)]\psi(i+1),$$

$$g^*(x_i) = [(\tfrac{3n}{2}+1)\lambda(i,1) + (2n-\tfrac{i-3}{2})\lambda(3,i)]\psi(i) + [(\tfrac{3n}{2}-1)\lambda(i,2) + \tfrac{2n+i-4}{2}\lambda(6,i)]\psi(i+1),$$

$$g^*(x_i) = \tfrac{3n}{2} \quad \text{if } i=4.$$

Theorem 8. *For $n \equiv 0 \pmod 4$, $n \geq 8$ and $2 \leq d \leq 3$, the generalized Petersen graph $P(n,2)$ has a d-antimagic labeling of type $(1,1,1)$.*

Proof. We consider two cases.

Case 1. $d = 2$.

Let $h_1 : V(P(n,2)) \cup E(P(n,2)) \to \{2i-1 : i = 1, 2, ..., 2n\} \cup \{2i : i = 1, 2, ..., 3n\}$ be the labeling of $P(n,2)$ such that

$$h_1(x_i) = 2g(x_i) - 1, \quad h_1(y_i) = 2g(y_i) - 1, \quad h_1(x_i y_i) = 2g(x_i y_i),$$
$$h_1(y_i y_{i+1}) = 2g(y_i y_{i+1}) \text{ and } h_1(x_i x_{i+2}) = 2g(x_i x_{i+2})$$

for $i \in I$ with indices taken modulo n.

We can see that all 5-sided faces have the same weight $22n+5$ and that both $\frac{n}{2}$-sided faces have the weight $4n^2+\frac{n}{2}$.

Construct the face labeling $h_2 : F(P(n,2)) \to \{4n+2i-1 : i \in I\} \cup \{6n+1, 6n+2\}$ where the 5-sided faces receive values $4n+1, 4n+3, ..., 6n-1$ and the $\frac{n}{2}$-sided faces receive values $6n+1, 6n+2$ and the labeling will be arranged so that the values $6n-1$ and $6n+1$ will be given to faces which have the common edge labeled by $6n$.

The labelings h_1 and h_2 together describe a labeling of type $(1,1,1)$ and $W_5(h_1,h_2) = \{26n+4+2i : i \in I\}$, $W_{\frac{n}{2}}(h_1,h_2) = \{4n^2+\frac{13n}{2}+1, 4n^2+\frac{13n}{2}+2\}$ are the sets of the weights of the 5-sided and the $\frac{n}{2}$-sided faces. The weights of the 5-sided faces constitute an arithmetic progression of difference 2 but the weights of the $\frac{n}{2}$-sided faces do not. Now, we swap the face value $6n-1$ with the edge value $6n$. Thus, the $\frac{n}{2}$-sided face will have the weight $4n^2+\frac{13n}{2}$ instead of $4n^2+\frac{13n}{2}+1$ and the weight of the 5-sided face will not be changed. This proves that there exists a labeling of type $(1,1,1)$ which is 2-antimagic.

Case 2. $d = 3$.

Define a vertex and an edge labeling h_3 of $P(n,2)$ as follows:

$$h_3(x_i) = 3[g(x_i) - n], \qquad h_3(y_i) = 3g(y_i) - 1,$$

$$h_3(y_i y_{i+1}) = 3g(y_i y_{i+1}) - 2, \qquad h_3(x_i y_i) = 3g(x_i y_i) - 2 \text{ and}$$

$$h_3(x_i x_{i+2}) = 3[g(x_i x_{i+2}) - n] - 1$$

for $i \in I$ with indices taken modulo n.

Observe that $h_3(V(P(n,2))) = \{3i-1 : i \in I\} \cup \{3i : i \in I\}$ and $h_3(E(P(n,2))) = \{3i-2 : i = 1,2,...,2n\} \cup \{3n+3i-1 : i \in I\}$.

The common weights (under the labeling h_3) for all the 5-sided faces and the $\frac{n}{2}$-sided faces are $24n+3$ and $3n^2+n$, respectively.

Let h_4 be a face labeling which assign labels from the set $\{3n+3i : i \in I\}$ to the 5-sided faces and labels $6n+1$, $6n+2$ to the $\frac{n}{2}$-sided faces so that labels $6n+1$ and $\frac{9n}{2}+3$ will be assigned to faces which have the common edge labeled by $\frac{9n}{2}-1$. Combining the labelings h_3 and h_4, we obtain labeling of type $(1,1,1)$ with

$$W_5(h_3,h_4) = \{27n+6, 27n+9, 27n+12, ..., 30n, 30n+3\} \text{ and}$$

$$W_{\frac{n}{2}}(h_3,h_4) = \{3n^2+7n+1, 3n^2+7n+2\}.$$

If we swap the face label $\frac{9n}{2}+3$ with the edge label $\frac{9n}{2}-1$ then the face weight $3n^2+7n+1$ will be changed to $3n^2+7n+5$ and our labeling of type $(1,1,1)$ will be 3-antimagic.

□

Theorem 9. *If $n \equiv 0 \pmod 4$, $n \geq 8$ and $d \in \{6,9\}$, then the graph $P(n,2)$ has a d-antimagic labeling of type $(1,1,1)$.*

Proof. First we treat the 6-antimagic case.

If the bijective function $h_5 : V(P(n,2)) \cup E(P(n,2)) \to \{2i-1 : i = 1,2,...,2n\} \cup \{2i : i = 1,2,...,3n\}$ is defined as

$$h_5(y_i) = 2g^*(y_i)-1, \quad h_5(x_i) = 2g^*(x_i)-1, \quad h_5(y_iy_{i+1}) = 2g(y_iy_{i+1}),$$

$$h_5(x_iy_i) = 2g(x_iy_i) \quad \text{and} \quad h_5(x_ix_{i+2}) = 2g(x_ix_{i+2})$$

for $i \in I$ with indices taken modulo n then it is a matter of routine checking to see that the weights of the 5-sided faces constitute the arithmetic progression $\{20n+3+4i : i \in I\}$ and that the weights for both the $\frac{n}{2}$-sided faces are equal to $4n^2 + \frac{n}{2}$.

Let $h_6 : F(P(n,2)) \to \{4n+2i-1 : i \in I\} \cup \{6n+1, 6n+2\}$ be the a face labeling and let the face values $6n+1$ and $5n-3$ be given to faces which have the common edge labeled by $5n+2$.

Now, we swap the face value $5n-3$ with the edge value $5n+2$. The other face values (other than the values $6n+1$ and $5n-3$) will be arranged in such a way that the labelings h_5 and h_6 combine to a 6-antimagic labeling which is of type $(1,1,1)$. At the same time, the weights of the 5-sided faces constitute the set $\{24n+2+6i : i \in I\}$ and the weights of the $\frac{n}{2}$-sided faces are $4n^2 + \frac{13n}{2} - 4$ and $4n^2 + \frac{13n}{2} + 2$.

In the 9-antimagic case we define the bijective function $h_7 : V(P(n,2)) \cup E(P(n,2)) \to \{3i-2 : i \in I\} \cup \{3i-1 : i = 1,2,...,2n\} \cup \{3i : i = 1,2,...,2n\}$ in the following way:

$$h_7(y_i) = 3g^*(y_i),$$

$$h_7(x_i) = 3[g^*(x_i) - n] - 2,$$

$$h_7(y_iy_{i+1}) = 3g(y_iy_{i+1}) - 1,$$

$$h_7(x_iy_i) = 3g(x_iy_i) - 1 \quad \text{and}$$

$$h_7(x_ix_{i+2}) = 3[g(x_ix_{i+2}) - n]$$

for $i \in I$ with indices taken modulo n.

It is simple to verify that, under this labeling, the set of weights of the 5-sided faces consists of an arithmetical progression of difference 6; $W_5(h_7) = \{21n+4+6i : i \in I\}$ and that the $\frac{n}{2}$-sided faces have the same weight $3n^2 + \frac{n}{2}$.

Consider the face labeling $h_8 : F(P(n,2)) \to \{3n+3i-2 : i \in I\} \cup \{6n+1, 6n+2\}$. If we give the face values $6n+1$ and $\frac{9n}{2} - 5$ to faces which have the common edge labeled by $\frac{9n}{2} + 3$ and if we swap the face value $\frac{9n}{2} - 5$ with the edge value $\frac{9n}{2} + 3$ then we can arrange the other face values of h_8 to faces of $P(n,2)$ so that labeling h_8 is 9-complementary to the labeling h_7.

Thus the labelings h_7 and h_8 combine to give a 9-antimagic labeling of type $(1,1,1)$. □

6. Conclusion

In this paper we proved that for $n \equiv 2 \pmod 4$, $n \geq 6$, $n \neq 10$, there exist face-magic labelings and 3-antimagic labelings of type $(1,1,1)$ for graphs of $P(n,2)$. We have tried to find a face-magic labelings and 3-antimagic labelings of type $(1,1,1)$ for the graph of the dodecahedron $P(10,2)$ but so far without success.

Theorem 9 states that $P(n,2)$ has 6-antimagic and 9-antimagic labelings of type $(1,1,1)$ when $n \equiv 0 \pmod 4$. How about the other case? We conjecture that

Conjecture 1. *There is a d-antimagic labeling of type* $(1,1,1)$ *for the generalized Petersen graph* $P(n,2)$ *for* $n \equiv 2 \pmod 4$, $n \geq 6$ *and* $d \in \{6,9\}$.

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