Independent Cycles in a Bipartite Graph

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Abstract

Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n \geq 2k$, where k is a positive integer. Let $\sigma'_2(G) = \min\{d(u) + d(v) : u \in V_1, v \in V_2, uv \notin E(G)\}$. Suppose $\sigma'_2(G) \geq 2k + 2$. In this paper we will show that if $n > 2k$, then G contains k independent cycles. If $n = 2k$, then it contains $k-1$ independent 4-cycles and a 4-path such that the path is independent of all the $k-1$ 4-cycles.

1 Introduction

All graphs considered here are simple graphs, i.e. graphs on finite vertices without loops or multiedges. Terminology and notation not defined here are refered to [2]. A *k-cycle* (*k-path*) denotes a cycle (path) of length k. Let $G = (V, E)$ be a graph. For $u \in V(G)$, let $N(u, G) = \{v : uv \in E(G)\}$ and $d(u, G) = |N(u, G)|$ denotes the degree of u in G. If H is a subgraph of G, set

$$N(u, H) = N(u, G) \cap V(H) \text{ and } d(u, H) = |N(u, H)|.$$

A vertex $u \in V(G)$ is called an *isolated vertex* of G if $d(u, G) = 0$ and $u \in V(G)$ is called an *endvertex* of G if $d(u, G) = 1$. A set of graphs is said to be *independent* if no two of them have a vertex in common. For

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a subset U of V, $G[U]$ denotes the subgraph of G induced by U. For two independent subgraphs G_1 and G_2 of G, $E(G_1, G_2)$ denotes the set of all edges of G between G_1 and G_2 and let $e(G_1, G_2) = |E(G_1, G_2)|$. Similarly, we define $E(G_1, G_2)$ and $e(G_1, G_2)$ if one of G_1 and G_2 is a subset of V or both are subsets of V and if G_1 and G_2 do not have any vertex of G in common. $l(C)$ ($l(P)$) denotes the length of a cycle C (path P).

Many articles([1],[3]-[6]) have devoted to the investigation of the maximum number of independent cycles in a graph. Corradi and Hajnal [3] proved that if $\delta(G) \geq 2k$, a graph G of order $n \geq 3k$ contains k independent cycles. Wang [5] studied a similar problem in a bipartite graph and proved the following results.

Theorem 1.1 *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = V_2| = n > 2k$, where k is a positive integer. If $\delta(G) \geq k + 1$, then G contains k independent cycles.*

Theorem 1.2 *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n = 2k$, where k is a positive integer. If $\delta(G) \geq k + 1$, then G contains $k - 1$ independent 4-cycles and a 4-path such that the path is independent of all the $k - 1$ 4-cycles.*

Let $\sigma'_2(G) = \min\{d(u) + d(v) : u \in V_1, v \in V_2, uv \notin E(G)\}$. In this paper we shall prove the following results.

Theorem 1.3 *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n > 2k$, where k is a positive integer. If $\sigma'_2 \geq 2k + 2$, then G contains k independent cycles.*

Theorem 1.4 *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n = 2k$, where k is a positive integer. If $\sigma'_2 \geq 2k + 2$, then G contains $k - 1$ independent 4-cycles and a 4-path such that the path is independent of all the $k - 1$ 4-cycles.*

Since $\delta(G) \geq k+1$ implies $\sigma_2' \geq 2(k+1)$, Theorems 1.3 and 1.4 generalize Theorems 1.1 and 1.2. On the other hand, the conditions of Theorems 1.1 and 1.2 imply that $\delta(G) \geq 2$ while Theorem 1.3 allows $\delta(G) \leq 1$. We have a similar conjecture as follows. Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 2k$, where k is a positive integer. If $\sigma_2' \geq 2k+2$, then G can be partitioned into k vertex disjoint 4-cycles.

2 Lemmas

Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n$ and let C be a cycle of G. $C + x$ is defined to be the subgraph induced by $V(C) \cup \{x\}$ if $x \in V(G)$. The following lemmas are useful in our proof.

Lemma 2.1 *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_2| \geq |V_1|$. If G is acycle, then there is $y \in V_2$ with $d(y, G) \leq 1$.*

Proof. Suppose, to the contrary, that for each $y \in V_2$, $d(y, G) \geq 2$. Then G contains at least $2|V_2|$ edges. Since $|V_1| + |V_2| - 1 < 2|V_2|$, G is not a forest and hence G must contain a cycle, a contradiction. □

Lemma 2.2 *([5]) Let C be a cycle of G and x a vertex of G not on C. If $d(x, C) \geq 2$, then either C is a 4-cycle or $C + x$ contains a cycle C' such that $l(C') < l(C)$.*

Proof. Let $C = x_1 x_2 \ldots x_{2p} x_1$ be a cycle of G with $p \geq 3$ and $x_i x, x_j x \in E(G) (i < j)$. Then $C + x$ contains two cycles: $C_1 = x x_i x_{i+1} \ldots x_j x$ and $C_2 = x x_j x_{j+1} \ldots x_i x$ (indices mod $2p$). It follows that either $l(C_1) \leq p+2$ or $l(C_2) \leq p+2$. Define

$$C' = \begin{cases} C_1, & \text{if } l(C_1) \leq p+2 \\ C_2, & \text{if otherwise.} \end{cases}$$

Then $l(C') < l(C)$. □

Lemma 2.3 *([5]) Let C be a 4-cycle of G. We have the following statements.*

(i) Let $x \in V_1$ and $y \in V_2$ be two vertices not on C. If $d(x,C)+d(y,C) \geq 3$, then $G[V(C) \cup \{x,y\}]$ contains a new 4-cycle C' and an edge e such that they are independent and $|V(e) \cap \{x,y\}| = 1$.

(ii) Let $P = x_1x_2x_3$ and $Q = y_1y_2y_3$ be two independent paths of G with $x_1 \in V_1$, $y_1 \in V_2$ such that C is independent of both P and Q. If $d(x_1,C)+d(x_3,C)+d(y_1,C)+d(y_3,C) \geq 5$, then $G[V(C \cup P \cup Q)]$ contains a 4-cycle C' and a 6-path P' such that P' is independent of C'.

(iii) Let P be a 4-path of G such that P is independent of C and $e(P,C) \geq 6$. Then either $G[V(C \cup P)]$ contains two independent 4-cycles or P has an endvertex with degree 0 in C.

(vi) Let P be a path of order $p \geq 6$ in G such that C is independent of P. If $e(P,C) \geq p+1$, then $G[V(C) \cup V(P)]$ contains two independent cycles.

Lemma 2.4 *Suppose that C is a 4-cycle of G. Let uv and xy be two independent edges of G such that they are independent of C. If $d(u,C) + d(v,C) + d(x,C) + d(y,C) \geq 5$, then $G[V(C) \cup \{u,v,x,y\}]$ contains a 4-cycle C' and a 4-path P' such that P' is independent of C'. Moreover, if $d(u,C)+d(v,C)+d(x,C)+d(y,C) \geq 7$, then $G[V(C) \cup \{u,v,x,y\}]$ contains two vertex-disjoint 4-cycles.*

Proof. Suppose that $C = z_1z_2z_3z_4z_1$ and that $\{z_1, x, u\} \subset V_1$.

Case 1. $d(u,C) + d(v,C) + d(x,C) + d(y,C) \geq 5$.

We assume, without loss of generality, that $d(u,C) + d(v,C) \geq 3$. If $d(u,C) + d(v,C) = 4$, we may assume that $yz_1 \in E(G)$. Then $G[V(C) \cup \{x,y,u,v\}]$ contains a 4-cycle $C' = uvz_3z_2u$ and a 4-path $P' = xyz_1z_4$ such that P' is independent of C'.

Thus we assume that $d(u,C)+d(v,C) = 3$. Then $d(x,C)+d(y,C) \geq 2$. Without loss of generality, assume that $d(u,C) = 2$ and $vz_1 \in E(G)$. If $d(x,C) = 2$, then $G[V(C) \cup \{x,y,u,v\}]$ contains a 4-cycle $C' = vuz_2z_1v$ and a 4-path $P' = z_3z_4xy$ such that C' and P' are independent. If $d(y,C) = 2$, then $G[V(C) \cup \{x, y, u, v\}]$ contains a 4-cycle $C' = z_1z_2uvz_1$ and a 4-path xyz_3z_4 such that P' and C' are imdependent. Now we only need to consider the case that $d(x, C) = d(y, C) = 1$. If $xz_2 \in E(G)$, then $G[V(C) \cup \{x, y, u, v\}]$ contains a 4-cycle $C' = uvz_1z_4u$ and a 4-path $P' = z_3z_2xy$ such that P' is independent of C'. If $xz_4 \in E(G)$, then $G[V(C) \cup \{x, y, u, v\}]$ contains a 4-cycle $C' = uvz_1z_2u$ and a 4-path $P' = z_3z_4xy$ such that C' and P' are independent.

Case 2. $d(u, C) + d(v, C) + d(x, C) + d(y, C) \geq 7$.

We assume, without loss of generality, that $d(u, C) + d(v, C) = 4$, $d(x, C) + d(y, C) \geq 3$, $d(x, C) = 2$ and $yz_1 \in E(G)$. Then $G[V(C) \cup \{x, y, u, v\}]$ contains two vertex-disjoint 4-cycles xyz_1z_2x and $z_3z_4uvz_3$. □

Lemma 2.5 *([5]) Let s and t be two integers such that $t \geq s \geq 2$ and $t \geq 3$. Let C_1 and C_2 be two independent cycles of G with lengths $2s$ and $2t$, respectively. If $\Sigma_{x \in V(C_2)} d(x, C_1) \geq 2t + 1$, then $G[V(C_1 \cup C_2)]$ contains two independent cycles C' and C'' such that $l(C') + l(C'') < 2s + 2t$.*

Lemma 2.6 *Let $P = x_1x_2 \dots x_p$ be a path which is independent of edge $e = y_1y_2$, where $p \geq 3$ is odd and $x_1, y_1 \in V_1$. Suppose that there exists a 4-cycle C, which is independent of both P and e, such that $d(x_1, C) + d(x_2, C) + d(x_p, C) + d(y_1, C) + 2d(y_2, C) \geq 7$. Then $G[V(P \cup \{y_1, y_2\} \cup V(C)]$ contains a 4-cycle C' and a path P' such that P' is independent of C' and $l(P') > l(P)$.*

Proof. Let $C = z_1z_2z_3z_4z_1$ and $z_1 \in V_1$. We shall distinguish the following cases.

Case 1. $d(y_2, C) = 2$.

If $d(x_1, C) + d(x_p, C) \geq 1$, say $x_1z_2 \in E(G)$, define $P' = z_2x_1x_2 \dots x_p$ and $C' = C - z_2 + y_2$. Then P' is independent of the cycle C' with $l(P') > l(P)$. So suppose that $d(x_1, C) + d(x_p, C) = 0$. Then $d(x_2, C) + d(y_1, C) \geq 3$. If $d(x_2, C) = 2$ and $d(y_1, C) \geq 1$, say $y_1z_4 \in E(G)$, define $P' = z_2z_1x_2x_3 \dots x_p$ and $C' = y_1z_4z_3y_2y_1$. Then P' is independent of C' with $l(P') > l(P)$. If $d(y_1, C) = 2$ and $d(x_2, C) \geq 1$, say $x_2z_1 \in E(G)$, define $P' = y_2 z_1 x_2 x_3 \dots x_p$ and $C' = C - z_1 + y_1$. Then P' is independent of C' with $l(P') > l(P)$.

Case 2. $d(y_2, C) = 1$.

We may assume, without loss of generality, that $y_2 z_3 \in E(G)$. If $d(y_1, C) = 2$, then $d(x_2, C) = 0$ for otherwise let $C' = y_1 y_2 z_3 z_2 y_1$ and $P' = z_4 z_1 x_2 x_3 \dots x_p$ if $x_2 z_1 \in E(G)$, or let $C' = C - z_3 + y_1$ and $P' = y_2 z_3 x_2 x_3 \dots x_p$ if $x_2 z_3 \in E(G)$, where P' is independent of C' with $l(P') > l(P)$. It follows that $d(x_1, C) + d(x_p, C) \geq 3$. We may assume, without loss of generality, that $z_2 x_1, z_2 x_p \in E$. Define $P' = z_1 z_2 x_1 \dots x_p$ and $C' = y_1 z_4 z_3 y_2 y_1$. Then C' is independent of P' with $l(P') > l(P)$.

So we assume that $d(y_1, C) = 1$, say $y_1 z_4 \in E(G)$. If $d(x_2, C) = 2$, define $C' = y_1 y_2 z_3 z_4 y_1$ and $P' = z_2 z_1 x_2 \dots x_p$. Then C' is independent of P' with $l(P') > l(P)$. Thus suppose that $d(x_2, C) \leq 1$. Then $d(x_1, C) + d(x_p, C) \geq 3$. Without loss of generality, we assume that $d(x_1, C) = 2$. Then $C' = y_1 y_2 z_3 z_4 y_1$ is independent of $P' = z_1 z_2 x_1 \dots x_p$ with $l(P') > l(P)$.

Finally we assume that $d(y_1, C) = 0$. If $d(x_2, C) = 1$ and $x_2 z_1 \in E(G)$, then $d(x_1, C) = d(x_p, C) = 2$. Define $C' = x_1 x_2 z_1 z_2 x_1$ and $P' = y_1 y_2 z_3 z_4 x_p \dots x_3$. If $d(x_2, C) = 1$ and $x_2 z_3 \in E(G)$, then $d(x_1, C) = d(x_p, C) = 2$. Define $C' = x_1 z_2 z_1 z_4 x_1$ and $P' = y_1 y_2 z_3 x_2 \dots x_p$. So suppose that $d(x_2, C) = 2$. By symmetry we assume that $d(x_1, C) = 2$ and $x_p z_2 \in E(G)$. Define $C' = x_1 x_2 z_1 z_4 x_1$ and $P' = y_1 y_2 z_3 z_2 x_p \dots x_3$. Then C' is independent of P' with $l(P') > l(P)$.

Case 3. $d(y_2, C) = 0$.

It follows that $d(x_1, C) + d(x_2, C) + d(x_p, C) + d(y_1, C) \geq 7$. Thus $d(x_2, C) \geq 1$ and $d(x_p, C) + d(y_1, C) \geq 3$. Since $d(x_p, C) + d(y_1, C) \geq 3$, we may assume, without loss of generality, that $z_2 y_1 \in E(G), z_2 x_p \in E(G)$. If $d(x_2, C) = 2$, define $P' = y_2 y_1 z_2 x_p x_{p-1} \dots x_3$ and $C' = C - z_2 + x_2$. Then P' is independent of C' with $l(P') > l(P)$. So we assume that $d(x_2, C) = 1$, say $x_2 z_1 \in E(G)$, and $d(x_1, C) = d(x_p, C) = d(y_1, C) = 2$. Define $C' = x_1 z_2 z_1 x_2 x_1$ and $P' = y_2 y_1 z_4 x_p x_{p-1} \dots x_3$. It follows that C' and P' are independent with $l(P') > l(P)$. □

3 Proofs of Theorems 1.3 and 1.4

Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n \geq 2k$ and $\sigma'_2 \geq 2(k+1)$, where k is a positive integer. We claim that G contains cycles. If $\delta(G) \leq 1$, then $e(G) \geq 2kn > 2n-1$. It follows that G is not a forest and hence G contains cycles. If $\delta(G) \geq 2$, G must contain cycles.

Let t be the greatest integer such that G contains t independent cycles. If $t \geq k$, then Theorems 1.3 and 1.4 hold. Thus we assume that $t < k$. We choose t cycles $C_1, C_2, \ldots, C_t$, say $H = \bigcup_{i=1}^{t} V(C_i)$, such that

$$|V(H)| \text{ is as small as possible.} \tag{1}$$

Let $D = G - H$. It follows that $|V(D)|$ is even.

We claim that $|V(D)| \geq 2$. Suppose, to the contrary, that $|V(D)| = 0$. Without loss of generality, we assume that C_t is a longest cycle in H. Since $4t < 4k \leq |V_1| + |V_2|$, we have $l(C_t) \geq 6$. By (1), each cycle C_i has no chord for $i \in \{1, 2, \ldots, t\}$. It follows that

$$\begin{aligned} e(C_t, H - V(C_t)) &\geq \frac{|V(C_t)|}{2} \times (2k+2) - 2|V(C_t)| \\ &= (k-1)|V(C_t)| \\ &> (t-1)|V(C_t)|. \end{aligned}$$

Thus there exists a cycle, say C_j ($j \neq t$), such that $e(C_t, C_j) \geq |V(C_t)| + 1$. By Lemma 2.5, $G[V(C_t \cup C_j)]$ contains two shorter independent cycles, contrary to (1). Thus $|V(D)| \geq 2$. Subject to (1), we choose $C_1, \ldots, C_t$ such that

$$\text{the length of longest paths in } D \text{ is maximum.} \tag{2}$$

Let $P = x_1 x_2 \ldots x_p$ be a fixed longest path in D.

We claim that $|V(P)| \geq 2$. Suppose, to the contrary, that $|V(P)| = 1$. Since $|V(D)| \geq 2$, let $y \in V(D) - V(P)$. Thus we assume that $y \in V_2, x_1 \in V_1$. Then $x_1 y \notin E(G)$ and

$$\begin{aligned} d(x_1, G) + d(y, G) &= d(x_1, H) + d(y, H) \\ &= d(x_1, \bigcup_{i=1}^{t} V(C_i)) + d(y, \bigcup_{i=1}^{t} V(C_i)) \\ &\geq 2(k+1). \end{aligned}$$

So there exists some C_i such that

$$d(x_1, C_i) + d(y, C_i) \geq 3.$$

By Lemma 2.3, $G[V(C_i) \cup \{x_1, y\}]$ contains a cycle and a path of length at least 2 which is independent of the cycle, a contradiction.

We assume $x_1 \in V_1$. Subject to (1) and (2), we choose $C_1, C_2, \ldots, C_t$ such that

$$d(x_2, D) \text{ is minimum.} \tag{3}$$

Let $D_0 = D - V(P)$. Subject to (1), (2) and (3), we choose $C_1, C_2, \ldots, C_t$ such that

$$\text{the length of the longest path in } D_0 \text{ is maximum.} \tag{4}$$

Let $Q = y_1 y_2 \ldots y_q$ be a fixed longest path in D_0. If $|V(Q)| \geq 2$, subjuect to (1),(2), (3) and (4), we choose $C_1, C_2, \ldots, C_t$ such that

$$d(y_2, D) \text{ is minimum.} \tag{5}$$

Claim 1. D has the following properties:

(i) If either $n > 2k$ or $t < k - 1$ then $|V(D)| \geq 6$;

(ii) If both $n = 2k$ and $t = k - 1$ then $|V(D)| = 4$.

Proof. (i) By contradiction, suppose that $|V(D)| \leq 4$ where either $n > 2k$ or $t < k - 1$. We may assume that $l(C_1) \leq l(C_2) \leq \cdots \leq l(C_t)$. It follows that $l(C_t) \geq 6$. By lemma 2.5 and by (1), we have $e(C_i, C_t) \leq l(C_t)$ for all $i \in \{1, 2, \ldots, t-1\}$. By Lemma 2.2 and by (1), $e(D, C_t) \leq |V(D)|$. Since $l(C_t) \geq 6$, we obtain

$$\begin{aligned} 2(k+1) \times \frac{l(C_t)}{2} &\leq \sum_{x \in V(C_t)} d(x, G) \\ &= \sum_{i-1}^{t-1} e(C_i, C_t) + e(D, C_t) + 2l(C_t) \\ &\leq (t+1) l(C_t) + |V(D)|, \end{aligned}$$

which implies $|V(D)| \geq (k - t) l(C_t) \geq 6$, a contradiction.

The proof for (ii) is similar. □

Claim 2. $p \geq 3$ and $d(x_2, D) = 2$.

Proof. By contradiction, suppose that $p = 2$ (note that $p \geq 2$). We claim that D_0 contains at least one edge. Suppose, to the contrary, that $d(x, D_0) = 0$ for every vertex $x \in D_0$. Since $|D| \geq 4$, we can choose $x \in V_1 \cap V(D_0)$, $y \in V_2 \cap V(D_0)$. It follows that $d(x,D) + d(y,D) = 0$. Hence $d(x,H) + d(y,H) = d(x, \bigcup_{i=1}^{t} C_i) + d(y, \bigcup_{i=1}^{t} C_i) \geq 2(k+1)$. Since $t \leq k-1$, there exists some C_j such that

$$d(x, C_j) + d(y, C_j) \geq 3.$$

By Lemma 2.2 and by (1), C_j is a 4-cycle. By Lemma 2.3 $G[V(C_j) \cup \{x,y\}]$ contains a cycle C' and a path P_0 of length at least 2 which is independent of P. Let $H' = (H - V(C_j)) \cup V(C'), D' = G - H'$ and $D'_0 = (D_0 - V(C')) \cup V(P_0)$. Then H' satisfifies (1)-(3) and contradicts (4). Thus D contains two independent edges, say x_1x_2, x_3x_4 where $x_1, x_3 \in V_1$. It follows that $\sum_{i=1}^{4} d(x_i, H) \geq 4(k+1) - 4 = 4(k-1) + 4$. Then there exists some $C_j \in H$ such that $\sum_{i=1}^{4} d(x_i, C_j) \geq 5$. By Lemma 2.4, $G[V(C_j) \cup \{x_1, x_2, x_3, x_4\}]$ contains a 4-cycle and a 4-path such that they are independent, contrary to that $p = 2$. Thus $p \geq 3$.

Now we are ready to prove that $d(x_2, D) = 2$. By contradiction, suppose that $d(x_2, D) \geq 3$. Then there exists $z_1 \in V(D) \cap V_1$ such that $z_1x_2 \in E(G)$ and $z_1 \notin V(P)$. Since P is the longest path in $G[D]$ and D contains no cycle, $d(z_1, D) = 1$. Let (A, B) be the bipartition of $D - (V(P) \cup \{z_1\})$ with $A \subset V_1, B \subset V_2$. Then $|B| > |A|$. By Lemma 2.1 there exists a vertex $z_2 \in B$ such that $d(z_2, A) \leq 1$. Since P is a longest path in D, $z_1z_2 \notin E(G)$. Since D is acycle, $d(z_2, P) \leq 1$. Thus $d(z_2, D) \leq 2$. Therefore

$$d(z_1, H) + d(z_2, H) \geq 2(k+1) - 2 - 1 = 2(k-1) + 1.$$

It follows that there exists some cycle C_j in H such that $d(z_1, C_j) + d(z_2, C_j) \geq 3$. By Lemma 2.2 and by (1) we have $l(C_j) = 4$. By Lemma 2.3 and by the choice of P, $G[V(C_j) \cup \{z_1, z_2\}]$ contains a new 4-cycle C' and an edge e such that $|V(e) \cap \{z_1, z_2\}| = 1$. Let $H' = (H - V(C_j)) \cup V(C'), D' = G - H'$ and $D'_0 = (D_0 - V(C')) \cup V(e)$. If $z_1 \in V(e)$, then there is a longer path than P, a contradiction. If $z_2 \in V(e)$, then P is still the longest path in D', but $d(x_2, D') < d(x_2, D)$, contrary to (3). □

Claim 3. If $|V(D_0)| \geq 2$, then $q \geq 2$ and $d(y_2, D_0) \leq 2$.

Proof. It follows from $|V(D_0)| \geq 2$ that $D_0 \cap V_1 \neq \emptyset$ and $D_0 \cap V_2 \neq \emptyset$.

First we show that $q \geq 2$. By contradiction, suppose that $q = 1$. Let $x \in V_1 \cap D_0$ and $y \in V_2 \cap D_0$. Then $d(x, D_0) = d(y, D_0) = 0$. Since D is acyclic, $d(x, P) \leq 1$ and $d(y, P) \leq 1$. It follows that

$$d(x, H) + d(y, H) \geq 2(k+1) - 2 = 2k > 2t.$$

Then there exists some cycle C in H such that $d(x, C) + d(y, C) \geq 3$. By (1) and by Lemma 2.2 C must be a 4-cycle. By Lemma 2.3 $G[V(C) \cup \{x, y\}]$ contains a 4-cycle C' and edge e which is independent of C'. Let $H' = (H - V(C)) \cup V(C')$, $D' = G - H'$ and $D_0' = (D_0 - V(C')) \cup V(e)$. H' satifies (1)-(3) and contrdicts (4). Thus $q \geq 2$.

Assume that $q = 2$. Let $Q = y_1 y_2$ is a longest path in D_0. It follows that $d(y_2, D_0) = 1 \leq 2$. Thus Claim 3 follows. Now we assume that $q \geq 3$. Let $y_1 \in V_a$ and $y_2 \in V_b$, where $\{a, b\} = \{1, 2\}$. By contradiction, suppose that $d(y_2, D_0) \geq 3$. Then there exists a vertex $u_1 \in V_a \cap (D_0 - V(Q))$ such that $u_1 y_2 \in E(G)$. By the choice of Q, $d(u_1, D_0) = 1$. Since D is acyclic, either $d(u_1, P) = 0$ or $d(y_1, P) = 0$. We assume, without loss of generality, that $d(u_1, P) = 0$. Let (A, B) be the bipartition of $D_0 - (V(Q) \cup \{u_1\})$ with $A \subset V_a, B \subset V_b$. By the choice of Q, $|B| > |A|$. By Lemma 2.1 there exists a vertex $u_2 \in B$ such that $d(u_2, A) \leq 1$. Since D is acyclic, $d(u_2, Q) \leq 1$ and $d(u_2, P) \leq 1$. Thus $d(u_1, D) \leq 1$ and $d(u_2, D) \leq 3$. If $d(u_2, D) = 3$, then $d(u_2, A) = d(u_2, P) = d(u_2, Q) = 1$. Let $w_2 \in A$ and $w_2 u_2 \in E(G)$ and let (A_1, B_1) be the bipartition of $D_0 - (V(Q) \cup \{u_1, u_2, w_2\})$ with $A_1 \subset V_a, B_1 \subset V_b$. It follows that $|B_1| > |A_1|$. By Lemma 2.1, there exists $u_3 \in B_1$ such that $d(u_3, A_1) \leq 1$. Since D is acycle, $d(u_3, Q) \leq 1$ and $d(u_2, P) \leq 1$. Thus $d(u_3, D) \leq 3$. If $d(u_3, D) = 3$, then $d(u_3, A_1) = d(u_3, Q) = d(u_3, P) = 1$. Thus we get another cycle C' in D, contrary to the choice of t. Therefore we assume that $d(u_2, D) \leq 2$. It follows that

$$d(u_1, H) + d(u_2, H) \geq 2(k+1) - 2 - 1 = 2(k-1) + 1.$$

Then there exists some cycle C_i in H such that $d(u_1, C_i) + d(u_2, C_i) \geq 3$. By Lemma 2.2 and by (1), $l(C_i) = 4$. By Lemma 2.3 and by the choice of Q, $G[V(C_i) \cup \{u_1, u_2\}]$ contains a new 4-cycles C' and an edge e such that $|V(e) \cap \{u_1, u_2\}| = 1$. Let $H' = (H - V(C_i)) \cup V(C'), D' = G - H'$ and $D_0 = (D_0 - V(C')) \cup V(e)$. Then H' satisfies (1)-(4). If $d(u_1, C_i) = 2$, then Q is still the longest path in D_0'. But $d(y_2, D_0') < d(y_2, D_0)$, contrary to

(5). If $d(u_1, C_i) = 1$, then we get a longer path Q' in D'_0, a contradiction. □

Claim 4. $p \geq |V(D)| - 1$.

Proof. By contradiction, suppose that that $p \leq |V(D)| - 2$. By Claim 3 and by the choice of P and Q, $p \geq q \geq 2$, $d(x_2, D) = 2$ and $d(y_2, D_0) \leq 2$. We shall distinguish the following two cases:

Case 4.1. p is even.

Let $R = \{x_1, x_p, y_1, y_2\}$. Since D contains no cycle, $\sum_{u \in R} d(u, D) \leq 6$. By the choice of P, $e(\{x_1, x_p\}, \{y_1, y_2\}) = 0$. Thus

$$e(R, H) \geq 4(k+1) - 6 = 4(k-1) + 2.$$

So there exists some C_i in H such that $e(R, C_i) \geq 5$. By (1) and by Lemma 2.2, $l(C_i) = 4$. Let $C_i = u_1u_2u_3u_4u_1$ with $u_1 \in V_1$. Assume that $x_1 \in V_1, y_1 \in V_a$.

We first consider $a = 2$. Since

$$(d(x_1, C_i) + d(y_1, C_i)) + (d(x_p, C_i) + d(y_2, C_i)) \geq 5,$$

without loss of generality, we assume that $d(x_1, C_i) + d(y_1, C_i) \geq 3$. By the choice of P, $d(x_1, C_i) = 2$ and $d(y_1, C_i) = 1$, say $u_1y_1 \in E(G)$. It follows that $d(y_2, C_i) + d(x_p, C_i) \geq 2$. If $x_pu_1 \in E(G)$, then let $C' = C_i - u_1 + x_1$ and $P' = y_q \dots y_2y_1u_1x_px_{p-1} \dots x_2$ with $l(P') > l(P)$, contrary to (2). Thus $d(x_p, C_i) \leq 1$. It follows that $d(y_2, C_i) \geq 1$, say $y_2u_2 \in E(G)$. Hence $C' = y_1y_2u_2u_1y_1$ is a 4-cycle, $P' = u_3u_4x_1 \dots x_p$ is independent of C' with $l(P') > l(P)$, contrary to the choice of P. We can symmetrically obtain a contradiction for $a = 1$.

Case 4.2. p is odd.

Let $\{z_1, z_2\} = \{y_1, y_2\}$ and $z_1 \in V_1, z_2 \in V_2$. By the choice of P, $e(\{z_1, z_2\}, \{x_1, x_2, x_p\}) = 0$. Thus

$$(d(x_1, G) + d(z_2, G)) + (d(x_2, G) + d(z_1, G)) + (d(x_p, G) + d(z_2, G)) \geq 6(k+1).$$

Since $d(x_1, D) = d(x_p, D) = 1$ and $d(x_2, D) = 2$ and since $d(z_1, D) + 2d(z_2, D) \leq 7$, it follows that $d(x_1, H) + d(x_2, H) + d(x_p, H) + d(z_1, H) +$

$2d(z_2,H) \geq 6(k+1) - 11 = 6(k-1)+1$. Then there exists some cycle C_i in H such that

$$d(x_1,C_i) + d(x_2,C_i) + d(x_p,C_i) + d(z_1,C_i) + 2d(z_2,C_i) \geq 7.$$

It follows that there exists some $u \in \{x_1,x_2,x_p,z_1,z_2\}$ such that $d(u,C_i) \geq 2$. By (1) and Lemma 2.2, C_i must be a 4-cycle. By Lemma 2.6 we get a path P' longer than P and a 4-cycle C' in $G[V(P) \cup \{y_1,y_2\} \cup V(C_i)]$, contrary to the choice of P. □

We are ready to complete the proofs of Theorems 1.3 and 1.4. Since D is acyclic, by Claim 4 $\sum_{i=1}^{p} d(x_i,D) \leq 2(p-1)+1$. We shall distinguish the following two cases.

Case 1. $|V(D)| \geq 6$.

First we suppose that p is even. By Claim 4 $p = |V(D)| \geq 6$. Thus $\sum_{i=1}^{p} d(x_i,H) \geq p(k+1) - 2(p-1) - 1 = p(k-1)+1$. It follows that there exists some C_j in H such that $\sum_{i=1}^{p} d(x_i,C_j) \geq p+1$. By Lemmas 2.2 and 2.3, $l(C_j) = 4$ and $G[V(P) \cup V(C_j)]$ contains two independent 4-cycles, contrary to the maximality of t.

Next we suppose that p is odd and $p = |V(D)| - 1 \geq 7$. Let $\{u\} = D - V(P)$ and let $x_1 \in V_1$, $u \in V_2$. Since D is acyclic, $d(u,P) \leq 1$. Define

$$P' = \begin{cases} ux_3x_4x_5x_6x_7 & \text{if } ux_3 \in E(G) \text{ and } p = 7, \\ x_px_{p-1}x_{p-2}x_{p-3}x_{p-4}x_{p-5} & \text{if } ux_3 \in E(G) \text{ and } p \geq 9, \\ x_1x_2x_3x_4x_5u & \text{if } ux_5 \in E(G), \\ x_1 \dots x_6 & \text{if } N(u,P) \cap \{x_3,x_5\} = \emptyset. \end{cases}$$

It follows that $|V(P')| = 6$ and $e(P',D) = 11$. Thus

$$\sum_{x \in V(P')} d(x,H) \geq 6(k+1) - 11 = 6(k-1)+1.$$

Then there exists some C_j in H such that $\sum_{x \in V(P')} d(x,C_j) \geq 7 = |V(P')| + 1$. By Lemmas 2.2 and 2.3, $l(C_j) = 4$ and $G[V(P \cup C_j)]$ contains two independent cycles, contrary to the maximality of t.

Finally we suppose that $p = 5$. Let $z_0 \in V(D) - V(P)$. Since $d(z_0,D) \leq 1$, $d(x_1,H) + d(z_0,H) \geq 2(k+1) - 2 = 2k > 2(k-1)+1$. It follows that there exists some C_i in H, say $C_i = C_1$, such that $d(x_1, C_1) + d(z_0, C_1) \geq 3$. By (1) and by Lemma 2.2, C_1 is a 4-cycle. By the choice of P, $d(x_1, C_1) = 2$ and $d(z_0, C_1) = 1$. Let $H_1 = H - V(C_1)$ and $z_1 \in V(C_1)$ such that $z_1 z_0 \in E(G)$. Since $d(z_0, D \cup V(C_1)) \leq 2$ and $d(x_5, D \cup V(C_1)) \leq 3$,

$$d(x_5, H_1) + d(z_0, H_1) \geq 2(k+1) - 5 = 2(k-2) + 1.$$

It follows that there exists some C_j in H_1, say $C_j = C_2$, such that $d(x_5, C_2) + d(z_0, C_2) \geq 3$. By Lemmas 2.2 and 2.3 again, C_2 is a 4-cycle, $d(x_5, C_2) = 2$ and $d(z_0, C_2) = 1$. Let $z_2 \in V(C_2)$ such that $z_0 z_2 \in E(G)$. Let $H' = (H - V(C_1 \cup C_2)) \cup (C_1 - z_1 + x_1) \cup (C_2 - z_2 + x_5), D' = G - V(H')$ and $U = \{x_2, x_4, z_1, z_2\}$. Clearly, H' consists of t independent cycles satisfying (1). Then $d(u, D') = 1$ for all $u \in U$ for otherwise D' contains a path of order 6, contrary to that $p = 5$. Therefore $\sum_{u \in U} d(u, H') \geq 4(k+1) - 4 = 4(k-1) + 4$. It follows that there exists a cycle C' in H' such that $\sum_{u \in U} d(u, H') \geq 5$. By Lemma 2.2 and by (1), C' is a 4-cycle. By Lemma 2.3 $G[V(C' \cup D')]$ contains a 4-cycle C'' and a 6-path P' such that C'' and P' are independent, contrary to that $p = 5$. Hence Theorems 1.3 and 1.4 follows.

Case 2. $|V(D)| = 4$.

By Claim 1, both $n = 2k$ and $t = k - 1$. By Claim 4, $p = 3$. Let $P = x_1 x_2 x_3$ and $x_4 \in D - V(P)$. Since P is a longest path, $d(x_4, P) = 0$. Since $d(x_1, H) + d(x_4, H) \geq 2(k+1) - 1 = 2k+1$, there exists a cycle $C \subseteq H$ such that $d(x_1, C) + d(x_4, C) \geq 3$. By Lemmas 2.2 and 2.3 and by the choice of P, C must be a 4-cycle, $d(x_1, C) = 2$ and $d(x_4, C) = 1$. Without loss of generality assume that $u \in V(C)$ and $ux_4 \in E(G)$. Clearly $ux_2 \notin E(G)$ otherwise Theorem 1.4 holds. Let $H' = (H - V(C)) \cup (C - u + x_1)$ and $D' = G - V(H')$. Then $x_2 x_3$ and $x_4 u$ are two independent edges in D' and

$$d(x_2, H') + d(x_3, H') + d(x_4, H') + d(u, H') \geq 4(k+1) - 4 = 4k > 4t.$$

It follows that there exists C'' in H' such that $d(x_2, C'') + d(x_3, C'') + d(x_4, C'') + d(u, C'') \geq 5$. By Lemma 2.4, Theorem 1.4 follows. □

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