The Linear 2-Arboricity of Outerplanar Graphs

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Abstract

The linear 2-arboricity $la_2(G)$ of a graph G is the least integer k such that G can be partitioned into k edge-disjoint forests, whose component trees are paths of length at most 2. We prove that $la_2(G) \leq \lfloor (\Delta(G)+4)/2 \rfloor$ if G is an outerplanar graph with maximum degree $\Delta(G)$.

1 Introduction

All graphs considered in this paper are finite simple graphs. For a graph G, we use $V(G)$, $E(G)$, $\Delta(G)$, and $\delta(G)$ to denote, respectively, its vertex set, edge set, maximum degree, and minimum degree. An *edge-partition* of a graph G is a decomposition of G into subgraphs $G_1, G_2, \cdots, G_m$ such that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_m)$ and $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$. A *linear k-forest* is a graph whose components are paths of length at most k. The *linear k-arboricity* of G, denoted by $la_k(G)$, is the least integer m such that G can be edge-partitioned into m linear k-forests. Clearly, $la_k(G) \geq la_{k+1}(G)$ for any $k \geq 1$. For extremities, $la_1(G)$ is the edge chromatic number, or chromatic index, $\chi'(G)$ of G; $la_\infty(G)$ representing the case when component paths have unlimited lengths is the ordinary linear arboricity $la(G)$ of G.

The linear k-arboricity of a graph was first introduced by Habib and

Péroche [8]. It was further studied by Bermond et al. [2], Jackson and Wormald [9], and Aldred and Wormald [1]. The linear 2-arboricities of cycles, trees, complete graphs, and complete bipartite graphs have been determined in [5] and [7]. Thomassen [11] proved that $la_k(G) \leq 2$ for a cubic graph G, where $k \geq 5$, and this result is best possible. Chang [3] and Chang et al. [4] investigated the algorithmic aspects of the linear k-arboricity. For a planar graph G of girth g, the authors in [10] established the following results.

(1) $la_2(G) \leq \lceil (\Delta(G)+1)/2 \rceil + 12$;

(2) $la_2(G) \leq \lceil (\Delta(G)+1)/2 \rceil + 6$ if $g \geq 4$;

(3) $la_2(G) \leq \lceil (\Delta(G)+1)/2 \rceil + 2$ if $g \geq 5$;

(4) $la_2(G) \leq \lceil (\Delta(G)+1)/2 \rceil + 1$ if $g \geq 7$.

Wu [13] proved that if G is a series-parallel graph with $\Delta(G) \geq 3$, then $la(G) = \lceil \Delta(G)/2 \rceil$. It implies that $la(G) = \lceil \Delta(G)/2 \rceil$ for an outerplanar graph G with maximum degree at least 3. In this paper, we are going to prove that $la_2(G) \leq \lfloor (\Delta(G)+4)/2 \rfloor$ for an outerplanar graph G, and the bound is sharp.

2 Results

A planar graph is called *outerplanar* if there is an embedding of G into the Euclidean plane such that all the vertices are incident to the unbounded face. An *outerplane* graph is a particular embedding of an outerplanar graph. For an outerplane graph G, all faces are called *inner* faces, except the unbounded one is called the *outer* face. The boundary edges of the outer face are called *outer edges*. The *degree* of a face is the number of edge-steps in the closed boundary walk of f. Note that each cut-edge is counted twice. A vertex (or face) of degree k is called a *k-vertex* (or *k-face*). For $x \in V(G) \cup F(G)$, the degree of x in G is denoted by $d_G(x)$. An outerplane graph is called *maximal* if each of its inner faces is of degree 3. It is easy to prove that every 2-connected maximal outerplane graph G has exactly $2|V(G)|-3$ edges.

It is well-known that every outerplanar graph contains a vertex of degree at most 2. Furthermore, we have the following result.

Lemma 1 ([12]) *Every outerplanar graph G satisfies one of the following.*

(1) $\delta(G) \leq 1$.

(2) *G contains two adjacent 2-vertices.*

(3) *G contains a 3-cycle $xyzx$ with $d_G(x) = 2$ and $3 \leq d_G(y) \leq 4$.*

Lemma 2 ([6]) *Let G be an outerplanar graph. Then $\chi'(G) = \Delta(G)$ if and only if G is not an odd cycle.*

Lemma 3 *Let $k \geq 3$ be an integer and G be an outerplanar graph. Then G has an edge-partition into a forest T and a subgraph H satisfying $\Delta(T) \leq \Delta(G) - k$ and $\Delta(H) \leq k$.*

Proof. Without loss of generality, we may assume that G is connected. We first prove the lemma for $k = 3$ by induction on $|V(G)| + |E(G)|$. When $|V(G)| + |E(G)| \leq 5$, it holds trivially. Let G be an outerplanar graph with $|V(G)| + |E(G)| \geq 6$. If $\Delta(G) \leq 3$, we let $H = G$ and $T = \emptyset$. If $\Delta(G) = 4$, then G has a proper 4-edge coloring with the color classes E_i, $1 \leq i \leq 4$, by Lemma 2. Take $T = E_1$ and $H = G - E_1$. Suppose that $\Delta(G) \geq 5$. Then $\Delta(G) - 3 \geq 2$. By Lemma 1, we have to consider three cases.

Case 1. *There is a 1-vertex u with $uv \in E(G)$.*

Suppose that $G' = G - uv$. By the induction hypothesis, G' has an edge-partition into a forest T' and a subgraph H' satisfying $\Delta(T') \leq \Delta(G') - 3$ and $\Delta(H') \leq 3$. If $d_{H'}(v) \leq 2$, we let $H = H' + uv$ and $T = T'$. If $d_{H'}(v) = 3$, we let $T = T' + uv$ and $H = H'$.

Case 2. *G contains two adjacent 2-vertices u and v.*

The induction hypothesis guarantees that $G' = G - uv$ has an edge-partition into a forest T' and a subgraph H' satisfying $\Delta(T') \leq \Delta(G') - 3$ and $\Delta(H') \leq 3$. We let $H = H' + uv$ and $T = T'$. Since u and v are of degree 2 in G and $d_H(x) = d_{H'}(x) \leq \Delta(H') \leq 3$ for all $x \in V(G) \setminus \{u, v\}$, it follows that $\Delta(H) \leq 3$.

Case 3. *G contains a 3-cycle $xyzx$ such that $d_G(x) = 2$ and $3 \leq d_G(y) \leq$ 4.*

Let $G' = G - xy$. By the induction hypothesis, G' has an edge-partition into a forest T' and a subgraph H' satisfying $\Delta(T') \le \Delta(G') - 3$ and $\Delta(H') \le 3$. If $d_{H'}(y) \le 2$, we let $H = H' + xy$ and $T = T'$. Suppose that $d_{H'}(y) = 3$. This implies that $d_G(y) = d_{G'}(y) + 1 = d_{H'}(y) + 1 = 4$. We let $T = T' + xy$ and $H = H'$. Note that $d_T(x) \le d_G(x) = 2$, $d_T(y) = 1$, and $\Delta(G) - 3 \ge 2$.

Now suppose $k \ge 4$. If $\Delta(G) \le k$, we let $H = G$ and $T = \emptyset$. So we assume $\Delta(G) \ge k + 1 \ge 5$. By the foregoing argument, we first partition $E(G)$ into a forest T^* and a subgraph H^* satisfying $\Delta(T^*) \le \Delta(G) - 3$ and $\Delta(H^*) \le 3$. Since $\chi'(T^*) = \Delta(T^*)$, T^* has a proper edge coloring with m color classes E_i for $i = 1, 2, \ldots, m = \Delta(T^*)$. We let $H = H^* \cup (E_1 \cup E_2 \cup \cdots \cup E_{k-3})$ and $T = T^* \setminus (E_1 \cup E_2 \cup \cdots \cup E_{k-3})$. □

Lemma 3 is best possible in the sense that there exist outerplanar graphs G which cannot be edge-partitioned into a forest T and a subgraph H with $\Delta(H) \le 2$. To illustrate this sharpness, we give a recursive construction of infinitely many such graphs. First let G_0 be a complete graph on three vertices. Assume that the graph G_{i-1}, $i > 0$, has been defined. We then define G_i in the following way. For each outer edge $e = xy$ of G_{i-1}, we add a distinct vertex v_e in the outer face. Join v_e to x and y respectively by two new edges. The resultant graph G_i is clearly a 2-connected maximal outerplane graph of even order. Moreover, if $x_1, x_2, \ldots, x_{2n}$ is the boundary walk of the outer face, then it can be so arranged that $x_2, x_4, \ldots, x_{2n}$ are exactly all the 2-vertices of G_i. We claim that any G_i, $i \ge 2$, cannot be edge-partitioned into a forest T and a subgraph H with $\Delta(H) \le 2$. Suppose on the contrary that such a partition existed for G_i. If H contains the two edges incident to a 2-vertex, then we can move one of them to T. So we may assume that $d_T(v) \ge 1$ for each vertex v of G_i, and hence $d_H(u) \le 1$ for each 2-vertex u of G_i. Now $|E(H)| = |E(G_i)| - |E(T)| \ge (2|V(G_i)| - 3) - (|V(G_i)| - 1) = |V(G_i)| - 2$. However, $2|E(H)| \le \frac{1}{2}|V(G_i)| + 2 \cdot \frac{1}{2}|V(G_i)| = \frac{3}{2}|V(G_i)|$ by considering the sum of degrees. The above inequlities imply that $|V(G_i)| - 2 \le \frac{3}{4}|V(G_i)|$. Yet $|V(G_i)| \ge 12$ when $i \ge 2$. We have arrived at a contradiction.

In the sequel, we use $[xyz]$ to denote a 3-face whose boundary contains the vertices x, y, and z. We call a 3-face f of an outerplane graph *special* if the boundary of f contains a 2-vertex.

Lemma 4 *If G is a connected outerplane graph satisfying $|V(G)| \geq 2$ and $\Delta(G) \leq 4$, then G contains one of the following configurations:*

(C1) *A 1-vertex u adjacent to a k-vertex v, where $k \leq 3$;*

(C2) *Two adjacent 2-vertices u and v;*

(C3) *Two 1-vertices u and v both adjacent to a 4-vertex x;*

(C4) *A special 3-face $[uxy]$ with $d_G(u) = 2$ and $d_G(x) = 3$;*

(C5) *A 1-vertex u adjacent to a 4-vertex v such that v lies on the boundary of some special 3-face;*

(C6) *Two special 3-faces $[u_1v_1y]$ and $[u_2yv_2]$ with $d_G(u_1) = d_G(u_2) = 2$ and $v_1v_2 \in E(G)$, where u_1, u_2, v_1, v_2, y are distinct;*

(C7) *Three special 3-faces $[u_1xy]$, $[u_2yz]$, and $[u_3zv]$ such that $d_G(u_1) = d_G(u_2) = d_G(u_3) = 2$ and $u_1, u_2, u_3, x, y, z, v$ are distinct.*

Proof. Suppose that the lemma is false. Let G be a connected outerplane graph of order at least two and $\Delta(G) \leq 4$ that contains none of (C1) to (C7). We perform the following operations simultaneously on G.

(OP1) Delete all 1-vertices of G.

(OP2) If u is a 2-vertex with two nonadjacent neighbors x and y, we delete the vertex u and add the edge xy to G.

(OP3) If both x and y are incident to the unique special 3-face $[uxy]$ such that u is a 2-vertex, then we delete the vertex u from G.

(OP4) If a 4-vertex y is incident to two special 3-faces $[uxy]$ and $[vyz]$ such that both u and v are 2-vertices and x and z are nonadjacent, then we delete the vertices u, v, y and add the edge xz to G.

Let H denote the resulting graph when operations (OP1) to (OP4) are done. It is easy to see that H is an outerplane graph, hence $\delta(H) \leq 2$. However, we will derive a contradiction to show that $d_H(v) \geq 3$ for all vertices $v \in V(H)$.

Any vertex v of H is obviously a vertex of G of degree at least 2. Assume that $d_G(v) = 2$. Let x and y be two neighbors of v. Since G contains neither

(C1) nor (C2), we have $d_G(x) \geq 3$ and $d_G(y) \geq 3$. By (OP2), we know that xy is an edge of G and $[uxy]$ is a special 3-face. Since G excludes (C4), we have $d_G(x) = d_G(y) = 4$. Both x and y are not adjacent to any 1-vertex by the exclusion of (C5). By (OP3), at least one of x and y is incident to two special 3-faces. Since G excludes (C6) and (C7), at most one of x and y is incident to two special 3-faces. Thus exactly one, say y, is incident to two special 3-faces, e.g., $[vxy]$ and $[uyz]$ with $d_G(u) = 2$. Since G excludes (C6), we have $xz \notin E(G)$. Note that z is not incident to two special 3-faces by the exclusion of (C7). However, v, u, y should have been removed from G by (OP4), this contradicts the fact that $v \in V(H)$.

Suppose that $d_G(v) = 3$. Then v is not incident to any special 3-face by (C4). Let x be an arbitrary neighbor of v in G. Then $d_G(x) \geq 2$ by the exclusion of (C1). Assume that $d_G(x) = 2$. Let u be the neighbor of x in G distinct from v. We have $d_G(u) \geq 3$ and $uv \notin E(G)$ since G excludes (C1), (C2), and (C4). The operation (OP2) implies that the edge uv was added to G after the deletion of x. If $d_G(x) = 3$, then the edge vx still belongs to $E(H)$. Now suppose that $d_G(x) = 4$. A 4-vertex was deleted from G by (OP4) if and only if it is incident to two different special 3-faces, each of which contains two 4-vertices and one 2-vertex on the boundary. Thus $x \in V(H)$ and $xv \in E(H)$. Therefore $d_H(v) = d_G(v) = 3$.

Finally suppose that $d_G(v) = 4$. Let $\alpha(v)$ and $\beta(v)$ denote, respectively, the number of 1-vertices adjacent to v in G and the number of special 3-faces incident to v in G. In view of (OP4), (C3), and (C5), we derive $\alpha(v) + \beta(v) \leq 1$. Suppose that u is a neighbor of v in G. If ever performed, the operation (OP2) on u does not affect the degree of v; any one of the operations (OP1), (OP3), or (OP4) on u decreases the degree of v by at most 1. Therefore $d_H(v) \geq d_G(v) - 1 = 3$. □

Lemma 5 *If G is an outerplane graph with $\Delta(G) \leq 4$, then $la_2(G) \leq 3$.*

Proof. It suffices to prove that G has a (probably improper) 3-edge coloring such that every color class induces a subgraph whose components are paths of length at most 2. We proceed to show this assertion by induction on $|V(G)| + |E(G)|$. Without loss of generality, we may assume that G is connected. If $|V(G)|+|E(G)| \leq 5$, the result is straightforward. Let G be an outerplane graph with $\Delta(G) \leq 4$ and $|V(G)|+|E(G)| \geq 6$. Let H denote a proper subgraph of G to be defined later. By the induction hypothesis, H has a 3-edge coloring ϕ such that every color class induces a subgraph whose components are paths of length at most 2. Let the color set be $C = \{1,2,3\}$. For an edge $e \in E(G) \setminus E(H)$, let $S(e) \subseteq C$ denote the set of colors used on those edges of H that are adjacent to e in G. By Lemma 4, we need to consider the following cases for defining H and extending ϕ to an edge coloring of G.

If either (C1) or (C2) holds, we define $H = G - uv$. Then we color uv with a color $a \in C \setminus S(uv)$. This can be done since $|S(uv)| \leq 2 < |C|$.

If (C3) holds, we let $H = G - \{u,v\}$. Thus $S(ux) = S(vx)$ and $|S(ux)| = |S(vx)| \leq 2$. We color both ux and vx with the same color $a \in C \setminus S(vx)$.

If (C4) holds, let x_1 denote the neighbor of x in G distinct from u and y. Obviously, we may suppose that $d_G(y) = 4$ and let y_1 and y_2 be the neighbors of y distinct from u and x. We let $H = G - ux$. If $|S(ux)| \leq 2$, we color ux with a color in $C \setminus S(ux)$. If $|S(ux)| = 3$, we may suppose $\phi(xx_1) = 1$, $\phi(xy) = 2$, and $\phi(uy) = 3$. If at least one of 2 and 3 does not belong to the color set $\{\phi(yy_1), \phi(yy_2)\}$, we color ux with that color. Otherwise, we change the color of uy to 1 and then color ux with 3.

If (C5) holds, we let x, y, z denote the neighbors of v in G distinct from u such that $[vxy]$ is a special 3-face with $d_G(x) = 2$. In view of (C4), we may suppose that $d_G(y) = 4$ and let y_1 and y_2 be the neighbors of y distinct from v and x. Let $H = G - u$. If $|S(uv)| \leq 2$, we color uv with a color $a \in C \setminus S(uv)$. Otherwise, let $\phi(vx) = 1$, $\phi(vy) = 2$, and $\phi(vz) = 3$. If $\phi(xy) \neq 1$, we color uv with 1. Suppose that $\phi(xy) = 1$. It follows that $1 \notin \{\phi(yy_1), \phi(yy_2)\}$. We exchange the colors of vx and vy, then color uv with 2.

If (C6) holds, we let $H = G - \{u_1, u_2, y\}$. Let w_1 be the neighbor of v_1 in H distinct from v_2, and w_2 be the neighbor of v_2 in H distinct from v_1. Erase the color of the edge v_1v_2. If $\phi(v_1w_1) = \phi(v_2w_2) = 1$, we color yu_1 and yu_2

with 1, u_1v_1, u_2v_2, and yv_2 with 2, and v_1v_2 and yv_1 with 3. If $\phi(v_1w_1) = 1$ and $\phi(v_2w_2) = 2$, we color yv_2 and u_2v_2 with 1, yv_1 and u_1v_1 with 2, and yu_1, yu_2, and v_1v_2 with 3.

If (C7) holds, we may suppose $d_G(x) = d_G(v) = 4$ by (C4). We define $H = G - \{u_1, u_2, u_3, y, z\}$. Let x_1 and x_2 be the neighbors of x in H, and v_1 and v_2 be the neighbors of v in H. First color xy and xu_1 with a color $a \in C \backslash \{\phi(xx_1), \phi(xx_2)\}$, then color vz and vu_3 with a color $b \in C \backslash \{\phi(vv_1), \phi(vv_2)\}$. If $a = b = 1$, we color yu_1, zu_2, and zu_3 with 2, and yu_2 and yz with 3. If $a = 1$ and $b = 2$, we color zu_2 and zu_3 with 1, yu_1 and yu_2 with 2, and yz with 3. □

Lemma 6 *If a graph G can be edge-partitioned into m subgraphs $G_1, G_2, \ldots, G_m$, then $la_2(G) \leq \sum_{i=1}^{m} la_2(G_i)$.*

Lemma 7 ([5]) *For a forest T, we have $la_2(T) \leq \lceil (\Delta(T) + 1)/2 \rceil$.*

Lemma 8 ([2]) *For a graph G, we have $la_2(G) \leq \Delta(G)$.*

Theorem 9 *If G is an outerplanar graph, then $la_2(G) \leq \lfloor (\Delta(G) + 4)/2 \rfloor$ and the bound can be attained.*

Proof. If $\Delta(G) \leq 4$, the result follows from Lemma 8. Thus suppose that $\Delta(G) \geq 5$. By Lemma 3, G has an edge-partition into a forest T and a subgraph H satisfying $\Delta(T) \leq \Delta(G) - 4$ and $\Delta(H) \leq 4$. It is obvious that H is an outerplanar graph. By Lemmas 5, 6, and 7, we have $la_2(G) \leq la_2(T) + la_2(H) \leq \lceil (\Delta(T) + 1)/2 \rceil + 3 \leq \lceil (\Delta(G) - 4 + 1)/2 \rceil + 3 = \lfloor (\Delta(G) + 4)/2 \rfloor$. The upper bound can be attained, for instance, by the complete graph on four vertices with one edge deleted. □

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