The Index of Tricyclic Hamiltonian Graphs with $\Delta(G) = 3$

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Abstract

If G is a tricyclic Hamiltonian graph of order n with maximum degree 3 then G has one of two forms, $X(q,r,s,t)$ and $Y(q,r,s,t)$, where $q+r+s+t=n$. We find the graph G with maximal index by first identifying the graphs of each form having maximal index.

1 Introduction

We consider only undirected finite graphs without loops or multiple edges. A graph with n vertices is tricyclic hamiltonian if it is connected and has $n+2$ edges, and hence consists of a cycle and two chords. If the maximal vertex degree $\Delta(G)$ of a tricyclic hamiltonian graph G is 3, then there are two possible forms $X(q,r,s,t)$ and $Y(q,r,s,t)$ illustrated in Figure 1, where a,b,c,d, denote vertices and q,r,s,t denote numbers of edges.

The index of a graph G is the largest eigenvalue of a (0,1)-adjacency matrix of G. In references [3] and [4] the tricyclic Hamiltonian graphs of order n with maximal index and minimal index have been determined . Let Φ_n denotes the class of all tricyclic Hamiltonian graphs G with n vertices($n \geq 5$); the maximal vertex degree $\Delta(G)$ of G is 4 or 3 according as the two chords do or do not have a vertex in common. In reference [4] it has been shown that if $G \in \Phi_n$

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has maximal index then $\Delta(G) = 4$. Here we deal with the maximal index of a graph G in Φ_n with $\Delta(G) = 3$. In reference [4] we found the equation

$$\begin{aligned}\Omega = \; & 32sh^4\theta\{1 - chn\theta\} \\ & +32sh^3\theta\{sh(q+r)\theta + sh(r+s)\theta + sh(s+t)\theta + sh(t+q)\theta\} \\ & +8sh^2\theta\{2chn\theta + 4shq\theta shs\theta + 4shr\theta sht\theta \\ & \quad -ch(q+r-s-t)\theta - ch(q+t-r-s)\theta\} \\ & -16shq\theta shr\theta shs\theta sht\theta.\end{aligned} \tag{1}$$

The index μ_1 of $X(q,r,s,t)$ is $2ch\theta_1$, where θ_1 is the largest solution of $\Omega = 0$ regarded as an equation in θ. Next we give some lemmas which will be used later.

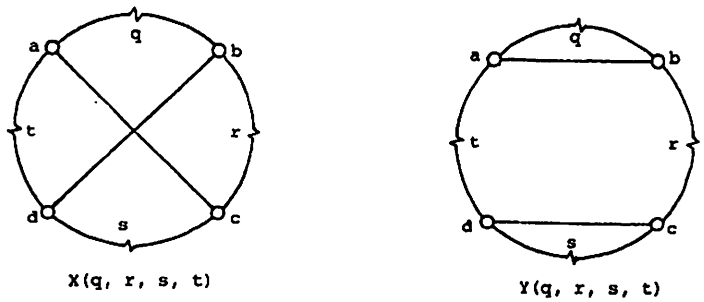

Figure 1. The two types of tricyclic Hamiltonian graphs with $\Delta(G) = 3$.

Lemma 1.1 [4] *$\partial\Omega/\partial\theta$ is non-positive at $\theta = \theta_1$.*

Lemma 1.2 [4] *Suppose that $\theta = \theta_1$ and n, r, t are fixed. If $s > q$, then $\partial\Omega/\partial s > 0$; and if $s < q$ then $\partial\Omega/\partial s < 0$.*

Lemma 1.3 [1] *For any graph G, and any $uv \in E(G)$, let C_{uv} be the set of cycles of G containing the edge uv. Then*

$$P_G = P_{G-uv} - P_{G-u-v} - 2\sum_{C\in C_{uv}} P_{G-C}.$$

Lemma 1.4 [1] *Let G and H be (disjoint)graphs and let F be the graph obtained from $G\bigcup H$ by introducing an edge between vertex j of G and vertex k of H. Then*

$$P_F(x) = P_G(x)P_H(x) - P_{G-j}(x)P_{H-k}(x).$$

2 Main Results

We shall use implicitly the facts that the characteristic polynomial of an n-vertex path P_n is $U_n(\frac{1}{2}x)$, and the characteristic polynomial of an cycle C_n is

$2T_n(\frac{1}{2}x) - 2$ [2,p.73]. Here T_n, U_n are Chebyshev polynomials of the first and second kind respectively. Thus if $x = 2cos\theta$ and $0 < \theta < \pi$ then $U_n(\frac{1}{2}x) = sin(n+1)\theta/sin\theta$ and $T_n(\frac{1}{2}x) = cosn\theta$. Next we study the maximal index of the two forms of tricyclic Hamiltonian graph, $X(q,r,s,t)$ and $Y(q,r,s,t)$.

Lemma 2.1 *If $q \geq t$, then $\mu_1(X(q,r,s,t)) < \mu_1(X(q+1,r,s,t-1))$.*

Proof. We write $X = X(q,r,s,t), X' = X(q+1,r,s,t-1)$ with chords $ac, a'c$ repectively(cf. Fig.1). On identifying a graph with its characteristic polynomial, we obtain from lemma 1.3:

$$\begin{aligned} X &= (X-ac) - (X-a-c) - 2(P_{q+r-1} + P_{s+t-1} + P_{q-1}P_{s-1} + P_{r-1}P_{t-1}), \\ X' &= (X'-a'c) - (X'-a'-c) - 2(P_{q+r} + P_{s+t-2} + P_qP_{s-1} + P_{r-1}P_{t-2}), \end{aligned}$$

where P_k denotes a k-vertex path. Obviously the graph $X - ac$ is the same as the graph $X' - a'c$, and $X-a-c, X'-a'-c$ are two trees for which we may use Lemma 1.4 to obtain

$$\begin{aligned} X-a-c &= P_{q+r-1}P_{s+t-1} - P_{q-1}P_{s-1}P_{r-1}P_{t-1}, \\ X'-a'-c &= P_{q+r}P_{s+t-2} - P_qP_{s-1}P_{r-1}P_{t-2}. \end{aligned}$$

Thus we have $X - X' = s_1 + s_2 - s_3$, where

$$\begin{aligned} s_1 &= (P_{q+r}P_{s+t-2} - P_qP_{s-1}P_{r-1}P_{t-2}) - (P_{q+r-1}P_{s+t-1} - P_{q-1}P_{s-1}P_{r-1}P_{t-1}), \\ s_2 &= 2(P_{q+r} + P_{s+t-2} + P_qP_{s-1} + P_{r-1}P_{t-2}), \\ s_3 &= 2(P_{q+r-1} + P_{s+t-1} + P_{q-1}P_{s-1} + P_{r-1}P_{t-1}). \end{aligned}$$

Suppose $s+t \leq q+r$. On using the relation $U_aU_b - U_{a+b} = U_{a-1}U_{b-1}$, and defining $V_m = U_{m+1} - U_m$ (where $U_0 = 1, U_{-1} = 0$), we can obtain

$$X - X' = U_{r-1}U_{s-1}U_{q-t} - U_{q+r-s-t} + 2(V_{q+r-1} - V_{s+t-2} + U_{s-1}V_{q-1} - U_{r-1}V_{t-2}).$$

This polynomial is positive on $[2, \infty)$, since a n-cycle C_n is a subgraph of $X(q,r,s,t)$, $\mu_1(X) > \mu_1(C_n) = 2$, the result follows.

Lemma 2.2 *If $s \geq q$, then $\mu_1(X(q,r,s,t)) < \mu_1(X(q-1,r,s+1,t))$.*

Proof. We fix r,t and regard q,s as real variables. By equation (1), since $\Omega = 0$, we have $(\partial\Omega/\partial\theta)(\partial\theta/\partial s) + \partial\Omega/\partial s = 0$. By lemmas 1.1 and 1.2, it follows that when $\theta = \theta_1$ we can deduce that if $s > q$ then $\partial\theta/\partial s > 0$. So θ can be regarded as an increasing function on s, and since $\mu_1(X) = 2ch\theta_1$ is also an increasing function on $\theta_1(\theta > 0)$, our result follows.

Theorem 2.3 *If the graph $X(q,r,s,t)$ has maximal index, where $q+r+s+t = n$ is fixed, then $X(q,r,s,t)$ is isomorphic to $X(1,1,n-3,1)$.*

Proof. We use the fact that

$$X(q,r,s,t) \cong X(t,q,r,s) \cong X(t,s,r,q),$$

and then apply lemmas 2.1 and 2.2 repeatly to obtain the result as desired.

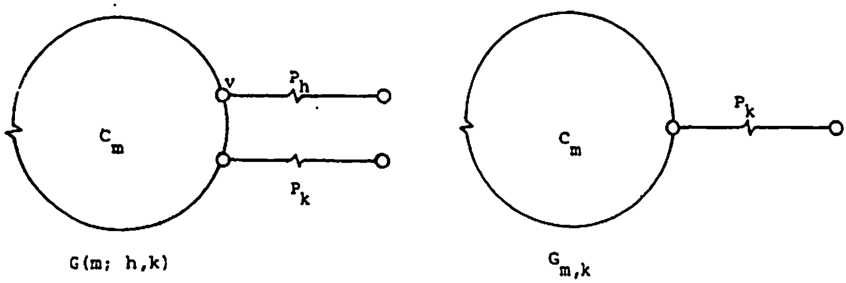

Figure 2. The graphs $G(m; h, k)$ and $G(m, k)$.

Next we consider the form $Y(q, r, s, t)$. Here we give two graphs illustrated in Figure 2 which will be used later. Here $G(m; h, k)$ is a unicyclic graph which consists of an m cycle together with pendant paths of lengths $h-1, k-1$ attached at adjacent vertices of the cycle, and $G(m, k) = G(m; 1, k)$ as shown in Figure 2 .

Lemma 2.4 *If $s \geq r$, then $\mu_1(Y(q, r, s, t)) < \mu_1(Y(q, r-1, s+1, t))$.*

Proof. Let $Y = Y(q, r, s, t), Y' = Y(q, r-1, s+1, t)$ with chords cd and $c'd$ respectively. On identifying a graph with its characteristic polynomial, we obtain from Lemma 1.3:

$$\begin{aligned} Y \; &= (Y - cd) - (Y - c - d) - 2(P_{s-1} + P_{s-1}P_{q-1} + G(q+1, t, r)), \\ Y' \; &= (Y' - c'd) - (Y' - c' - d) - 2(P_s + P_sP_{q-1} + G(q+1, t, r-1)). \end{aligned}$$

Obviously the graph $Y - cd$ is the same as $Y' - c'd$, and

$$\begin{aligned} Y - c - d \; &= P_{s-1}G(q+1, t, r), \\ Y' - c' - d \; &= P_sG(q+1, t, r-1). \end{aligned}$$

Also, by lemma 1.4 we have

$$\begin{aligned} G(q+1, t, r) \; &= G(q+1, t)P_{r-1} - P_{q+t-1}P_{r-2}, \\ G(q+1, t, r-1) \; &= G(q+1, t)P_{r-2} - P_{q+t-1}P_{r-3}. \end{aligned}$$

Thus $Y - Y' = s_1 + s_2 + s_3$, where

$$\begin{aligned} s_1 \; &= P_s(G(q+1, t)P_{r-2} - P_{q+t-1}P_{r-3}) - P_{s-1}(G(q+1, t)P_{r-1} - P_{q+t-1}P_{r-2}), \\ s_2 \; &= 2(P_s + P_sP_{q-1} - P_{s-1} - P_{s-1}P_{q-1}), \\ s_3 \; &= 2(G(q+1, t)P_{r-2} - P_{q+t-1}P_{r-3} - G(q+1, t)P_{r-1} + P_{q+t-1}P_{r-2}). \end{aligned}$$

Now $G(q+1, t) = C_{q+1}P_{t-1} - P_qP_{t-2}$. On simplifying the corresponding expression involving Chebyshev polynomial, and defining $V_m = U_{m+1} - U_m$ (where

$U_{-1} = 0$) we obtain

$$\begin{aligned} s_1 &= U_qU_{t-2}U_{s-r} + U_{q+t-1}U_{s-r+1} - (2T_{q+1} - 2)U_{s-r}U_{t-1}, \\ s_2 &= 2V_{s-1}(U_{q-1} + 1), \\ s_3 &= 2(U_{q+t-1}V_{r-3} - ((2T_{q+1} - 2)U_{t-1} - U_qU_{t-2})V_{r-2}). \end{aligned}$$

We obtain $Y - Y' = U_qU_{t-2}U_{s-r} + U_{q+t-1}U_{s-r+1} - (2T_{q+1} - 2)U_{s-r}U_{t-1} + 2V_{s-1}(U_{q-1}+1) + 2(U_{q+t-1}V_{r-3} - ((2T_{q+1} - 2)U_{t-1} - U_qU_{t-2})V_{r-2})$. Since this function is positive on $(2, \infty)$, and $\mu_1(Y(q,r,s,t)) > 2$, the result follows.

Lemma 2.5 *If $s \geq q$, then $\mu_1(Y(q,r,s,t)) < \mu_1(Y(q-1,r,s+1,t))$.*

Proof. We fix r, t and regard q, s as real variables where $q + r + s + t = n$. Let $Y = Y(q,r,s,t), Y' = Y(q-1,r,s+1,t)$. Suppose that Y' is obtained from Y by replacing the edge ab, cd with $ab', c'd$. We apply lemma 1.3 to Y and Y' to obtain

$$\begin{aligned} Y &= (Y - cd) - (Y - c - d) - 2(P_{s-1} + P_{s-1}P_{q-1} + G(q+1,t,r)), \\ Y' &= (Y' - c'd) - (Y' - c' - d) - 2(P_s + P_sP_{q-2} + G(q,t,r)). \end{aligned}$$

Also we apply lemma 1.3 to $Y - cd$ and $Y' - c'd$ in respect of the edges ab and ab', to obtain

$$\begin{aligned} Y - cd &= C_n - P_{q-1}P_{t+s+r-1} - 2(P_{t+s+r-1} + P_{q-1}), \\ Y' - c'd &= C_n - P_{q-2}P_{t+s+r} - 2(P_{t+s+r} + P_{q-2}). \end{aligned}$$

We also have

$$\begin{aligned} Y - c - d &= P_{s-1}G(q+1,t,r), \\ Y' - c' - d &= P_sG(q,t,r). \end{aligned}$$

By lemma 1.4 we obtain

$$\begin{aligned} G(q+1,t,r) &= G(q+1,t)P_{r-1} - P_{q+t-1}P_{r-2}, \\ G(q,t,r) &= G(q,t)P_{r-1} - P_{q+t-2}P_{r-2}, \\ G(q,t) &= C_qP_{t-1} - P_{q-1}P_{t-2}, \\ G(q+1,t) &= C_{q+1}P_{t-1} - P_qP_{t-2}. \end{aligned}$$

Noe we obtain $Y - Y' = s_1 + s_2 + s_3$, where

$$\begin{aligned} s_1 &= (P_{q-2}P_{t+s+r} - P_{q-1}P_{t+s+r-1}) + 2(P_{t+s+r} + P_{q-2} - P_{q-1} - P_{t+s+r-1}), \\ s_2 &= P_sP_{r-1}G(q,t) - P_sP_{q+t-2}P_{r-2} - P_{s-1}P_{r-1}G_{q+1,t} + P_{s-1}P_{q+t-1}P_{r-2}), \\ s_3 &= 2(P_s + P_sP_{q-2} + G(q,t,r) - P_{s-1} - P_{s-1}P_{q-1} - G(q+1,t,r)). \end{aligned}$$

We define V_m as before, and on simplifying the corresponding expression involving Chebyshev polynomial, routine calculations yield the following equations:

$$\begin{aligned} s_1 &= U_{t+s+r-q} + 2(U_{t+s+r} + U_{q-2} - U_{t+s+r-1} - U_{q-1}), \\ s_2 &= U_{r-1}U_{t-1}(2T_{s-q} - V_{s-1}) + U_{r-1}U_{t-2}U_{s-q-1} - U_{r-2}U_{q+t-s-2}, \\ s_3 &= 2(V_{s-1} - U_{s-q} + 2U_{r-1}U_{t-1}(T_q - T_{q+1}) + U_{r-1}U_{t-2}V_{q-1} + U_{r-2}V_{q+t-2}). \end{aligned}$$

Thus we obtain $Y - Y' = U_{t+s+r-q} + 2(U_{t+s+r} + U_{q-2} - U_{t+s+r-1} - U_{q-1}) + U_{r-1}U_{t-1}(2T_{s-q} - V_{s-1}) + U_{r-1}U_{t-2}U_{s-q-1} - U_{r-2}U_{q+t-s-2} + 2(V_{s-1} - U_{s-q} + 2U_{r-1}U_{t-1}(T_q - T_{q+1}) + U_{r-1}U_{t-2}V_{q-1} + U_{r-2}V_{q+t-2})$. Since this function is positive on $[2,\infty)$, and $\mu_1(Y(q,r,s,t)) > 2$, then we get the result as desired.

Theorem 2.6 *If $Y(q,r,s,t)$ has maximal index where $q+r+s+t=n$ is fixed, then $Y(q,r,s,t)$ is isomorphic to $Y(2,1,n-4,1)$.*

Proof. This is a direct corollary of Lemmas 2.4 and 2.5.

Theorem 2.7 *If $G \in \Phi_n$ with $\Delta(G) = 3$, and if G has the maximal index, then $G \cong X(1,1,n-3,1)$.*

Proof. By theorem 2.3 and 2.6, we have only to show that $\mu_1(X(1,1,n-3,1)) > \mu_1(Y(2,1,n-4,1))$. Suppose that the vertices of $X(1,1,n-3,1)$ and $Y(2,1,n-4,1)$ have been numbered as $1,2,\cdots,n$ respectively: $X(1,1,n-3,1)$ has the two chords $13, 24$, $Y(2,1,n-4,1)$ has the two chords $13, 4n$. Then by Lemma 1.3 we have

$$\begin{aligned} X &= (X-24) - (X-2-4) - 2(P_{n-3} + P_1 + P_{n-4} + 1), \\ Y &= (Y-4n) - (Y-4-n) - 2(P_{n-5} + P_1P_{n-5} + C_3). \end{aligned}$$

and $X - 2 - 4 = P_{n-2}, Y - 4 - n = C_3P_{n-5}$. Thus we have

$$Y - X = P_{n-2} - C_3P_{n-5} + 2(P_{n-3} + P_1 + P_{n-4} + 1 - P_{n-5} - P_1P_{n-5} - C_3).$$

Since $\mu_1(P_{n-2}) = 2cos\frac{\pi}{n-1} < \mu_1(C_3 \bigcup P_{n-5}) = \mu_1(C_3) = 2$, we have $P_{n-2} - C_3P_{n-5} > 0$; and since also the function $P_{n-3} + P_1 + P_{n-4} + 1 - P_{n-5} - P_1P_{n-5} - C_3$ is positive on $(2,\infty)$, we have $Y - X > 0$ on $(2,\infty)$, and the result follows.

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