Minor clique free extremal graphs

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Abstract

In this note, we prove that the largest non contractible to K^p graph of order n with $\left\lceil \frac{2n+3}{3} \right\rceil \leq p \leq n$, is the Turán's graph $T_{2p-n-1}(n)$. Furthermore, a new upperbound for this problem is determined.

1 Introduction

The study of the maximum number of edges $ex(n; F)$ of a graph of order n not containing F as a subgraph is one of the best known extremal problems in Graph Theory. In this sense, the most important result is Turán's Theorem [17], proved in 1941, that answers the question when F is the complete graph K^p.

Various non-trivial extensions of the Turán's problem have been analyzed, but the results are rather fragmentary. Two of these extensions are the following:

(1) The study of the maximum number of edges $ex(n; MK^p)$ of a graph of order n not containing K^p as a minor.

(2) The study of the maximum number of edges $ex(n; TK^p)$ of a graph of order n not containing K^p as a topological minor.

It is clear that these extremal functions are pairwise related. In effect, if a graph G contains K^p as a topological minor then G contains it as a minor. The other implication holds for $p \leq 4$ (See [7]).

Dirac [8] proved that $ex(n; MK^5) = 3n - 6$ and conjectured the same exact value for the function $ex(n; TK^5)$. Finally, this famous conjecture was verified by Mader in [14]. New exact values of the function $ex(n; MK^p)$ have been found by Mader [13] for $6 \leq p \leq 7$ and by Jorgensen [11] for $p = 8$, but they are still unknown for the function $ex(n; TK^p)$. It seems

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that these values may be also the same, because up to now it has not been possible to find a graph G of order n with at least $ex(n; MK^p) + 1$ edges not containing K^p as a topological minor.

There are also some works where the two extremal functions have been bounded. An upperbound for the function $ex(n; MK^p)$ was first given by W. Mader. He proved that there is a constant c_p for each $p \geq 2$, such that $ex(n; MK^p) \leq c_p n$, and if c_p is the infimum of all the constants for a p fixed, then $c_p \leq \lceil 2c_{p-1} \rceil$. He proved in [13] that $c_p \leq 8p \log_2(p)$. W.F. de la Vega, in [6] and A. Thomason [15] proved that $c_p \geq \frac{1}{4} p \sqrt{\log_2(p)}$ and this last author also showed that $c_p \leq 2.68 p \sqrt{\log_2(p)}$. These extremal results have been improved by A. Thomason [16]. With regards to the function $ex(n; TK^p)$, it was conjectured by W. Mader [12] and P. Erdös and A. Hajnal [9] that $ex(n; TK^p) \leq cp^2 n$ and finally, B. Bollobás and A. Thomason [1] have shown that $ex(n; TK^p) \leq 256 p^2 n$.

All these works are devoted to find out results when p is a fixed integer. However, in recent works of M. Cera, A. Diánez and A. Márquez [2] and [3] and M. Cera, A. Diánez and P. García–Vázquez [4] a new technique based in showing the existence of a complete matching in an adequate bipartite graph has been developed and it has permitted them to find out new solutions for the function $ex(n; TK^p)$. In this case, results have been found working with n and p related by the expression $\left\lceil \frac{2n+2}{3} \right\rceil \leq p \leq n$. Besides, it is curious that the solutions may be expressed in terms of the Turán's graph.

The purpose of this paper is to find out new exact values of the function $ex(n; MK^p)$ in the same infinity sector as the one described above. In fact, using the relationship between these functions both, we will prove that the solutions are also coincident in this case. Besides we will characterize the extremal family $EX(n; MK^p)$ formed by graphs of order n with $ex(n; MK^p)$ edges that are not contractible to K^p. Finally we will get an upperbound of this function that, asymptotically, is worse than the one proved in [1], but it is a good upperbound when n and p are related.

2 Notations

For any graph G we say that G contains K^p as a minor if K^p can be obtained from a subgraph of G by a sequence of edge-contractions. We say that G contains K^p as a topological minor, if it is possible to find p vertices $\{v_1 \ldots, v_p\}$ and $\binom{p}{2}$ pairwise vertex disjoint paths joining these vertices in G.

We denote by $T_r(n)$ the r-partite Turán's graph with n vertices, i.e., the unique complete r-partite graph of order n whose classes are as equal as possible. The number of edges of this graph, known by the Turán's number, is denoted by $t_r(n)$.

Given a graph G and a subset of vertices of G, V, we denote by $G[V]$ the induced subgraph in G by the set V. Finally, we denote by $E(G)$ the set of edges of G and by $e(G)$ the cardinality of this set.

Notations and terminologies not explicitly specified here can be found in [7].

3 Exact values and extremal family for contractions to K^p

In this section we will show that for values of n and p such that $\left\lceil \frac{2n+2}{3} \right\rceil \leq p \leq n$, the solutions for the extremal problem $ex(n; MK^p)$ for contractions to complete graphs are the same as the solutions for the extremal problem $ex(n; TK^p)$ for containing K^p as topological minor. So, we will conclude deducing that these solutions are the same as the ones pointed by the Turán's Theorem for the function $ex(n; K^{2p-n})$.

We know that the two extremal functions described above are pairwise related, as we observe in the following result:

Proposition 1 *(See [7]) Every topological minor of a graph is also its minor.*

So, by applying Proposition 1, we may deduce various consequences, described in this corollary:

Corollary 1 *Let n and p positive integers, with $n \geq p$. Then:*

1. $ex(n; MK^p) \leq ex(n; TK^p)$.
2. *If* $ex(n; MK^p) = ex(n; TK^p)$, *then* $EX(n; MK^p) \subseteq EX(n; TK^p)$.

On one hand, there exists the following result shown in [5] which relates the extremal function $ex(n; MK^p)$ with the Turán's graph $T_{2p-n-1}(n)$.

Theorem 1 *(See [5]) Let n and p be positive integers such that $p \leq n \leq 2p-3$. Then the Turán's graph $T_{2p-n-1}(n)$ is not contractible to K^p.*

On the other hand solutions for the extremal problem for the function $ex(n; TK^p)$ are gotten in [2] and they also are related with the same Turán's graph $T_{2p-n-1}(n)$, as we may observe.

Theorem 2 *(See [2]) Let n and p be positive integers, with* $\left\lceil \frac{2n+2}{3} \right\rceil \le p \le n$. *Then:*

$$ex\,(n; TK^p) = t_{2p-n-1}(n).$$

Taking into account Theorems 1 and 2, we may deduce that $ex(n; MK^p) \ge ex(n; TK^p) = t_{2p-n-1}(n)$, and conclude with the following result.

Theorem 3 *Let n and p be positive integers, with* $\left\lceil \frac{2n+2}{3} \right\rceil \le p \le n$. *Then:*

$$ex\,(n; MK^p) = ex\,(n; TK^p) = ex(n; K^{2p-n}).$$

In order to characterize the extremal family $EX(n; MK^p)$ formed by the graphs with n vertices and $ex(n; MK^p)$ edges that are not contractible to K^p, we will also apply the known results about the extremal family $EX(n; TK^p)$ for containing K^p as a topological minor.

On one hand, in Theorem 3 we have proved that it is verified that $ex(n; MK^p) = ex(n; TK^p)$, for $\left\lceil \frac{2n+2}{3} \right\rceil \le p \le n$. So, by applying Corollary 1, we deduce that

$$EX(n; MK^p) \subseteq EX(n; TK^p).$$

On the other hand, the extremal family $EX(n; TK^p)$ is characterized in [3], and they may be summarized in the following result.

Theorem 4 *(See [3]) Let n and p be positive integers, with* $\left\lceil \frac{2n+3}{3} \right\rceil \le p \le n-5$ *Then:*

$$EX\,(n; TK^p) = \{T_{2p-n-1}(n)\}\,.$$

Finally, by Theorem 1, we know that the graph $T_{2p-n-1}(n)$ is not contractible to K^p. So the other contention is proved and, consequently, we deduce the following result.

Theorem 5 *Let n and p be positive integers, with* $\left\lceil \frac{2n+3}{3} \right\rceil \le p \le n-5$. *Then*

$$EX\,(n; MK^p) = EX\,(n; TK^p) = EX(n; K^{2p-n}).$$

4 An upperbound for the function ex(n; MK^p)

For values of n and p such that $p < \left\lceil \frac{2n+2}{3} \right\rceil$ the solution of the extremal problem $ex(n; MK^p)$ is unknown yet. In this section we are going to approach the exact value of this function with an upperbound.

Previously, we recall a technical lemma shown in [2] that relates the number of vertices of maximum degree 2 in a graph with the number of them being independent.

Lemma 1 *Let k be a nonnegative integer and H a graph with maximum degree 2 and at least $3k+1$ vertices of maximum degree. Then there exist at least $k+1$ nonadjacent vertices with degree 2.*

Since $ex(n; MK^p) \leq ex(n; TK^p)$, all exact values or upperbounds for the function $ex(n; TK^p)$ permit us to deduce an upperbound for the function
$ex(n; MK^p)$. In this sense, the following theorem, shown in [4], may be useful in order to get this goal.

Theorem 6 *Let n and p be positive integers with $n - p \geq 11$ and $\left\lceil \frac{7n+7}{12} \right\rceil \leq p < \left\lceil \frac{2n+1}{3} \right\rceil$. It is verified:*

1. $ex(n; TK^p) = \binom{n}{2} - (5n - 6p + 2)$.

2. $EX(n; TK^p) = \left\{ (2n - 3p + 1)\overline{K^4} + \overline{F} : \ F \in \mathcal{F}_{12p-7n-4} \right\}$,

where $\mathcal{F}_{12p-7n-4}$ is the family of graphs of order $12p-7n-4$ whose vertices have degree 2.

If it would be verified that $ex(n; MK^p) = ex(n; TK^p)$, then, by applying Corollary 1, we would have that $EX(n; MK^p) \subseteq EX(n; TK^p)$.

Given G a graph belonging to the family $EX(n; TK^p)$, denoting by H its complement graph, we have that H is a graph of order n whose connected components are $2n - 3p + 1$ copies of K^4 and a certain graph F with $12p - 7n - 4$ vertices formed by one or several disjoint cycles. Since F has $3\left(4p - \frac{7n+7}{3}\right) + 1$ vertices of maximum degree 2, by applying Lemma 1, there exist, at least, $4p - \frac{7n+7}{3} + 1$ nonadjacent vertices with degree 2 in F.

We are going to select an appropriate subset $U = \{v_1, \ldots, v_{n-p}\}$ of vertices of H, according to the next two cases.

(1) If $2(2n-3p+1) < n-p$, then we have that

$$4p - \frac{7n+7}{3} + 1 = 5p - 3n - 2 + \frac{2n-3p+2}{3} > 5p - 3n - 2.$$

So, there exist, at least, $5p-3n-2$ nonadjacent vertices of degree 2 in F. In this case, let U be the resultant subset obtained by choosing two vertices in each copy of K^4 and $5p-3n-2$ nonadjacent vertices of degree 2 in F.

(2) For $2(2n-3p+1) \geq n-p$, let U be the resultant subset by choosing two vertices in $2p-n-1$ copies of K^4 and one vertex in $3n-5p+2$ copies of K^4.

Hence, it is easy to check that certain edge-contractions in G permit us to obtain a complete graph K^p whose set of vertices is $V(G) \setminus U$.

Summarizing, given any graph $G \in EX(n; TK^p)$, we can check that G contains K^p as a minor. But since $EX(n; MK^p) \subseteq EX(n; TK^p)$, we have that $EX(n; MK^p) = \emptyset$ and this is not possible.

So, in this infinity sector, solutions for both extremal problems are not coincident, and we only may deduce the following upperbound.

Theorem 7 *Let n and p be positive integers with $n-p \geq 11$ and $\left\lceil \frac{7n+7}{12} \right\rceil \leq p < \left\lceil \frac{2n+1}{3} \right\rceil$. Then*

$$ex\,(n; MK^p) \leq ex\,(n; TK^p) - 1.$$

In the following theorem, we are going to get a relationship between the values $ex(n-1; MK^{p-1})$ and $ex(n; MK^p)$. For this, we will prove that given G a graph belonging to the family $EX(n-1; MK^{p-1})$, it is possible to find a graph G^* with n vertices that are not contractible to K^p.

Theorem 8 *Let n and p be positive integers, with $n \geq p$. It is verified that*

$$ex\left(n-1; MK^{p-1}\right) \leq ex\,(n; MK^p) - (n-1).$$

Proof. Let G be a graph belonging to the family $EX(n-1; MK^{p-1})$. This graph has $n-1$ vertices, $ex(n-1; MK^{p-1})$ edges and is not contractible to K^{p-1}. Let $\{v_1, \ldots, v_p\}$ be a subset of vertices of G and let v be a vertex not belonging to G. Let's consider the graph $G^* = G + v = G + K^1$.

Since G is not contractible to K^{p-1}, it is evident that the graph with vertices $\{v_1, \ldots, v_p\}$ obtained by certain sequence of edges-contractions in G needs at least, two disjoint edges in order to be K^p, because otherwise this graph would contain a K^{p-1} and, therefore, G would be contractible to K^{p-1}. So, G^* is not contractible to a complete graph K^p with vertices $\{v_1, \ldots, v_p\}$.

But, on the other hand, it is evident that G^* is not contractible to a K^p with vertices $\{v_1, \ldots, v_p, v\} - \{v_j\}$, because otherwise G would be contractible to a K^{p-1} with vertices $\{v_1, \ldots, v_p\} - \{v_j\}$ and this is not possible.

Hence G^* is not contractible to K^p and, therefore,

$$ex(n; MK^p) \geq e(G^*) = ex(n-1; MK^{p-1}) + (n-1).$$

□

The previous result permits us to obtain a new upperbound valid for all n and p with $n - p \geq 11$ and $9 \leq p < \left\lceil \frac{7n+7}{12} \right\rceil$, as it is observed in this corollary.

Corollary 2 *Let n and p be positive integers, with $n - p \geq 11$ and $9 \leq p < \left\lceil \frac{7n+7}{12} \right\rceil$. Then*

$$ex(n; MK^p) \leq \binom{n}{2} - \left\lceil \frac{18n - 18p + 4}{5} \right\rceil.$$

Proof. Given n and p as in the hypothesis. Let's consider the following two positive integers:

$$\begin{aligned} p^* &= \left\lceil \frac{7(n-p)+7}{5} \right\rceil \\ n^* &= p^* + (n-p) \end{aligned}$$

It is easy to check that $p^* = \left\lceil \frac{7n^*+7}{12} \right\rceil$.

So, applying Theorem 7, we have that

$$ex\left(n^*; MK^{p^*}\right) \leq \binom{n^*}{2} - (5n^* - 6p^* + 3).$$

And applying Theorem 8 successively, we have that

$$\begin{aligned} ex(n; MK^p) &\leq ex\left(n+1; MK^{p+1}\right) - n \leq \ldots \leq \\ &\leq ex\left(n^*; MK^{p^*}\right) - \sum_{i=n}^{n^*-1} i \\ &= ex\left(n^*; MK^{p^*}\right) - \left[\binom{n^*}{2} - \binom{n}{2}\right] \\ &\leq \binom{n}{2} - (5n^* - 6p^* + 3). \end{aligned}$$

Hence,

$$
\begin{aligned}
ex(n; MK^p) &\leq \binom{n}{2} - (5n^* - 6p^* + 3) \\
&= \binom{n}{2} - (5(n-p) - p^* + 3) \\
&= \binom{n}{2} - \left\lceil \frac{18n - 18p + 4}{5} \right\rceil .
\end{aligned}
$$

□

5 Conclusions

Up to now, solutions for the extremal problem $ex(n; MK^p)$ were known only for $p \leq 8$. In this work we have solved this problem for infinity values of n and p related by the expression $\left\lceil \frac{2n+2}{3} \right\rceil \leq p \leq n-2$. We have also got a new upper bound for the function $ex(n; MK^p)$. All results are summarized in Tables 1 and 2.

Values of n and p	$ex(n; MK^p)$	Reference
$p \leq 7$	$(p-2)n - \binom{p-1}{2}$	[13]
$p = 8$	$6n - 20$, if 5 divides n $6n - 21$, otherwise	[11]
$\left\lceil \frac{2n+2}{3} \right\rceil \leq p \leq n$	$t_{2p-n-1}(n)$	Theorem 3

Table 1: Exact values for the function $ex(n; MK^p)$.

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Values of n and p	$ex(n; MK^p)$	Reference
$\left\lceil \frac{7n+7}{12} \right\rceil \leq p < \left\lceil \frac{2n+1}{3} \right\rceil$ $n - p \geq 11$	$\binom{n}{2} - (5n - 6p + 3)$	Theorem 7
$p < \left\lceil \frac{7n+7}{12} \right\rceil$ $n - p \geq 11$	$\binom{n}{2} - \left\lceil \frac{18n - 18p + 4}{5} \right\rceil$	Corollary 2

Table 2: Upper bounds for the function $ex(n; MK^p)$.

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