# A LABELING PROBLEM ON THE PLANE GRAPHS $P_{a,b}$

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**Abstract:**

This Paper concerns a labeling problem of the plane graphs $P_{a,b}$. We discuss the magic labeling of type (1,1,1) and consecutive labeling of type(1,1,1) of the graphs $P_{a,b}$.

## 1. Introduction And Definitions:

We shall consider non-trivial finite connected planar graphs without loops and multiple edges. A graph G is said to be *Plane* if it is drawn on the Euclidean Plane in such a way that the edges do not cross each other except at the vertices of the graph. For a plane graph G it makes sense to consider its faces, including the unique face of infinite area. Let G= (V,E,F) be such a plane graph with the vertex set V(G), the edge set E(G) and the face set F(G), where $|V(G)|$, $|E(G)|$ and $|F(G)|$ are the number of vertices, edges and faces respectively.

A *labeling of type* (a,b,c) assigns labels from the set $\{1,2,3,\ldots,a|V(G)|+b|E(G)|+c|F(G)|\}$ to the vertices, edges and faces of G such that each vertex receives a labels, each edge receives b labels and each face receives c labels and each number is used exactly once as a label. We restrict a,b and c to be no greater than one. Labelings of type (1,0,0), (0,1,0) and (0,0,1) are also called *vertex*, *edge* and *face* labelings, respectively.

The *weight* of a face under a labeling is the sum of labels of the face itself together with labels of vertices and edges surrounding that face. A labeling is said to be *consecutive* if, for every number s, the weights of all s-sided faces constitute a set of consecutive integers. We allow different sets for different s.

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A labeling is said to be *magic* if, for every number s, all s-sided faces have the same weight. We allow different weights for different s. Two labelings f and f' are said to be *complementary* if, for every integer s, the sum of the f-weight and the f'-weight of each s-sided face is a constant depending on s.

The notions of consecutive and magic labeling of plane graphs were defined by Ko – Wei Lih [1]. However, the subject can be traced back to the thirteenth century (Classical Chinese Mathematics) when similar notions were investigated by Yang Hui(1275) and later by Chang Chhao (1670) and Pao Chhi – Shou (1880), see [1].

Consecutive and magic labeling for wheels, friendship graphs, prisms and some of the platonic polyhedra are given in [1]. Consecutive and magic labeling for fans, planar bi-pyramids and ladders are described in [2] and for mobius ladders can be found in [3]. Baca has described the magic labeling of type (1,1,1) for the hexagonal planar graphs $H_n^m$ (honey comb) (see[4]) and for the grid graphs (see[5]). Consecutive labelings of type(1,1,1) of plane graphs $P_{m,n}$ are given in [6]. In [7], it is shown that if $m \geq 1$, $n \geq 3$, $n \neq 5$ and $n \neq 6$, then the plane graph $G_n^m$ has consecutive labelings of type (1,1,1). Consecutive labeling of type(1,0,0) for subdivision of ladders and lotus inside a circle are described in [8].

A certain generalization of the notion of consecutive labeling of graph is an antimagic labeling. The concept of an antimagic labeling was introduced by Hartsfield and Ringel [10]. An *antimagic graph* is a graph whose edges can be labeled with the integers $1,2,3,\ldots,|E(G)|$, so that the sum of the labels at any given vertex is different from the sum of the labels at any other vertex, that is, no two vertices have the same sum.

Bodendiek and Walther [11] defined the concept of an (a,d)- *antimagic graph* as a special case of an antimagic graph. For the special graphs called *Parachutes*, (a,d)- antimagic labelings are described in [12,13]. Note that an (a,0) -antimagic labeling and an (a,1)- antimagic labeling of the graph G of a convex polytope is equivalent to magic labeling and consecutive labeling of a dual graph $G^*$, respectively.

Magic labeling of type (1,1,1) for the plane graphs consisting of 3-sided faces, 4- sided faces, 5- sided faces and one external infinite face are described in [14,15]. Magic labeling of type (1,1,1) for two classes of plane graphs in which the face sets contain 3-sided faces, 5- sided faces, 6- sided faces and one external infinite face are described in [9].

Kathiresan has described the gracefulness of a new class of graphs denoted by $P_{a,b}$ (see[9]). This paper describes the magic labelings of type (1,1,1) for plane graph $P_{a,b}$ if b is odd, b≥3, a≥2 and consecutive labelings of type (1,1,1) for $P_{a,b}$ if b is even, b≥2, a≥3.

We shall use the function $\alpha(n) = [1+(-1)^n]/2$ to simplify the later notations.

**2. Some Lemmas:**

Let u and v be the two fixed vertices. We connect u and v by means of b≥2 internally disjoint paths of length a≥2 each. The resulting graph embedded in the plane is denoted by $P_{a,b}$ (see[9]). Let $v_0^i, v_1^i, \ldots, v_a^i$ be the vertices of the $i^{th}$ copy of the path of length a where i = 1,2,..., b, $v_0^i = u$ and $v_a^i = v$ for all i. We observe that the number of vertices of this graph is ab-b+2 and the number of edges of this graph is ab. This graph contains b 2a-sided faces including the external face.

**Lemma 2.1**

The graph $P_{a,b}$ has a magic labeling of type (1,0,0) when $a \equiv 1 \pmod 2$.

**Proof:**

Define a mapping $f_1$: $V(P_{a,b}) \to \{1,2,3,\ldots,ab-b+2\}$ in the following way.

$f_1(u) = 1$, $f_1(v) = 2$

$f_1(v_{2j+1}^i) = 2bj+2+i$ ; $1 \le i \le b$, $0 \le j \le (a-3)/2$

$f_1(v_{2j}^i) = 2bj+3-i$ ; $1 \le i \le b, 1 \le j \le (a-1)/2$

Under the vertex labeling $f_1$ the common weight for all 2a – sided faces is $a^2b - 2ab + 5a + b - 2$.

**Lemma 2.2**

The graph $P_{a,b}$ has a consecutive labeling of type (1,0,0) when $a \equiv 0 \pmod 2$ and $b \equiv 1 \pmod 2$.

**Proof:**

Define a mapping $f_2$: $V(P_{a,b}) \to \{1,2,3,\ldots,ab-b+2\}$ in the following way.

$f_2(u) = 1$, $f_2(v) = 2$

$f_2(v_1^i) = \alpha(i+1)\,[(i+5)/2] + \alpha(i)\,[(b+5+i)/2]$ ; $1 \le i \le b$

$f_2(v_{2j+1}^i) = 2bj+2+i$ ; $1 \le i \le b$, $1 \le j \le (a-2)/2$

$f_2(V_{2j}^{i}) = 2bj+3-i \quad ; \quad 1 \le i \le b, 1 \le j \le (a-2)/2.$

The weights for all 2a - sided faces successively assume consecutive values
$(2a^2b+10a-4ab+b-3)/2, (2a^2b+10a-4ab+b-1)/2, (2a^2b+10a-4ab+b+1)/2, \ldots, (2a^2b+10a-4ab+3b-5)/2.$

**Lemma 2.3**

The graph $P_{a,b}$ has a magic labeling of type(1,0,0) when $a \equiv 0 \pmod 2$, $a \neq 2$ and $b \equiv 0 \pmod 2$.

**Proof:**

Define a mapping $f_3 : V(P_{a,b}) \to \{1,2,\ldots,ab-b+2\}$ in the following way.

$f_3(u) = 1, f_3(v) = 2$

$f_3(V_1^{i}) = \alpha(i+1)\,[(i+5)/2] + \alpha(i)\,[(b+4+i)/2] \quad ; \quad 1 \le i \le b$

$f_3(V_{2j+1}^{i}) = 2bj+2+i \quad ; \quad 1 \le i \le b, 1 \le j \le (a-4)/2$

$f_3(V_{2j}^{i}) = 2bj+3-i \quad ; \quad 1 \le i \le b, 1 \le j \le (a-2)/2$

$f_3(V_{a-1}^{i}) = \alpha(i+1)\,[(2ab-4b+5+i)/2] + \alpha(i)\,[(2ab-3b+4+i)/2] \quad ;$

$1 \le i \le b.$

Under the vertex labeling $f_3$ the common weight for all 2a- sided faces is $a^2b-2ab+5a+b-2$.

**Lemma 2.4**

The graph $P_{2,b}$ has a consecutive labeling of type(1,0,0) when $b \equiv 0 \pmod 4$.

**Proof:**

Define a mapping $f_4 : V(P_{2,b}) \to \{1,2,\ldots,b+2\}$ in the following way

$f_4(u) = 1, f_4(v) = (b/4)+2$

$f_4(V_1^{2i-1}) = i+1 \quad ; \quad 1 \le i \le (b/4)$

$f_4(V_1^{2i-1}) = i+2 \quad ; \quad (b/4)+1 \le i \le (b/2)$

$f_4(V_1^{2i}) = (b/2)+2+i \quad ; \quad 1 \le i \le (b/2).$

The weights for all 4-sided faces successively assume consecutive values from $(3b+32)/4$ to $(7b+28)/4$.

**Lemma 2.5**

The graph $P_{2,b}$ has no magic labeling of type(1,0,0) for $b \geq 3$.

**Proof:**

Suppose that a bijection $\varphi : V(P_{2,b}) \rightarrow \{1,2,\ldots,b+2\}$ is a magic labeling of type (1,0,0). Then $\varphi(u) + \varphi(v_1^i) + \varphi(v) + \varphi(v_1^{i+1})$ is equal to $\varphi(u) + \varphi(v_1^{i+1}) + \varphi(v) + \varphi(v_1^{i+2})$ for $i=1,2,\ldots,b-2$. Hence $\varphi(v_1^i) = \varphi(v_1^{i+2})$, which is a contradiction.

**Lemma 2.6**

The graph $P_{2,b}$ has no consecutive labeling of type (1,0,0) when $b \equiv 2 \pmod 4$

**Proof:**

Suppose that $\varphi : V(P_{2,b}) \rightarrow \{1,2,\ldots,b+2\}$ is a consecutive labeling of type (1,0,0) with face weights $c, c+1, c+2, \ldots, c+b-1$. The sum of face weights is $bc + (b-1)b/2$ and the sum of all the vertex labels used to calculate the weights of faces is $b\varphi(u) + b\varphi(v) + 2\sum_{i=1}^{b}\varphi(v_1^i)$ .

Thus we get $b\varphi(u) + b\varphi(v) + 2\sum_{i=1}^{b}\varphi(v_1^i) = bc + (b-1)b/2$ . Since $b \equiv 2 \pmod 4$,

we have $(b-1)b/2$ is odd and $b[\varphi(u)+\varphi(v)-c] + 2\sum_{i=1}^{b}\varphi(v_1^i)$ is even.

Consequently, the consecutive labeling $\varphi$ cannot exist.

**Lemma 2.7**

The graph $P_{a,b}$ has a magic labeling of type (0,1,0) when $a \equiv 0 \pmod 2$.

**Proof:**

Define a mapping $f_5 : E(P_{a,b}) \rightarrow \{1,2,\ldots,ab\}$ in the following way

$$f_5(u\, v_1^i) = i \qquad ;1 \leq i \leq b$$

$$f_5(v_{2j-1}^i\, v_{2j}^i) = 2bj+1-i \qquad ;1 \leq i \leq b,\ 1 \leq j \leq a/2$$

$$f_5(v_{2j}^i\, v_{2j+1}^i) = 2bj + i \qquad ;1 \leq i \leq b,\ 1 \leq j \leq (a-2)/2.$$

Under the edge labeling $f_5$ the common weight for all 2a -sided faces is $a^2b + a$.

**Lemma 2.8**

The graph $P_{a,b}$ has a consecutive labeling of type (0,1,0) when $a \equiv 1 \pmod 2$ and $b \equiv 1 \pmod 2$.

**Proof:**

Define a mapping $f_6: E(P_{a,b}) \to \{1,2,\ldots,ab\}$ in the following way

$$f_6(u\, v_1^i) = \alpha(i+1)\,[(i+1)/2] + \alpha(i)[(b+1+i)/2] \quad ;1 \le i \le b$$

$$f_6(v_{2j-1}^i v_{2j}^i) = 2bj+1-i \quad ;1 \le i \le b,\ 1 \le j \le (a-1)/2$$

$$f_6(v_{2j}^i v_{2j+1}^i) = 2bj + i \quad ;1 \le i \le b,\ 1 \le j \le (a-1)/2.$$

The weights for all 2a-sided faces successively assume consecutive values $(2a^2b + 2a - b+1)/2$, $(2a^2b + 2a - b+3)/2$, $(2a^2b + 2a - b+5)/2, \ldots,$ $(2a^2b + 2a + b-1)/2$.

**Lemma 2.9**

The graph $P_{a,b}$ has a magic labeling of type(0,1,0) when $a \equiv 1 \pmod 2$ and $b \equiv 0 \pmod 2$.

**Proof:**

Define a mapping $f_7: E(P_{a,b}) \to \{1,2,\ldots,ab\}$ in the following way.

$$f_7(u\, v_1^i) = \alpha(i+1)\,[(i+1)/2] + \alpha(i)\,[(b+i)/2] \quad ;1 \le i \le b$$

$$f_7(v_{2j-1}^i v_{2j}^i) = 2bj+1-i \quad ;1 \le i \le b,\ 1 \le j \le (a-1)/2$$

$$f_7(v_{2j}^i v_{2j+1}^i) = 2bj + i \quad ;1 \le i \le b,\ 1 \le j \le (a-3)/2$$

$$f_7(v_{a-1}^i v) = \alpha(i+1)\,[(2ab - 2b + 1 + i)/2] + \alpha(i)\,[(2ab - b + i)/2]; \quad 1 \le i \le b.$$

Under the edge labeling $f_7$ the common weight for all 2a -sided faces is $a^2b + a$.

## 3. Main Results:

**Theorem 3.1**

The graph $P_{a,b}$ has a magic labeling of type (1,1,1) when $a \equiv 1 \pmod 2$ and $b \equiv 1 \pmod 2$.

**Proof:**

Label the vertices and edges of $P_{a,b}$ by $f_1$ and $|V(P_{a,b})| + f_6$ respectively. From lemma 2.1 and lemma 2.8 it easily follows that in the resulting labeling of type (1,1,0) the weights of 2a- sided faces constitute a set of

consecutive integers. Hence, if $f_8$ is the complementary face labeling with values in the set
$\{|V(P_{a,b})| + |E(P_{a,b})| +1,..,|V(P_{a,b})| + |E(P_{a,b})| + |F(P_{a,b})|\}$,
then the labeling $f_1$, $|V(P_{a,b})| + f_6$ and $f_8$ combine to a magic labeling of type (1,1,1).

**Theorem 3.2**

The graph $P_{a,b}$ has a magic labeling of type (1,1,1) when $a \equiv 0 \pmod 2$ and $b \equiv 1 \pmod 2$.

**Proof:**

Label the vertices and edges of $P_{a,b}$ by $f_2$ and $|V(P_{a,b})| + f_5$ respectively. From lemma 2.2 and lemma 2.7 it easily follows that in the resulting labeling of type (1,1,0) the weights of 2a- sided faces constitute a set of consecutive integers. Hence, if $f_9$ is complementary face labeling with values in the set
$\{|V(P_{a,b})| + |E(P_{a,b})| +1, \ldots,|V(P_{a,b})| + |E(P_{a,b})| + |F(P_{a,b})|\}$,
then the labeling $f_2$, $|V(P_{a,b})| + f_5$ and $f_9$ combine to a magic labeling of type (1,1,1).

**Theorem 3.3**

The graph $P_{a,b}$ has a consecutive labeling of type (1,1,1) when $a \equiv 1 \pmod 2$ and $b \equiv 0 \pmod 2$.

**Proof:**

Label the vertices and edges of $P_{a,b}$ by $f_1$ and $|V(P_{a,b})| + f_7$ respectively. From lemma 2.1 and lemma 2.9 it easily follows that in the resulting labeling of type (1,1,0) the weights of 2a- sided faces have a common weight . Hence, if $f_{10}$ is face labeling with values in the set
$\{|V(P_{a,b})| + |E(P_{a,b})| +1,\ldots,|V(P_{a,b})| + |E(P_{a,b})| + |F(P_{a,b})|\}$,
then the labelings $f_1$, $|V(P_{a,b})| + f_7$ and $f_{10}$ combine to a consecutive labeling of type (1,1,1).

**Theorem 3.4**

The graph $P_{a,b}$ has a consecutive labeling of type (1,1,1) when $a \equiv 0 \pmod 2$, $a \neq 2$ and $b \equiv 0 \pmod 2$.

**Proof:**

Label the vertices and edges of $P_{a,b}$ by $f_3$ and $|V(P_{a,b})| + f_5$ respectively. From lemma 2.3 and lemma 2.7 it easily follows that in the resulting labeling of type (1,1,0) the weights of 2a- sided faces have a common weight. Hence, if $f_{11}$ is the face labeling with values in the set
$\{|V(P_{a,b})| + |E(P_{a,b})| + 1,\ldots,|V(P_{a,b})| + |E(P_{a,b})| + |F(P_{a,b})|\}$,

then the labeling $f_3$, $|V(P_{a,b})| + f_5$ and $f_{11}$ combine to a consecutive labeling of type (1,1,1)

**Theorem 3.5**

The graph $P_{2,b}$ has a magic labeling of type (1,1,1) when $b \equiv 0 (\mathrm{mod}\ 4)$.

**Proof:**

Label the vertices and edges of $P_{2,b}$ by $f_4$ and $|V(P_{2,b})| + f_5$ respectively. From lemma 2.4 and lemma 2.7 it easily follows that in the resulting labeling of type (1,1,0) the weights of 2a- sided faces constitute a set of consecutive integers. Hence, if $f_{12}$ is the complementary face labeling with values in the set $\{|V(P_{2,b})| + |E(P_{2,b})| + 1, \ldots, |V(P_{2,b})| + |E(P_{2,b})| + |F(P_{2,b})|\}$, then the labeling $f_4$, $|V(P_{2,b})| + f_5$ and $f_{12}$ combine to a magic labeling of type (1,1,1).

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**References:**

[1]. Lih Ko-Wei, On magic and consecutive labelings of plane graphs, *Utilitas Math.*24(1983),165-197.

[2]. Baca,M., On magic and consecutive labelings for the special classes of plane graphs,*Utilitas Math.*32(1987),59-65.

[3]. Baca,M.,On magic labelings of Mobius ladders, *J.Franklin Inst* .326 (1989),885-888.

[4]. Baca,M., On magic labelings of honeycomb,*Discrete Math.* 105 (1992), 305-311

[5]. Baca,M.,On magic labelings of grid graphs , *Ars Combinatoria* 33(1992) 295-299

[6]. Baca,M., On consecutive labeling of plane graphs, *J.Franklin Inst* .328 (1991), 249-253.

[7]. Baca, M. and Hollander,L.,Labelings of a certain class of convex polytope J.*Franklin Inst* . 329 (1992),539-547.

[8]. Kathiresan, KM., Muthuvel, S. and Nagasubbu, V.N., Consecutive labelings for two classes of plane graphs ,*Utilitas Math* .55 (1999),237-241.

[9]. Kathiresan, KM., Two classes of graceful graphs, *Ars Combinatoria* 55 (2000),129-132.

[10]. Hartsfield, N. and Ringel,G., Pearls in graph theory, *Academic press,* Boston –SanDiego-Newyork-London (1990).

[11]. Bodendiek, R. and Walther,G., On number theoretical methods in graph labelings , *Res. Exb.Math* .21 (1995),3-25.

[12]. Bodendiek,R.and Walther,G., On (a,d)-antimagic parachutes, *Ars combinatoria* 42 (1996) ,129-149.

[13]. Bodendiek,R.and Walther,G., On (a,d)-antimagic parachutes II, Ars *combinatoria* 46 (1997) ,33-63.
[14]. Baca, M..,On magic labelings of type (1,1,1) for three classes of plane graph, *Math. Slovaca* 39 (1989),233-239.
[15]. Baca, M..,Labelings of two classes of plane graphs, *Acta Math.Appl. Sinica* 9 (1993),82-87.