On the enumeration of noncrossing partitions with fixed points

A. Sapounakis and **P. Tsikouras**
Department of Informatics
University of Piraeus
80, Karaoli and Dimitriou
18534 Pireaus, Greece.

e-mail: pgtsik@unipi.gr

February 4, 2004.

Abstract

New results on the enumeration of noncrossing partitions with m fixed points are presented, using an enumeration polynomial $P_m(x_1, x_2, \ldots, x_m)$. The double sequence of the coefficient $a_{m,k}$ of each x_i^k in P_m is endowed with some important structural properties, which are used in order to determine the coefficient of each $x_i^k x_j^l$ in P_m.

1 Introduction

A partition $\pi = B_1/B_2/\cdots/B_m$ of a totally ordered set X is called *noncrossing partition* (n.c.p.) iff there do not exist four elements $a < b < c < d$ of X such that $a, c \in B_i$, $b, d \in B_j$ and $i \neq j$. We denote the set of all n.c.p. of X with $NC(X)$.

Among many authors that have worked on n.c.p. are Kreweras [4], Poupard [5], Edelman [3] and more recently Speicher [9], Stanley [10], Athanasiadis [1] and Simion [7], [8].

A class of n.c.p. that is of particular interest has been introduced in [6] . Namely, $\pi \in NC(X)$ is called *noncrossing partition with fixed points* the elements of a given set $A \subseteq X$ iff every block of π contains exactly one element of A. The set of all these n.c.p. is denoted with $NC(X, A)$. Our aim is to evaluate the cardinality $|NC(X, A)|$.

Since it is the distribution of the elements of A in X, rather than the elements themselves, that determines the cardinality $|NC(X, A)|$, it is more convenient to restrict the problem to the equivalent case where $A = [m] = \{1, 2, \ldots m\}$ and $X \subseteq [1, +\infty)$, so that a function f_m of m variables is defined with $f_m(x_1, x_2, \ldots, x_m) = |NC(X, [m])|$, where $x_i = |X \cap (i, i+1)|$ for every $i \in [m-1]$ and $x_m = |X \cap (m, +\infty)|$ (see [6]). A recursive formula for the evaluation of $f_m(x_1, x_2, \ldots, x_m)$ has been given.

Furthermore, the following result has been proved.

Proposition 1.1 *For every $m \geq 2$, there exists a polynomial $P_m(x_1, x_2, \ldots, x_m)$ of degree $m-2$, with positive rational coefficients, such that*

$$f_m(x_1, x_2, \ldots, x_m) = \prod_{\nu=1}^{m} (x_\nu + 1) P_m(x_1, x_2, \ldots, x_m)$$

for every sequence (x_i), $i \in [m]$, in $\mathbb{N}$.

The polynomial $P_m(x_1, x_2, \ldots, x_m)$ is called *enumerating polynomial* of the set $NC(X, [m])$. So, we concentrate our efforts on the evaluation of the coefficient $[x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}]P_m$ of each term $x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$ of P_m, where $k_i \in \{0, 1, \ldots, m-2\}$ for every $i \in [m]$ and $\sum_{i=1}^{m} k_i \leq m-2$.

In [6] the double sequence $a_{m,k} = [x_i^k]P_m$ has been determined with the aid of the Stirling numbers of the first kind.

The main result of this paper is the evaluation of the coefficients $[x_i^k x_j^l]P_m$ for every $i, j \in [m]$ and every k, l with $k+l \leq m-2$. This is established, using the properties of the sequence $a_{m,k}$ which are presented in the beginning of the next section.

2 The coefficients of $x_i^k x_j^l$ in P_m

In [6] we have proved that for every $i \in [m]$ and $k \in \{0, 1, \ldots, m-2\}$

$$a_{m,k} = [x_i^k]P_m = \frac{(-1)^{m-k}}{(m-1)!} \sum_{\rho=k+2}^{m} s(m, \rho) \qquad (1)$$

where $s(m, \rho)$ are the Stirling numbers of the first kind.

Notice that from this result we can easily obtain that $a_{m,0} = 1$ and $a_{m,m-2} = \frac{1}{(m-1)!}$.

The above coefficients are going to be used throughout this paper, so it is useful to present here some recursive relations which are necessary for their evaluation. The proofs of these relations are based on the fact that

$$s(m, r) = (-1)^{m-r}(m-1)!(a_{m,r-1} + a_{m,r-2})$$

which can be easily proved using (1), and on some basic recursive relations for the Stirling numbers of the first kind (see [2]).

Proposition 2.1 *The following relations hold for $\nu \leq m-2$.*

i) $a_{m+1,\nu} = a_{m,\nu} + \frac{1}{m}a_{m,\nu-1}$.

ii) $a_{m+1,\nu+1} = \sum_{r=\nu+2}^{m} \frac{1}{r}a_{r,\nu}$.

iii) $a_{m+1,\nu+1} = \frac{1}{m} \sum_{r=\nu+1}^{m} [\binom{r-1}{\nu+1} + \binom{r}{\nu+1}]a_{m,r-1}$.

Using the triangular, vertical and horizontal relations of proposition 2.1, we can form a table of the values of the coefficients $[x_i^k]P_m = a_{m,k}$.

$m\backslash k$	0	1	2	3	4	5
2	1					
3	1	$\frac{1}{2}$				
4	1	$\frac{5}{6}$	$\frac{1}{6}$			
5	1	$\frac{13}{12}$	$\frac{3}{8}$	$\frac{1}{24}$		
6	1	$\frac{77}{60}$	$\frac{71}{120}$	$\frac{7}{60}$	$\frac{1}{120}$	
7	1	$\frac{29}{20}$	$\frac{29}{36}$	$\frac{31}{144}$	$\frac{1}{36}$	$\frac{1}{720}$

The coefficients $[x_i^k]P_m$, for $m \leq 7$, $k \leq 5$.

In order to evaluate the coefficients $[x_i^k x_j^l]P_m$ we first need to find the formula for the function $f_m(x_1, x_2, \ldots, x_m)$ in the case that exactly two of its variables are nonzero. We first consider the case of consecutive nonzero variables. Since $f_m(y_1, y_2, \ldots, y_m) = f_m(x_1, x_2, \ldots, x_m)$ when $(y_1, y_2, \ldots, y_m)$ is a cyclic permutation of $(x_1, x_2, \ldots, x_m)$ (see [6]), we may restrict ourselves in the case where $x_1, x_2 \neq 0$.

Proposition 2.2 *For every $m \in \mathbb{N}$, with $m \geq 2$ we have that*

$$f_m(x_1, x_2, 0, \ldots, 0) = \binom{x_1+x_2+m}{m} - \binom{x_1+m-1}{m} - \binom{x_2+m-1}{m}.$$

Proof. For $m = 2$ the proof is obvious; so, let $m \geq 3$. Then, we are in the case that

$$f_m(x_1, x_2, 0, \ldots, 0) = |NC(X, [m])|$$

with $X = [m] \cup Y$, $Y \subset (1,2) \cup (2,3)$, $|Y \cap (1,2)| = x_1$ and $|Y \cap (2,3)| = x_2$.

We partition the set $NC(X, [m])$ into sets $A_{\mu,\nu}$ ($\mu, \nu \in \mathbb{N}$, $\mu \leq x_1$, $\nu \leq x_2$) defined as follows:

Each set $A_{\mu,\nu}$ consists of all $\pi \in NC(X,[m])$ with the property that the block of the element 2 contains $x-\mu$ elements of $(1,2)$ and $y-\nu$ elements of $(2,3)$.

Since $f_m(x_1,0,\dots,0) = \binom{x_1+m-1}{m-1}$ (see [6]), we get

$$|A_{\mu,\nu}| = f_{m-1}(\mu+\nu,0,\dots,0) = \binom{\mu+\nu+m-2}{m-2}$$

and hence, using the formula

$$\sum_{\mu=0}^{x_1}\sum_{\nu=0}^{x_2}\binom{\mu+\nu+m}{m} = \binom{x_1+x_2+m+2}{m+2} - \binom{x_1+m+1}{m+2} - \binom{x_2+m+1}{m+2}$$

we finally obtain

$$\begin{aligned} f_m(x_1,x_2,0,\dots,0) &= \sum_{\mu=0}^{x_1}\sum_{\nu=0}^{x_2}\binom{\mu+\nu+m-2}{m-2} \\ &= \binom{x_1+x_2+m}{m} - \binom{x_1+m-1}{m} - \binom{x_2+m-1}{m}. \end{aligned}$$ ∎

In order now to proceed to the case where x_i, x_j are nonconsecutive elements of $(x_1,x_2,\dots,x_m)$ we define their *cyclic distance* $\rho = min\{|i-j|-1, m-|i-j|-1\}$. Without loss of generality, we may assume that the nonzero variables are x_1 and $x_{\rho+2}$.

Proposition 2.3 *For every $m \in \mathbb{N}$, with $m \geq 4$ and for every $\rho \in \mathbb{N}^*$, with $2\rho \leq m-2$ we have that*

$$f_m(x_1,0,\dots,0,x_{\rho+2},0,\dots,0) = \binom{x_1+x_{\rho+2}+m}{m} - \binom{x_1+m-1}{m} - \binom{x_{\rho+2}+m-1}{m} + \sum_{\delta=2}^{\rho+1}\sum_{k=\delta}^{m-\delta}\binom{x_1+k-1}{k}\binom{x_{\rho+2}+m-k-1}{m-k}.$$

Proof. We use induction on ρ.

For $\rho = 1$, we are in the case that $X = [m] \cup Y$, $Y \subset (1,2) \cup (3,4)$, $|Y \cap (1,2)| = x_1$ and $|Y \cap (3,4)| = x_3$.

We partition the set $NC(X,[m])$ into sets A_α and $B_{\alpha,\beta,\nu}$ with $\alpha,\beta,\nu \in N$, $\alpha \leq x_1$ and $\nu \leq \beta \leq x_3 - 1$, defined as follows: Each set A_α consists of all $\pi \in NC(X,[m])$ with the property that the block of the element 2 contains $x_1 - \alpha$ elements of $(1,2)$ and no element of $(3,4)$. Each set $B_{\alpha,\beta,\nu}$ consists of all $\pi \in NC(X,[m])$ with the property that the block of 2 contains $x_1-\alpha$ elements of $(1,2)$ and $x_3-\beta$ elements of $(3,4)$ so that ν out of the remaining β elements are greater than the elements of this block.

We then have that

$|A_\alpha| = f_{m-1}(\alpha,x_3,0,\dots,0)$ and $|B_{\alpha,\beta,\nu}| = f_{m-2}(\alpha+\nu,0,\dots,0)$
for every $\alpha,\beta,\nu \in \mathbb{N}$ with $\alpha \leq x_1$, $\nu \leq \beta \leq x_3 - 1$.

We thus have, using proposition 2.2 as well as various known combinatorial relations, that

$f_m(x_1, 0, x_3, 0, \dots, 0)$

$$= \sum_{\alpha=0}^{x_1} f_{m-1}(\alpha, x_3, 0, \dots, 0) + \sum_{\alpha=0}^{x_1} \sum_{\beta=0}^{x_3-1} \sum_{\nu=0}^{\beta} f_{m-2}(\alpha+\nu, 0, \dots, 0)$$

$$= \sum_{\alpha=0}^{x_1} \left[\binom{\alpha+x_3+m-1}{m-1} - \binom{\alpha+m-2}{m-1} - \binom{x_3+m-2}{m-1} \right] + \sum_{\alpha=0}^{x_1} \sum_{\beta=0}^{x_3-1} \sum_{\nu=0}^{\beta} \binom{\alpha+\nu+m-3}{m-3}$$

$$= \binom{x_1+x_3+m}{m} - \binom{x_1+m-1}{m} - \binom{x_3+m-1}{m} + \sum_{k=2}^{m-2} \binom{x_1+k-1}{k} \binom{x_3+m-k-1}{m-k}$$

proving the formula for $\rho = 1$.

Suppose now that $\rho \geq 2$ and that the formula holds for $\rho - 1$. We will prove it for ρ.

Here, $X = [m] \cup Y$, with $Y \subset (1,2) \cup (\rho+2, \rho+3)$, $|Y \cap (1,2)| = x_1$ and $|Y \cap (\rho+2, \rho+3)| = x_{\rho+2}$.

We partition the set $NC(X, [m])$ into sets T, $A_{\alpha,\mu}$, $B_{\beta,\nu}$, $C_{\alpha,\beta,\mu,\nu}$, with $\alpha, \beta, \mu, \nu \in \mathbb{N}$, $\mu \leq \alpha \leq x_1 - 1$, $\nu \leq \beta \leq x_{\rho+2} - 1$, defined as follows: The set T consists of all $\pi \in NC(X, [m])$ with the property that the block of the element 3 contains no element of Y. Each set $A_{\alpha,\mu}$ consists of all $\pi \in NC(X, [m])$ with the property that the block of the element 3 contains $x_1 - \alpha$ elements of $(1,2)$, contains no element of $(\rho+2, \rho+3)$ and μ out of the remaining α elements of $(1,2)$ that do not belong to the block of 3 are smaller than the elements of this block. Each set $B_{\beta,\nu}$ consists of all $\pi \in NC(X, [m])$ with the property that the block of the element 3 contains $x_{\rho+2} - \beta$ elements of $(\rho+2, \rho+3)$, contains no element of $(1,2)$ and ν out of the remaining β elements of $(\rho+2, \rho+3)$ that do not belong to the block of 3 are greater than the elements of this block. Finally, each set $C_{\alpha,\beta,\mu,\nu}$ consists of all $\pi \in NC(X, [m])$ with the property that the block of the element 3 contains $x_1 - \alpha$ elements of $(1,2)$, $x_{\rho+2} - \beta$ elements of $(\rho+2, \rho+3)$ and μ (resp. ν) out of the remaining α (resp. β) elements of $(1,2)$ (resp. $(\rho+2, \rho+3)$) that do not belong to the block of 3 are smaller (resp. greater) than the elements of this block.

We have that

$|T| = f_{m-1}(x_1, 0, \dots, 0, x_{\rho+2}, 0, \dots, 0)$, with $\rho-1$ zeros between x_1 and $x_{\rho+2}$.

$|A_{\alpha,\mu}| = \sum_{\alpha=0}^{x_1-1} \sum_{\mu=0}^{\alpha} f_{m-2}(\mu, 0, \dots, 0, x_{\rho+2}, 0, \dots, 0)$, with $\rho-2$ zeros between μ and $x_{\rho+2}$.

$|B_{\beta,\nu}| = \sum_{\beta=0}^{x_{\rho+2}-1} \sum_{\nu=0}^{\beta} f_{\rho-1}(\beta-\nu, 0, \dots, 0) f_{m-\rho}(x_1, \nu, 0, \dots, 0)$ and

$|C_{\alpha,\beta,\mu,\nu}| = \sum_{\alpha=0}^{x_1-1} \sum_{\beta=0}^{x_{\rho+2}-1} \sum_{\mu=0}^{\alpha} \sum_{\nu=0}^{\beta} f_{\rho-1}(\beta-\nu, 0, \dots, 0) f_{m-\rho-1}(\mu+\nu, 0, \dots, 0)$.

Using the induction hypothesis, proposition 2.2 and some combinatorial calculus, we obtain that

$$|T| = \binom{x_1+x_{\rho+2}+m-1}{m-1} - \binom{x_1+m-2}{m-1} - \binom{x_{\rho+2}+m-2}{m-1} + \sum_{\delta=2}^{\rho} \sum_{k=\delta}^{m-1-\delta} \binom{x_1+k-1}{k}\binom{x_{\rho+2}+m-k-2}{m-k-1} \tag{2}$$

$$|A_{\alpha,\mu}| = \binom{x_1+x_{\rho+2}+m-1}{m} - \binom{x_{\rho+2}+m-1}{m} - x_1\binom{x_{\rho+2}+m-2}{m-1} - \binom{x_1+m-2}{m} - \frac{x_1(x_1+1)}{2}\binom{x_{\rho+2}+m-3}{m-2} + \sum_{\delta=2}^{\rho-1} \sum_{k=\delta+2}^{m-\delta} \binom{x_1+k-2}{k}\binom{x_{\rho+2}+m-k-1}{m-k} \tag{3}$$

$$|B_{\beta,\nu}| = \sum_{k=0}^{m-\rho-1} \binom{x_1+k-1}{k}\binom{x_{\rho+2}+m-k-1}{m-k} - \binom{x_{\rho+2}+m-2}{m} \tag{4}$$

$$|C_{\alpha,\beta,\mu,\nu}| = \sum_{k=2}^{m-\rho} \binom{x_1+k-2}{k}\binom{x_{\rho+2}-m-k-1}{m-k} + x_1\binom{x_{\rho+2}+m-3}{m-2}. \tag{5}$$

Since

$$f_m(x_1, 0, \ldots, 0, x_{\rho+2}, 0, \ldots, 0) = |T| + |A_{\alpha,\mu}| + |B_{\beta,\nu}| + |C_{\alpha,\beta,\mu,\nu}|,$$

relations (2), (3), (4) and (5) give with simple calculations the required formula. ∎

Note: Notice that, using proposition 2.2 the formula of proposition 2.3 can be restated as follows:

$$f_m(x_1, 0, \ldots, 0, x_{\rho+2}, 0, \ldots, 0) = f_m(x_1, x_{\rho+2}, 0, \ldots, 0) + \sum_{\delta=2}^{\rho+1} \sum_{k=\delta}^{m-\delta} \binom{x_1+k-1}{k}\binom{x_{\rho+2}+m-k-1}{m-k}.$$

It would be interesting to find a combinatorial interpretation of the above double sum, in order to reobtain the previous formula through a bijective argument.

Using propositions 2.2 and 2.3 (and the notation $a_{m,k} = [x_i^k]P_m$ introduced earlier) we are now ready to evaluate the coefficients $[x_i^k x_j^l]P_m$, by considering the corresponding two cases.

Proposition 2.4 *The coefficient* $[x_i^k x_{i+1}^l]P_m$ *for* $k, l \geq 1$ *is the same for every* $i \in [m]$ *and it is given by the formula*

$$[x_i^k x_{i+1}^l]P_m = \sum_{\nu=k+1}^{m-l-1} a_{\nu+1,k}(a_{m-\nu+1,l} - a_{m-\nu,l}).$$

Proof. The coefficient $[x_i^k x_{i+1}^l]P_m(x_1, x_2, \ldots, x_m)$ is indeed the same for every $i \in [m]$, since $[x_i^k, x_{i+1}^l]P_m(x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_m)$ $= [x_i^k, x_{i+1}^l]P_m(0, 0, \ldots, x_i, x_{i+1}, \ldots, 0) = [x_1^k x_2^l]P_m(x_1, x_2, 0, \ldots, 0)$.

In order now to evaluate the coefficient $[x_1^k x_2^l]P_m(x_1, x_2, 0, \ldots, 0)$ we use proposition 2.2 and the relation

$$\binom{x+m-1}{m-1} = (x+1)\sum_{k=0}^{m-2} a_{m,k}x^k \tag{6}$$

for every $x \in \mathbb{N}^*$, $m \geq 2$ (which is also obtained from proposition 2.2, for $x_2 = 0$), getting

$$
\begin{aligned}
f_m(x_1, x_2, 0, \ldots, 0) &= \tbinom{x_1+x_2+m}{m} - \tbinom{x_1+m-1}{m} - \tbinom{x_2+m-1}{m} \\
&= \sum_{\nu=0}^{m} \tbinom{x_1+\nu}{\nu}\tbinom{x_2+m-\nu-1}{m-\nu} - \tbinom{x_1+m-1}{m} - \tbinom{x_2+m-1}{m} \\
&= \sum_{\nu=1}^{m-2} \tbinom{x_1+\nu}{\nu}\tbinom{x_2+m-\nu-1}{m-\nu} + (x_2+1)\tbinom{x_1+m-1}{m-1} \\
&= \sum_{\nu=1}^{m-2} \tbinom{x_1+\nu}{\nu}\left[\tbinom{x_2+m-\nu}{m-\nu} - \tbinom{x_2+m-\nu-1}{m-\nu-1}\right] + (x_2+1)\tbinom{x_1+m-1}{m-1} \\
&= \sum_{\nu=1}^{m-2} \left[(x_1+1)\sum_{k=0}^{\nu-1} a_{\nu+1,k} x_1^k\right]\left[(x_2+1)\sum_{l=0}^{m-\nu-1}(a_{m-\nu+1,l} - a_{m-\nu,l})x_2^l\right] + \\
&\quad (x_1+1)(x_2+1)\sum_{\lambda=0}^{m-2} a_{m,\lambda} x_1^\lambda \\
&= (x_1+1)(x_2+1)\left[\sum_{\nu=1}^{m-2}\left[\sum_{k=0}^{\nu-1}\sum_{l=1}^{m-\nu-1} a_{\nu+1,k}(a_{m-\nu+1,l} - a_{m-\nu,l})x_1^k x_2^l\right] + \right. \\
&\quad \left. \sum_{\lambda=0}^{m-2} a_{m,\lambda} x_1^\lambda\right] \\
&= (x_1+1)(x_2+1)\left[\sum_{\substack{k\geq 0, l\geq 1 \\ k+l\leq m-2}}\left[\sum_{\nu=k+1}^{m-l-1} a_{\nu+1,k}(a_{m-\nu+1,l} - a_{m-\nu,l})\right] x_1^k x_2^l + \right. \\
&\quad \left. \sum_{\lambda=0}^{m-2} a_{m,\lambda} x_1^\lambda\right] \\
&= (x_1+1)(x_2+1)\left[\sum_{\substack{k,l\geq 1 \\ k+l\leq m-2}}\left[\sum_{\nu=k+1}^{m-l-1} a_{\nu+1,k}(a_{m-\nu+1,l} - a_{m-\nu,l})\right] x_1^k x_2^l + \right. \\
&\quad \left. \sum_{\lambda=1}^{m-2} a_{m,\lambda}(x_1^\lambda + x_2^\lambda) + 1\right].
\end{aligned}
$$

So, we realize that for $k, l \geq 1$, the coefficient of $x_1^k x_2^l$ and hence of $x_i^k x_{i+1}^l$ for every $i \in [m]$ in P_m is indeed equal to

$$\sum_{\nu=k+1}^{m-l-1} a_{\nu+1,k}(a_{m-\nu+1,l} - a_{m-\nu,l}). \qquad \blacksquare$$

Note: The following formula for $[x_i^k x_{i+1}^l]P_m$, using Stirling numbers of the first kind, holds:

$$[x_i^k x_{i+1}^l]P_m = \sum_{\nu=k+1}^{m-l-1} \frac{(-1)^{m-k-l}}{\nu!(m-\nu)!} \sum_{h=1}^{k+1} s(\nu+1, h)\left[\sum_{r=1}^{l} s(m-\nu, r)\right].$$

Its proof uses proposition 2.4 as well as relation (1).

Finally, we deal with the case where x_i, x_j are nonconsecutive elements of $(x_1, x_2, \ldots, x_m)$.

Proposition 2.5 *The coefficient* $[x_i^k x_j^l]P_m$ *for* $k, l \geq 1$ *is the same for every nonconsecutive* x_i, x_j *with cyclic distance* ρ *and it is given by the relation*

$$[x_i^k x_j^l]P_m = \sum_{\nu=k+1}^{m-l-1} a_{\nu+1,k}(a_{m-\nu+1,l} - a_{m-\nu,l}) + \sum_{\delta=2}^{\rho+1}\sum_{\nu=\delta}^{m-\delta} \frac{1}{\nu(m-\nu)} a_{\nu,k-1}a_{m-\nu,l-1}.$$

Proof. Without loss of generality suppose that $x_i = x_1$ and $\rho = |1-j|-1 = j-2$. Following a line of argument similar to that of the proof of proposition 2.4, we realize that it is enough to determine $[x_1^k x_{\rho+2}^l]P_m(x_1, 0, \ldots, 0, x_{\rho+2}, 0, \ldots, 0)$ and hence it is enough to deal with $f_m(x_1, 0, \ldots, 0, x_{\rho+2}, 0, \ldots, 0)$, which has been determined in proposition 2.3.

Using the formula

$$\binom{x+\lambda-1}{\lambda} = \frac{x+1}{\lambda}\sum_{\mu=0}^{\lambda-2} a_{\lambda,\mu}x^{\mu+1} \tag{7}$$

for every $x, \lambda \in \mathbb{N}^*$, $\lambda \geq 2$ (obtained from relation (6) and proposition 2.1,i), we get

$$\begin{aligned}
&\sum_{\delta=2}^{\rho+1}\sum_{\nu=\delta}^{m-\delta}\binom{x_1+\nu-1}{\nu}\binom{x_{\rho+2}+m-\nu-1}{m-\nu} \\
&= \sum_{\delta=2}^{\rho+1}\sum_{\nu=\delta}^{m-\delta}\left[\frac{x_1+1}{\nu}\Big(\sum_{k=0}^{\nu-2} a_{\nu,k}x_1^{k+1}\Big)\frac{x_{\rho+2}+1}{m-\nu}\Big(\sum_{l=0}^{m-\nu-2} a_{m-\nu,l}x_{\rho+2}^{l+1}\Big)\right] \\
&= (x_1+1)(x_{\rho+2}+1)\sum_{\substack{k,l\geq 0\\ k+l\leq m-4}}\left[\sum_{\delta=2}^{\rho+1}\sum_{\nu=\delta}^{m-\delta}\frac{1}{\nu(m-\nu)}a_{\nu,k}a_{m-\nu,l}\right]x_1^{k+1}x_{\rho+2}^{l+1} \\
&= (x_1+1)(x_{\rho+2}+1)\sum_{\substack{k,l\geq 1\\ k+l\leq m-2}}\left[\sum_{\delta=2}^{\rho+1}\sum_{\nu=\delta}^{m-\delta}\frac{1}{\nu(m-\nu)}a_{\nu,k-1}a_{m-\nu,l-1}\right]x_1^k x_{\rho+2}^l.
\end{aligned}$$

So, proposition 2.3, with the use of proposition 2.4 and of the last relation give that

$$\begin{aligned}
&f_m(x_1, 0, \ldots, 0, x_{\rho+2}, 0, \ldots, 0) = \\
&\quad (x_1+1)(x_{\rho+2}+1)\Bigg[\sum_{\lambda=1}^{m-2} a_{m,\lambda}(x_1^l + x_{\rho+2}^l) + 1 + \\
&\quad \sum_{\substack{k,l\geq 1\\ k+l\leq m-2}}\Bigg[\sum_{\nu=k+1}^{m-l-1} a_{\nu+1,k}(a_{m-\nu+1,l} - a_{m-\nu,l}) + \\
&\quad \sum_{\delta=2}^{\rho+1}\sum_{\nu=\delta}^{m-\delta}\frac{1}{\nu(m-\nu)}a_{\nu,k-1}a_{m-\nu,l-1}\Bigg]x_1^k x_{\rho+2}^l\Bigg]
\end{aligned}$$

proving the required formula for $[x_i^k x_2^l]P_m$. ∎

It is an open problem to find some kind of combinatorial or algebraic interpretation of the coefficients of P_m, or at least of a large class of them.

References

[1] C.A. Athanasiadis, On non crossing and non nesting partitions for classical reflection groups. *The Electronic Journal of Combinatorics*, Vol. 5 (1998).

[2] L. Comtet, *Analyse Combinatoroire*, Presses Universitaires de France, Paris (1970).

[3] P.H. Edelman, Chain enumeration and non crossing partitions. *Discrete Math.*, Vol. 31 (1980), pp.171-180.

[4] G. Kreweras, Sur les partitions non croisées d' un cycle. *Discrete Math.*, Vol 1 (1971), pp. 335-350.

[5] Y. Poupard, Etude et denombrement paralleles des partitions non croisées d' un cycle et des decoupages d' un polygon convexe. *Discrete Math.* Vol. 2 (1971), pp. 441-451.

[6] A. Sapounakis and P. Tsikouras, Noncrossing partitions with fixed points. *The Australasian Journal of Combinatorics*, Vol. 28 (2003), pp. 263-272.

[7] R. Simion, Noncrossing partitions. *Discrete Math.*, Vol. 217 (2000), pp. 367-409.

[8] R. Simion, Combinatorial statistics on type-B analogues of non crossing partitions and restricted permutations. *The Electronic Journal of Combinatorics*, Vol. 7 (2000).

[9] R. Speicher, Multiplicative functions on the lattice of noncrossing partitions and free convolution. *Math. Ann.* 298 (1994), pp. 611-628.

[10] R. Stanley, Parking functions and noncrossing partitions. *The Electronic Journal of Combinatorics*, Vol. 4 (1997).