Metric-Locating-Dominating Sets in Graphs

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Abstract

If u and v are vertices of a graph, then $d(u,v)$ denotes the distance from u to v. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of vertices in a connected graph G. For each $v \in V(G)$, the k-vector $c_S(v)$ is defined by $c_S(v) = (d(v,v_1), d(v,v_2), \cdots, d(v,v_k))$. A dominating set $S = \{v_1, v_2, \ldots, v_k\}$ in a connected graph G is a metric-locating-dominating set, or an MLD-set, if the k-vectors $c_S(v)$ for $v \in V(G)$ are distinct. The metric-location-domination number $\gamma_M(G)$ of G is the minimum cardinality of an MLD-set in G. We determine the metric-location-domination number of a tree in terms of its domination number. In particular, we show that $\gamma(T) = \gamma_M(T)$ if and only if T contains no vertex that is adjacent to two or more end-vertices. We show that for a tree T the ratio $\gamma_L(T)/\gamma_M(T)$ is bounded above by 2, where $\gamma_L(G)$ is the location- domination number defined by Slater (Dominating and reference sets in graphs, *J. Math. Phys. Sci.* **22** (1988), 445–455). We establish that if G is a connected graph of order $n \geq 2$, then $\gamma_M(T) = n-1$ if and only if $G = K_{1,n-1}$ or $G = K_n$. The connected graphs G of order $n \geq 4$ for which $\gamma_M(T) = n-2$ are characterized in terms of seven families of graphs.

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1 Introduction

Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of vertices in a connected graph G, and let $v \in V(G)$. The k-vector (ordered k-tuple) $c_S(v)$ of v with respect to S is defined by

$$c_S(v) = (d(v, v_1), d(v, v_2), \cdots, d(v, v_k)),$$

where $d(v, v_i)$ is the distance between v and v_i $(1 \le i \le k)$. The set S is called a *locating set* if the k-vectors $c_S(v)$, $v \in V(G)$, are distinct. The *location number* $\text{loc}(G)$ of G is the minimum cardinality of a locating set in G. These concepts were studied in [1, 4, 8, 10].

A set S of vertices of a graph $G = (V, E)$ is a *dominating set* of G if every vertex in $V - S$ is adjacent to a vertex of S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [5, 6].

In this paper we merge the concepts of a locating set and a dominating set by defining the *metric-locating-dominating set*, denoted by an MLD-set, in a connected graph G to be a set of vertices of G that is both a dominating set and a locating set in G. We define the *metric-location-domination number* $\gamma_M(G)$ of G to be the minimum cardinality of an MLD-set in G. An MLD-set in G of cardinality $\gamma_M(G)$ is called a $\gamma_M(G)$-set.

Slater [9, 10] defined a *locating-dominating set*, denoted by an LD-set, in a connected graph G to be a dominating set D of G such that for every two vertices u and v in $V(G) - D$, $N(u) \cap D \neq N(v) \cap D$. The *location-domination number* $\gamma_L(G)$ of G is the minimum cardinality of an LD-set for G. An LD-set in G of cardinality $\gamma_L(G)$ is called a $\gamma_L(G)$-set. These concepts were studied in [2, 3, 7, 9, 10, 11] and elsewhere. If $N(u) \cap D \neq N(v) \cap D$, then $c_D(u) \neq c_D(v)$. Thus every LD-set is an MLD-set, implying that $\gamma_M(G) \le \gamma_L(G)$. The graph of Figure 1 illustrates that $\gamma_M(G) \neq \gamma_L(G)$ in general. Note that $\{v_1, v_2, w_1, w_2\}$ is a minimum cardinality MLD-set and that $\{v_1, v_2, w_{11}, w_{12}, w_{21}, w_{22}\}$ is a minimum cardinality LD-set.

2 Preliminary Results

In this section, we present a few preliminary results the proof of which are straightforward and are therefore omitted.

Observation 1 *Let S be an MLD-set in a connected graph G. If u and v are distinct vertices of G such that $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, then $|S \cap \{u, v\}| \ge 1$. In particular, if u and v are vertices of G such that $N(u) - \{v\} = N(v) - \{u\}$, then $|S \cap \{u, v\}| \ge 1$.*

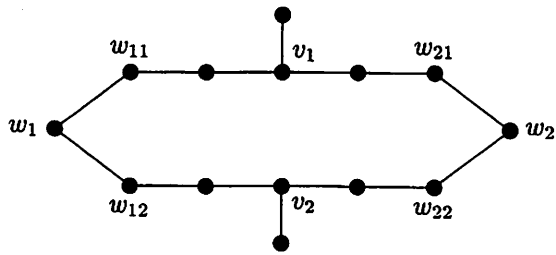

Figure 1: A graph G with $\gamma_L(G) = 6$ and $\gamma_M(G) = 4$.

Observation 2 *If G is a connected graph containing a support vertex v, then any MLD-set of G contains all the leaves adjacent to v or all but one of the leaves adjacent to v as well as the vertex v.*

Observation 3 *For every connected graph G of order $n \geq 2$,*

$$\gamma(G) \leq \gamma_M(G) \leq n - 1.$$

3 Trees

Our aim in this section is first to characterize the trees T for which $\gamma(T) = \gamma_M(T)$ and secondly to determine the metric-location-domination number of a tree in terms of its domination number.

Lemma 4 *If a tree T contains no strong support vertex, then every dominating set is an MLD-set.*

Proof. Let S be a dominating set of T. Let $u, v \in V(T) - S$. If $N(u) \cap S \neq N(v) \cap S$, then $c_S(u) \neq c_S(v)$. Suppose, then, that $N(u) \cap S = N(v) \cap S$. Then, since T is a tree, there is a unique vertex $w \in S$ such that $N(u) \cap S = N(v) \cap S = \{w\}$. Since w is not a strong support vertex, at least one of u and v cannot be a leaf. We may assume that $\deg v \geq 2$. Let $x \in N(v) - \{w\}$. Since $N(v) \cap S = \{w\}$, $x \notin S$. Thus, x is adjacent to a vertex $y \in S$. Since T is a tree, $w \neq y$. Thus, $d(v, y) = 2$, while $d(u, y) = 4$. Thus, once again, $c_S(u) \neq c_S(v)$. □

As an immediate consequence of Observation 3 and Lemma 4 we have the following result.

Corollary 5 *If T is a tree that contains no strong support vertex, then $\gamma(T) = \gamma_M(T)$.*

Theorem 6 *Let T be a tree. Then, $\gamma(T) = \gamma_M(T)$ if and only if T contains no strong support vertex.*

Proof. The sufficiency follows from Corollary 5. Next we consider the necessity. Suppose that T contains a strong support vertex v. Let S be an MLD-set of G of cardinality $\gamma_M(T)$. Then, by Observation 2, S contains all the end-vertices adjacent to v or all but one of the end-vertices adjacent to v as well as the vertex v. Deleting all the leaves adjacent to v from the set S and adding the vertex v to S, produces a dominating set of T of cardinality less than that of S, and so $\gamma(T) < |S| = \gamma_M(T)$. This proves the necessity. □

We are now in a position to determine the metric-location-domination number of a tree in terms of its domination number. Recall that the set of strong support vertices in a tree T is denoted by $S(T)$. For a tree T, we denote the total number of leaves in T that are adjacent to a strong support vertex by $\ell(T)$.

Theorem 7 *If T is a tree, then*

$$\gamma_M(T) = \gamma(T) + \ell(T) - |S(T)|.$$

Proof. If T contains no strong support vertex, then, by Corollary 5, $\gamma(T) = \gamma_M(T)$. Thus, since $\ell(T) = |S(T)| = 0$ in this case, $\gamma_M(T) = \gamma(T) + \ell(T) - |S(T)|$. Hence we may assume that $|S(T)| \geq 1$.

We show first that $\gamma_M(T) \leq \gamma(T) + \ell(T) - |S(T)|$. For each vertex $v \in S(T)$, let L_v denote the set of leaves adjacent to v and let $v' \in L_v$. Let

$$T' = T - \sum_{v \in S(T)} (L_v - \{v'\}).$$

Hence, T' is the tree obtained from T by deleting all but one leaf adjacent to each strong support vertex of T. Then, T' is a tree with no strong support vertex. Let S' be a $\gamma(T')$-set of T' that contains all the support vertices of T'. By Lemma 4, S' is an MLD-set of T' (and a dominating set of T). Hence,

$$S' \cup \left(\bigcup_{v \in S(T)} (L_v - \{v'\}) \right)$$

is an MLD-set of T, and so

$$\gamma_M(T) \leq |S'| + \sum_{v \in S(T)} (|L_v| - 1) = \gamma(T) + \ell(T) - |S(T)|.$$

We show next that $\gamma_M(T) \geq \gamma(T)+\ell(T)-|S(T)|$. Let D be an MLD-set of T of cardinality $\gamma_M(T)$ that contains as few leaves as possible. It follows from Observation 2, that for each $v \in S(T)$, D contains all except one leaf adjacent to v as well as the vertex v. Let v' be the leaf adjacent to v that does not belong to D. Then,

$$D' = D - \left(\bigcup_{v \in S(T)} (L_v - \{v'\} \right)$$

is a dominating set of T. Thus,

$$\gamma(T) \leq |D| - \sum_{v \in S(T)} (|L_v| - 1) = \gamma_M(T) - \ell(T) + |S(T)|.$$

The desired result now follows. □

Since the domination number of a tree can be computed in linear time, it follows from Theorem 7 that so too can the metric-location-domination number of a tree be computed in linear time.

4 $\gamma_L(G)$ versus $\gamma_M(G)$

Our aim in this section is to show that the ratio $\gamma_L(G)/\gamma_M(G)$ can be made arbitrarily large for general connected graphs G but is bounded above by 2 when G is a tree.

Since every LD-set is also an MLD-set, $\gamma_M(G) \leq \gamma_L(G)$ for all graphs G. However, for graphs in general there is no constant c such that

$$\frac{\gamma_L(G)}{\gamma_M(G)} \leq c.$$

To see this consider the following construction. For $k \geq 2$, take ℓ disjoint stars $K_{1,k}$ and subdivide each edge twice. Let these subdivided stars be $T_1, T_2, \ldots, T_\ell$ with centers $w_1, w_2, \ldots, w_\ell$, respectively. Let $v_{i1}, v_{i2}, \ldots, v_{ik}$ be the leaves of T_i $(1 \leq i \leq \ell)$. Let G be the graph obtained from $T_1, T_2, \ldots, T_\ell$ by identifying for each j $(1 \leq j \leq k)$ the ℓ vertices $v_{1j}, v_{2j}, \ldots, v_{\ell j}$ in a new vertex v_j and then adding a new vertex u_j and the edge $u_j v_j$. (For example, when $k = 2$ and $\ell = 3$ the graph G is illustrated in Figure 2.) Then, $\gamma_L(G) = k(\ell + 1)$ (for example, the set

$$\left(\bigcup_{i=1}^{\ell} N(w_i) \right) \cup \left(\bigcup_{j=1}^{k} \{v_j\} \right)$$

is a $\gamma_L(G)$-set) and $\gamma_M(G) = k + \ell$ (for example, the set $\{v_1, v_2, \ldots, v_k\} \cup \{w_1, w_2, \ldots, w_\ell\}$ is a $\gamma_M(G)$-set). Thus,

$$\frac{\gamma_L(G)}{\gamma_M(G)} = \frac{k + k/\ell}{1 + k/\ell} \longrightarrow k$$

as $\ell \to \infty$. By choosing k sufficiently large, this ratio can be made arbitrarily large. We show, however, that for trees this ratio is bounded by 2.

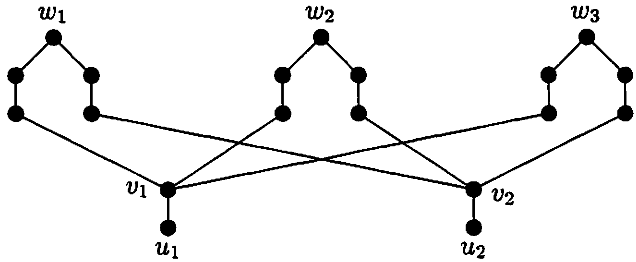

Figure 2: A graph G with $\gamma_L(G) = 8$ and $\gamma_M(G) = 5$.

For ease of presentation, we mostly consider *rooted trees*. For a vertex v in a (rooted) tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The maximal subtree at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

Lemma 8 *If T is a tree that contains no strong support vertex, then $\gamma_L(T) < 2\gamma(T)$.*

Proof. We proceed by induction on the order n of a tree T with no strong support vertex. If $n = 1$ or $n = 2$, then $\gamma_L(T) = \gamma(T) = 1$, and so the result holds in this case. This establishes the base case.

Suppose that if T' is a tree of order less than n, where $n \geq 3$, with no strong support vertex, then $\gamma_L(T') < 2\gamma(T')$, and let T be a tree of order n with no strong support vertex. We now root the tree T at a vertex r and let u be a vertex at maximum distance from r. Then, u is a leaf in T. Let v be the parent of u. Since T has no strong support vertex, $\deg v = 2$. Let w denote the parent of v.

Suppose w is adjacent with a leaf v'. Let $T' = T - \{u, v\}$. Then, $\gamma(T) = \gamma(T') + 1$ and $\gamma_L(T) \leq \gamma_L(T') + 1$. Applying the inductive hypothesis to the tree T', $\gamma_L(T') < 2\gamma(T')$. Thus, $\gamma_L(T) - 1 \leq \gamma_L(T') < 2\gamma(T') = 2\gamma(T) - 2$, and so $\gamma_L(T) < \gamma_L(T) + 1 < 2\gamma(T)$.

On the other hand, suppose that w is not adjacent to any leaf. Then each child of w has degree 2 and is adjacent to a leaf. Suppose $|C(w)| = k \geq 1$.

Let $T' = T - D[w]$. Then, $\gamma(T) = \gamma(T') + k$. Furthermore, any LD-set of T' can be extended to an LD-set of T by adding the set $C(w) \cup \{w\}$, and so $\gamma_L(T) \le \gamma_L(T') + k + 1$. Applying the inductive hypothesis to the tree T', $\gamma_L(T') < 2\gamma(T')$. Thus, $\gamma_L(T) - k - 1 \le \gamma_L(T') < 2\gamma(T') = 2\gamma(T) - 2k$, and so $\gamma_L(T) \le \gamma_L(T) + k - 1 < 2\gamma(T)$ since $k - 1 \ge 0$. This completes the proof of the lemma. □

As an immediate consequence of Corollary 5 and Lemma 8, we have the following result.

Corollary 9 *If T is a tree that contains no strong support vertex, then $\gamma_L(T) < 2\gamma_M(T)$.*

Lemma 10 *If T is a tree, then $\gamma_L(T) < 2\gamma_M(T)$.*

Proof. We proceed by induction on the order n of a tree T. If T is a star, then $\gamma_L(T) = \gamma_M(T)$ and the result follows. In particular, the result holds for $n = 1, 2, 3$. Suppose thus that $\operatorname{diam} T \ge 3$.

Suppose that if T' is a tree of order less than n, where $n \ge 4$, then $\gamma_L(T') < 2\gamma_M(T')$, and let T be a tree of order n. If T has no strong support vertex, then the result follows from Corollary 9. Suppose thus that T has a strong support vertex v. Let u be a leaf adjacent with v and let $T' = T - u$. Then, $\gamma_L(T) = \gamma_L(T') + 1$ and $\gamma_M(T) = \gamma_M(T') + 1$. Applying the inductive hypothesis to the tree T', gives $\gamma_L(T') < 2\gamma_M(T')$. Thus, $\gamma_L(T) - 1 = \gamma_L(T') < 2\gamma_M(T') = 2\gamma_M(T) - 2$, and so $\gamma_L(T) < \gamma_L(T) + 1 < 2\gamma_M(T)$. □

That the bound of Corollary 9 is asymptotically best possible may be seen as follows. For $k \ge 2$, let T be the tree obtained from a star $K_{1,k}$ by subdividing every edge three times. Then, $\gamma_L(T) = 2k$ and $\gamma_M(T) = k + 1$. Thus,

$$\frac{\gamma_L(T)}{\gamma_M(T)} = \frac{2}{1 + 1/k} \longrightarrow 2$$

as $k \to \infty$. As an immediate consequence of Lemma 10, we have the following result.

Corollary 11 *For any tree T,*

$$\frac{1}{2}\gamma_L(T) < \gamma_M(T) \le \gamma_L(T).$$

Furthermore, as an immediate consequence of Theorem 7 and Lemma 10, we have the following result.

Corollary 12 *For any tree T,*

$$\gamma_L(T) < 2(\gamma(T) + \ell(T) - |S(T)|),$$

where $S(T)$ is the set of strong support vertices of T and $\ell(T)$ is the number of leaves in T that are adjacent to a strong support vertex.

5 Graphs G with $\gamma_M(G) = n - 1$

Our aim in this section is to characterize connected graphs G of order $n \geq 2$ for which $\gamma_M(G) = n - 1$.

Theorem 13 *Let G be a connected graph of order $n \geq 2$. Then, $\gamma_M(G) = n - 1$ if and only if $G = K_{1,n-1}$ or $G = K_n$.*

Proof. If $G = K_{1,n-1}$, then, by Observation 2, $\gamma_M(G) = n - 1$, while if $G = K_n$, then, by Observation 1, $\gamma_M(G) = n - 1$. This establishes the sufficiency.

To prove the necessity, suppose that $\gamma_M(G) = n - 1$. If $\operatorname{diam} G \geq 3$, then let u and v be two vertices distance at least 3 apart in G. Then, $V(G) - \{u, v\}$ is an MLD-set of G, and so $\gamma_M(G) \leq n - 2$, a contradiction. Thus, $\operatorname{diam} G \leq 2$.

Suppose that $\delta(G) = 1$. Let u be an end-vertex of G and let $N(u) = \{v\}$. Since $\operatorname{diam} G \leq 2$, v is adjacent to every other vertex of G. If two vertices x and y in $N(v)$ are adjacent, then $V(G) - \{u, x\}$ is an MLD-set of G, and so $\gamma_M(G) \leq n - 2$, a contradiction. Thus, $N(v)$ is an independent set, i.e., $G = K_{1,n-1}$.

Suppose then that $\delta(G) \geq 2$. If u and v are distinct vertices of G such that $d(u, w) \neq d(v, w)$ for some vertex $w \in V(G) - \{u, v\}$, then $V(G) - \{u, v\}$ is an MLD-set of G, and so $\gamma_M(G) \leq n - 2$, a contradiction. Thus, for every two distinct vertices u and v of G we must have $d(u, w) = d(v, w)$ for every vertex $w \in V(G) - \{u, v\}$. If u and v are two vertices that are not adjacent, then for $x \in N(u)$ we have $d(x, v) = 1$, while $d(u, v) = 2$. So again $V(G) - \{u, x\}$ is an MLD-set, a contradiction. Hence, every two vertices of G must be adjacent, i.e., $G = K_n$. □

6 Graphs G with $\gamma_M(G) = n - 2$

In order to characterize those connected graphs G of order $n \geq 4$ with $\gamma_M(T) = n - 2$ we begin by defining seven families of graphs.

Let $\mathcal{F}_1$ be the family of double stars $S(m, k)$, $m, k \geq 1$, of order $m + k + 2 \geq 4$, where a double star $S(m, k)$ is obtained by appending m leaves to one of the vertices in a K_2 and k leaves to the other vertex of the K_2.

Let $\mathcal{F}_2$ be the family of graphs obtained from a complete graph of order at least 3 by appending any positive number of leaves to exactly one of its vertices.

Let $\mathcal{F}_3$ be the family of graphs obtained from $\overline{K_2} + K_m$, $m \geq 2$, by appending any positive number of leaves to exactly one of the vertices of degree m.

Let $\mathcal{F}_4$ be the family of graphs obtained from $K_2 + \overline{K_m}$, $m \geq 2$, by appending any positive number of leaves to exactly one of the vertices of degree $m + 1$.

Let $\mathcal{F}_5$ be the family of all complete bipartite graphs $K_{m,k}$ where $m, k \geq 2$.

Let $\mathcal{F}_6$ be the family of all complete multipartite graphs $\overline{K_m} + K_k$ where $m, k \geq 2$.

Let $\mathcal{F}_7$ be the family of graphs obtained from a complete graph K_m by deleting k edges incident with some vertex u where $2 \leq k \leq m - 3$.

Let $\mathcal{F} = \cup_{i=1}^{7} \mathcal{F}_i$.

Theorem 14 *Let $G = (V, E)$ be a connected graph of order $n \geq 4$. Then, $\gamma_M(G) = n - 2$ if and only if $G \in \mathcal{F}$.*

Proof. If $G \in \mathcal{F}$, then it is a straightforward task to show that $\gamma_M(G) = n - 2$.

Suppose now that $\gamma_M(G) = n - 2$. We begin by showing that $\operatorname{diam} G \leq 3$. Suppose there are vertices u and v such that $d(u, v) = 4$. Let u, x, y, z, v be a shortest u-v path. Then, $V - \{u, y, v\}$ is an MLD-set of G, and so $\gamma_M(G) \leq n - 3$, a contradiction. Thus, $\operatorname{diam} G \leq 3$.

We proceed by induction on $n \geq 4$ to show that if G is a connected graph of order n and $\gamma_M(G) = n - 2$, then $G \in \mathcal{F}$. If $n = 4$, then $G \cong P_4 \in \mathcal{F}_1$ or G is the graph in $\mathcal{F}_2$ obtained from K_3 by appending a leaf or $G \cong C_4 \in \mathcal{F}_5$ or $G \cong K_4 - e \in \mathcal{F}_6$ (where e is an edge of K_4). Hence, $G \in \mathcal{F}$.

Assume for any connected graph G' of order n', where $4 \leq n' < n$ and $\gamma_M(G') = n' - 2$, that $G' \in \mathcal{F}$. Suppose G is a connected graph of order n with $\gamma_M(G) = n - 2$. Before proceeding further, we prove the following claim.

Claim 1 *If $\delta(G) = 1$, then $G \in \cup_{i=1}^{4} \mathcal{F}_i$.*

Proof. Suppose $\delta(G) = 1$. Let v be a support vertex of G and let L_v be the collection of leaves adjacent to v. Let $G' = G - L_v$ and let n' be its order. By Observation 3, $\gamma_M(G') \leq n' - 1$. By construction, v is not adjacent to any leaf in G'.

If $\gamma_M(G') \leq n' - 3$, let S' be a $\gamma_M(G')$-set. Then, $S' \cup L_v$ is an MLD-set of G and hence $\gamma_M(G) \leq n - 3$, a contradiction. So, $\gamma_M(G') = n' - 2$ or $n' - 1$.

Suppose $\gamma_M(G') = n' - 1$. Then, by Theorem 13, G' is a star or a complete graph. G' is a star, then v must be a leaf of G' and hence $G \in \mathcal{F}_1$. If G' is a complete graph, then $G \in \mathcal{F}_2$.

Suppose now that $\gamma_M(G') = n' - 2$. By the inductive hypothesis, $G' \in \mathcal{F}$.

If $G' \in \mathcal{F}_1$, then v must be a leaf. However, then $\operatorname{diam} G = 4$, a contradiction. So, $G' \notin \mathcal{F}_1$.

Suppose $G' \in \mathcal{F}_2$. Let w be the vertex of maximum degree in G' and let x be a non-leaf neighbor of w. Then, $v \neq w$. Suppose first that v is a leaf of G'. Then, $V - \{v, w, x\}$ is an MLD-set of G, and so $\gamma_M(G) \leq n - 3$, a contradiction. Hence, v is a non-leaf adjacent with w. We may assume $v \neq x$. Let $v' \in L_v$ and let w' be a leaf adjacent with w. Then, $V - \{v', w', x\}$ is an MLD-set of G, a contradiction. So, $G' \notin \mathcal{F}_2$.

Suppose $G' \in \mathcal{F}_3$. Let w be the vertex of maximum degree in G', w' a leaf adjacent with w, y a vertex of degree 2 adjacent with w, and x the vertex not adjacent with w in G'. Then, $v \neq w$. If v is a leaf of G' or if $v = x$, then $\operatorname{diam} G = 4$, a contradiction. So we may assume $v = y$. Let $v' \in L_v$. Then, $V - \{v', w'. x\}$ is an MLD-set of G, a contradiction. So, $G' \notin \mathcal{F}_3$.

Suppose $G' \in \mathcal{F}_4$. Let w be the vertex of maximum degree in G', w' a leaf adjacent with w, y a vertex of degree 2 adjacent with w, and x the vertex of degree exceeding 2 adjacent with w in G'. Then, $v \neq w$. Let $v' \in L_v$. If v is a leaf of G', then $V - \{v', x, y\}$ is an MLD-set of G, a contradiction. If $v = x$, then $V - \{v', w', y\}$ is an MLD-set of G, a contradiction. If v is a vertex of degree 2 in G' adjacent with w, then we may assume $v \neq y$. Then, $V - \{v', w', y\}$ is an MLD-set of G, a contradiction. So, $G' \notin \mathcal{F}_4$.

Suppose $G' \in \mathcal{F}_5$. Then, $G' \cong K_{m,k}$ where $m, k \geq 2$. Suppose v belongs to the partite set of cardinality k in G'. We show first that $k = 2$. If $k > 2$, let u be a vertex in the same partite set as v, $v' \in L_v$, and w a vertex adjacent with v in G'. Then, $V - \{v', u, w\}$ is an MLD-set of G, a contradiction. Hence $k = 2$ and so $G \in \mathcal{F}_3$.

Suppose $G' \in \mathcal{F}_6$. Suppose first that v has degree $n - 1$ in G. Let u be a vertex of degree $n' - 1$ adjacent with v in G'. Let w be a vertex of degree less than $n' - 1$ in G', and let $v' \in L_v$. Then, $V - \{v', u, w\}$ is an MLD-set of G unless G' has exactly three partite sets, in which case $G \in \mathcal{F}_4$. On the other hand, suppose v has degree less than $n' - 1$ in G'. Let w be in the same partite set of G' as v, u a vertex of degree $n' - 1$ in G', and $v' \in L_v$. Then, $V - \{v', u, w\}$ is an MLD-set of G, a contradiction. So, v cannot have degree less than $n' - 1$ in G'.

Suppose $G' \in \mathcal{F}_7$. Suppose v is the vertex of degree at most $n' - 3$ in G'. Let $v' \in L_v$, u a vertex adjacent to v in G' and w a vertex not adjacent to v in G'. Then, $V - \{v', u, w\}$ is an MLD-set of G, a contradiction. Suppose v has degree at least $n' - 2$ in G'. Let x and y be two nonadjacent vertices in G' distinct from v and let $v' \in L_v$. Then, $V - \{v', x, y\}$ is an MLD-set of G, a contradiction. So, $G' \notin \mathcal{F}_7$. This completes the proof of Claim 1. □

We now return to the proof of Theorem 14. By Claim 1, if $\delta(G) = 1$, then $G \in \mathcal{F}$. Hence we may assume that $\delta(G) \geq 2$. Let S be a maximum independent set of G. Since G is not a complete graph, $|S| \geq 2$.

Claim 2 *If $G - S$ is a complete graph, then $G \in \mathcal{F}_6 \cup \mathcal{F}_7$.*

Proof. Since $\delta(G) \geq 2$, $V - S$ contains at least two vertices. If all edges between S and $V - S$ are present, then $G \in \mathcal{F}_6$. Hence we may assume that some vertex $u \in S$ is nonadjacent to some $v \in V - S$. Since S is a maximum independent set, v is adjacent with some $w \in S$.

Suppose now that $|S| \geq 3$. Let $z \in S - \{u, w\}$. Since $\delta(G) \geq 2$, z is adjacent with some $y \neq v$. Then, $V - \{u, w, y\}$ is an MLD-set of G, a contradiction. Hence, $|S| = 2$. Suppose w is nonadjacent to some vertex $y \in V - S$. If there is a common neighbor z of u and w, then $V - \{u, w, z\}$ is an MLD-set of G, a contradiction. It follows that $V - S$ can be partitioned into two sets U and W such that u is adjacent to every vertex of U and to no vertex of W, while w is adjacent to every vertex of W and to no vertex of U. Since $\delta(G) \geq 2$, $|U| \geq 2$ and $|V| \geq 2$. So, $V - \{v, w, y\}$ is an MLD-set of G, a contradiction. Thus, w is adjacent with every vertex in $V - S$. So $G \in \mathcal{F}_7$. □

By Claim 2, if $G - S$ is a complete graph, then $G \in \mathcal{F}$. Hence we may assume that $G - S$ is not complete. Let W be a maximum independent set in $G - S$. Then, $|W| \geq 2$.

Claim 3 *If $V - (S \cup W) \neq \emptyset$, then every vertex in $V - (S \cup W)$ is adjacent with either every vertex of S or every vertex of W.*

Proof. Suppose, to the contrary, that there exists a vertex $z \in V - (S \cup W)$ that is nonadjacent to some vertex $u \in S$ and nonadjacent to some vertex $v \in W$. Since S and W are maximum independent sets in G and $G - S$, respectively, z must be adjacent with some vertex $x \in S$ and some $y \in W$. Let $D = V - \{u, v, z\}$. Since D is not an MLD-set of G, $c_D(u) = c_D(v)$. So, $N(u) - \{v\} = N(v) - \{u\}$. Therefore, $N(v) \cap S \subseteq \{u\}$ and $N(u) \cap W \subseteq \{v\}$. Since S is a maximum independent set in G, $N(v) \cap S = \{u\}$.

If $|S| \geq 3$, let $w \in S - \{u, x\}$. Then, $V - \{w, u, z\}$ is an MLD-set of G, a contradiction. Therefore, $|S| = |W| = 2$.

Let $w \in V - \{u, v, x, y, z\}$. Since S is a maximum independent set, w is adjacent with u or x. If w is adjacent with both u and x, then $V - \{v, w, z\}$ is an MLD-set of G, a contradiction. Hence either $uw \in E(G)$ or $wx \in E(G)$ (but not both). Suppose $uw \in E(G)$. Then, $wx \notin E(G)$. Similarly, $wy \notin E(G)$. Since W is a maximum independent set in $G - S$,

$vw \in E(G)$. So if a vertex $w \in V - (S \cup W)$ is adjacent with u or v, then $N(w) \cap (S \cup W) = \{u, v\}$. Similarly, if a vertex $w \in V - (S \cup W)$ is adjacent with x or y, then $N(w) \cap (S \cup W) = \{x, y\}$. Moreover, if w_1 and w_2 are two vertices in $V - (S \cup W)$ for which $N(w_1) \cap (S \cup W) = \{u, v\}$ and $N(w_2) \cap (S \cup W) = \{x, y\}$, then $w_1 w_2 \notin E(G)$, for otherwise $V - \{u, y, w_2\}$ is an MLD-set of G. It follows that G is disconnected, contrary to our assumption that G is connected. Hence, every vertex in $V - (S \cup W)$ is adjacent with either every vertex of S or every vertex of W. □

Claim 4 $V = S \cup W$.

Proof. Suppose, to the contrary, that $V - (S \cup W) \neq \emptyset$. Then, by Claim 3, every vertex in $V - (S \cup W)$ is adjacent with either every vertex of S or every vertex of W. By our earlier assumptions, $|S| \geq 2$ and $|W| \geq 2$. Suppose some vertex $z \in V - (S \cup W)$ is adjacent with every vertex of W. Let $x \in W$. Since S is a maximum independent set of G, there is a vertex $y \in S$ adjacent with x. Let $u \in S - \{y\}$. Then, $V - \{u, x, z\}$ is an MLD-set of G unless $N(u) - \{x, z\} = N(z) - \{u, x\}$. In particular, $N(z) \cap S \subseteq \{u\}$. Since S is a maximum independent set of G, we must have $N(z) \cap S = \{u\}$. However, then $V - \{u, x, y\}$ is an MLD-set of G, a contradiction. Similarly, if some vertex of $V - (S \cup W)$ is adjacent with every vertex of S, then we obtain a contradiction. It follows that $V = S \cup W$. □

We now return to the proof of Theorem 14. By Claim 3, $V = S \cup W$. We show now that $G \in \mathcal{F}_5$. Suppose that some $u \in S$ is nonadjacent to some $v \in W$. Let y and w be two distinct neighbors of u and let z be a neighbor of w distinct from u. Then, $V - \{v, y, z\}$ is an MLD-set of G, a contradiction. Hence, every vertex of S is joined to every vertex of $V - S$, and so $G \in \mathcal{F}_5$. This completes the proof of Theorem 14. □

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