Tree Domination in Graphs

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Abstract. A subset S of $V(G)$ is called a *dominating set* if every vertex in $V(G) - S$ is adjacent to some vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets of G. A dominating set S is called a *tree dominating set* if the induced subgraph $\langle S \rangle$ is a tree. The *tree domination number* $\gamma_{tr}(G)$ of G is the minimum cardinality taken over all minimal tree dominating sets of G. In this paper, some exact values of tree domination number and some properties of tree domination are presented in Section [2]. Best possible bounds for the tree domination number, and graphs achieving these bounds are given in Section [3]. Relationships between the tree domination number and other domination invariants are explored in Section [4], and some open problems are given in Section [5].

1 Introduction

By a graph we mean a finite, undirected graph without loops or multiple edges. Let $G = (V(G), E(G))$ be a graph with $|V(G)| = n$ and $|E(G)| = m$. Let $\Delta(G)$ and $\delta(G)$ denote, respectively, the maximum degree and minimum degree of G. For any $v \in V$, $d_G(v)$ denote the degree number of v in G. A subset $S \subseteq V(G)$ is called a *dominating set* if every vertex in $V(G) - S$ is adjacent to some vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets of G. A dominating set S is called a *connected* (*acyclic*) *dominating set* if the induced subgraph $\langle S \rangle$ is connected (acyclic). The *connected* (*acyclic*

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) *domination number* $\gamma_c(G)(\gamma_a(G))$ of G is the minimum cardinality taken over all minimal connected (acyclic) dominating sets of G.

A vertex $v \in V(G)$ is called a *support vertex* if it is adjacent to a *pendant vertex* (that is, a vertex of degree one). Any vertex of degree greater than one is called an *internal vertex.* For a subset $S \subseteq V(G)$, $N(S)$ denotes the set of all vertices adjacent to some vertex in S and $N[S] = N(S) \cup S$.

Let C_n, P_n, $K_{1,n-1}$ and K_n denote, respectively, a cycle, path, star and complete graph of order n. Let $K_{r,s}$ denote the complete bipartite graph with disjoint vertex set A and B, where $|A| = r$ and $|B| = s$.

We say that two vertices are *identified* if they are replaced by a single vertex whose neighbour set is the union of the neighbour sets of the two vertices.

If $\langle S \rangle$ is both connected and acyclic, then $\langle S \rangle$ is a tree. A dominating set S is called a *tree dominating set* if the induced subgraph $\langle S \rangle$ is a tree. The *tree domination number* $\gamma_{tr}(G)$ of G is the minimum cardinality taken over all minimal tree dominating sets of G. It plays an important role in studying both the connected domination number, introduced by Sampathkumar and Walikar [1], and the acyclic domination number, introduced by Hedetniemiand, Hedetniemi and and Rall [2]. In this paper, some exact values of tree domination number and some properties of tree domination are presented in Section [2]. Best possible bounds for the tree domination number, and graphs achieving these bounds are given in Section [3]. Relationships between the tree domination number and other domination invariants are explored in Section [4], and some open problems are given in Section [5].

2 Some exact values of tree domination number and some properties of tree domination

By definition, we know that the tree domination number does not exist for many graphs, such as the corona graph $G \circ K_1$, for any graph G which contains a cycle. If the tree domination number does not exist for a given graph G, we define $\gamma_{tr}(G) = 0$. Since the tree domination number does not exist for any disconnected graph, we will only consider connected graphs in this paper. Let T be a tree with order n, and let l denote the number of leaves of T. Let $K_{r,s}$ denote the complete bipartite graph with $\min(r, s) \geq 2$. By observation, it is easy to determine the exact values of the tree domination number for several standard classes of graphs:

Class	$\gamma_{tr}(G)$
P_n	$n-2$
C_n	$n-2$

K_n	1
$K_{1,n-1}$	1
$K_{r,s}$	2
T	$n-l$

The next several results give some properties of tree domination in graphs.

Proposition 1. *For every connected graph G, $\gamma_{tr}(G) = 1$ if and only if $\Delta(G) = n - 1$.*

Proposion 2. *For every connected graph G of order $n > 2$, if $\gamma_{tr}(G) > 0$ then:*

(1) *no leaf is a member of any minimal tree dominating set;*

(2) *every support vertex is a member of every tree dominating set;*

(3) *every cutvertex is a member of every tree dominating set;*

(4) *if $\delta \geq 2$, then both vertices on every cut edge (bridge) are in every tree dominating set.*

By Proposition 2, we can construct many graphs with $\gamma_{tr}(G) = 0$

Corollary 3. *Let G be a connected graph and $v \in V(G)$. Then,*

(1) *if every internal vertex of G is a support and the induced subgraph of all internal vertices has at least a cycle, then $\gamma_{tr}(G) = 0$.*

(2) *let H be a connected graph and $u \in V(H)$. G' denote the graph obtained from G and H by identifying u and v . If G has no tree domination containing v, then $\gamma_{tr}(G') = 0$.*

(3) *let H be a connected graph and $u \in V(H)$. G' denote the graph obtained from G and H by joining an edge between u and v . If G has no tree domination containing v, then $\gamma_{tr}(G') = 0$.*

Note that all of our example of graphs having no tree dominating sets have cutvertices, that is, connectivity one. In fact, the following illustration, due to Alice McRae, shows that graphs can have arbitrarily large connectivity and still not have tree domination sets. The construction below is for 3-connected graphs.

Construct a complete graph K_{3n}, and partition the vertices of K_{3n} into n sets of three vertices each. For each set of three vertices, add a new vertex, and join it to each of these three vertices. It is easy to prove that the resulting 3-connected graph does not have a tree dominating set.

In order to increase the connectivity to some number k, start with a complete graph K_{kn}, partition the vertices into n sets of k vertices each, and to each set add a vertex which is joined to each of the k vertices.

3 Bounds of tree domination number

Obviously the tree domination number of a graph G can be zero. But for graphs G having tree dominating set, several lower and upper bounds can be given for $\gamma_{tr}(G)$.

Theorem 4. *Let G be a connected graph and l is the number of vertices of degree 1. If $\gamma_{tr}(G) > 0$, then $\gamma_{tr}(G) \geq 3n - 2m - l - 2$ and the bound is sharp.*

Proof. Let S be a minimum tree dominating set of G such that $|S| = \gamma_{tr}(G)$, and let t be the number of edges in G having one vertex in S and the other in $V(G) - S$. So

$$\begin{aligned} 2[m - (|S| - 1)] &= \textstyle\sum_{v_i \in V - S} d(v_i) + t \\ &\geq l + 2(n - |S| - l) + (n - |S|) \\ &= 3(n - |S|) - l \end{aligned}$$

That is,

$$\gamma_{tr}(G) = |S| \geq 3n - 2m - l - 2.$$

It is obvious that $\gamma_{tr}(P_n) = n - 2 = 3n - 2m - l - 2$, so the bound is sharp.

Theorem 5. *Let G be a connected graph with $\delta \geq 2$. If $\gamma_{tr}(G) > 0$, then $\gamma_{tr}(G) \geq \frac{(\delta+1)n-2m-2}{\delta-1}$ and the bound is sharp.*

Proof. Let S be a minimum tree dominating set of G such that $|S| = \gamma_{tr}(G)$, and let t be the number of edges in G having one vertex in S and the other in $V(G) - S$. So

$$\begin{aligned} 2[m - (|S| - 1)] &= \textstyle\sum_{v_i \in V - S} d(v_i) + t \\ &\geq \delta(n - |S|) + (n - |S|) \\ &= (\delta + 1)(n - |S|) \end{aligned}$$

That is,

$$\gamma_{tr}(G) = |S| \geq \frac{(\delta + 1)n - 2m - 2}{\delta - 1}.$$

It is obvious that $\gamma_{tr}(C_n) = n - 2 = \frac{(\delta+1)n-2m-2}{\delta-1}$, so the boundary is sharp.

From Theorem 5, we can conclude the following corollary.

Corollary 6. *Let G be a connected k-regular graph and $k \geq 2$. If $\gamma_{tr}(G) > 0$, then $\gamma_{tr}(G) \geq \frac{n-2}{k-1}$, and the bound is sharp.*

Next, a few sharp upper bounds of the tree domination number are presented.

Theorem 7. *For every connected graph G with $n \geq 3$, $\gamma_{tr}(G) \leq n-2$.*

Proof. Suppose to the contrary that $\gamma_{tr}(G) \geq n-1$. If $\gamma_{tr}(G) = n$, then G is a tree. So G has at least two vertices of degree 1, we may assume without loss of generality that $d(u) = d(v) = 1$. Let $G' = G - \{u, v\}$, hence G' is a tree dominating set of G with cardinality $n-2$, which is a contradiction.

If $\gamma_{tr}(G) = n-1$, then let S be a minimum tree dominating set of G such that $|S| = \gamma_{tr}(G)$. Let $G - S = \{w\}$, and $\langle S \rangle$ has at least two vertices of degree 1, say, u and v. We now claim that w is adjacent to both u and v. Otherwise, without loss of generality, we assume that w is not adjacent to u, then $S - \{u\}$ is a tree dominating set of G with cardinality $n-2$, which is a contradiction. But, if w is adjacent to both u and v, then $S - \{u\}$ is a tree dominating set of G with cardinality $n-2$, which is a contradiction.

Theorem 8. *For every connected graph G, $\gamma_{tr}(G) = n-2$ if and only if G is isomorphic to P_n or C_n.*

Proof. It can be easily verified that for all graphs stated in the theorem, $\gamma_{tr}(G) = n-2$.

Conversely, let G be any connected graph for which $\gamma_{tr}(G) = n-2$.

Let S be a minimum tree dominating set of G such that $|S| = n-2$ and $G - S = \{w_1, w_2\}$. Since $\langle S \rangle$ has at least two vertices of degree 1, say, u and v, we claim every vertex of degree 1 of $\langle S \rangle$ is adjacent to exactly one vertex of $G - S$. Otherwise, if u is not adjacent to $G - S$, then $S - \{u\}$ is a tree dominating set of G with cardinality $n-3$, which is a contradiction, if u is adjacent to both w_1 and w_2, then $S - \{v\}$ is a tree dominating set of G with cardinality $n-3$, which is a contradiction. In a similar way, we claim that every vertex of degree 1 of $\langle S \rangle$ is adjacent to different vertex of $G - S$. Hence, $\langle S \rangle$ has only two vertices with degree 1 and $\langle S \rangle$ is a path with $n-2$ vertices. So, G is isomorphic to P_n or C_n.

Let $P_{n-1} = u_1 u_2 \cdots u_{n-2} u_{n-1}$ be a path of order $n-1$. Let $\eta_1 = \{F \mid F$ be a simple graph and $V(F) = \{u_1, u_{n-1}, u_n\}\}$ and $\eta_2 = \{T \mid T$ is a tree with $V(T) = \{u_2, u_3, \cdots, u_{n-2}\}$ and three vertices of degree 1, $n \geq 7\}$. For any $T \in \eta_2$, let u_2, u_3, u_{n-2} denote the three vertices of degree 1 of T. Let $G_1 = P_{n-1} \cup \{u_n\} \cup \{u_1u_n, u_3u_n, u_1u_{n-1}\}$ and $G_2 = G_1 \cup \{u_2u_n\}$.

Now, we construct a few classes of graphs as follows:

$$
\begin{aligned}
H &= \{T \cup \{u_1, u_{n-1}, u_n\} \cup \{u_1u_{n-2}, u_2u_{n-1}, u_3u_n\} \\
&\cup\ F \mid T \in \eta_2, F \in \eta_1, \Delta(F) \leq 1\} \\
\xi_1^1 &= \{P_{n-1} \cup \{u_n\} \cup \{u_iu_n\} \cup F | i = 3, \cdots, n-3, F \in \eta_1, \Delta(F) \leq 1\} \\
\xi_1^2 &= \{P_{n-1} \cup \{u_n\} \cup \{u_2u_n\} \cup F | F \in \eta_1\} \\
\xi_2^1 &= \{P_{n-1} \cup \{u_n\} \cup \{u_iu_n, u_{i+1}u_n\} \cup F | 2 \leq i \leq n-3, F \in \eta_1, \Delta(F) \leq 1\} \\
\xi_2^2 &= \{P_{n-1} \cup \{u_n\} \cup \{u_iu_n, u_{i+2}u_n\} \cup F | 2 \leq i \leq n-4, F \in \eta_1, d_F(u_n) = 0\} \\
\xi_3 &= \{P_{n-1} \cup \{u_n\} \cup \{u_iu_n, u_{i+1}u_n, u_{i+2}u_n\} \\
&\cup\ F | i = 2, 3, \cdots, n-4, \quad F \in \eta, d_F(u_n) = 0\} \\
\xi_4 &= \{P_{n-1} \cup \{u_n\} \cup \{\cup_{1 \leq i \leq 2s} u_{t_i}u_n\} \cup F | 2 \leq t_1 < t_2 < \cdots < t_{2s} \\
&\leq\ n-2, \quad \cup_{1 \leq k \leq s}(u_{t_{2k-1}}u_{t_{2k}} \in E(P_{n-1})), \quad \cup_{1 \leq k \leq (s-1)}(u_{t_{2k}}u_{t_{2k+1}} \\
&\notin\ E(P_{n-1})), \quad F \in \eta, d_F(u_n) = 0, s \geq 2\}
\end{aligned}
$$

Let $\xi_1 = \xi_1^1 \cup \xi_1^2$ and $\xi_2 = \xi_2^1 \cup \xi_2^2$.

Theorem 9. *Let G be a connected graph. Then $\gamma_{tr}(G) = n - 3$ if and only if G is isomorphic to one graph of $H \cup (\cup_{1 \leq i \leq 4} \xi_i) \cup \{G_i | i = 1, 2\}$.*

Proof. It can be easily verified that for all graphs stated in the theorem, $\gamma_{tr}(G) = n - 3$.

Conversely, let G be any connected graph for which $\gamma_{tr}(G) = n - 3$.

Let S be a minimum tree dominating set of G such that $|S| = n - 3$. We may assume without loss of generality that $S = \{v_1, v_2, \cdots, v_{n-3}\}$, $V(G) - S = \{v_{n-2}, v_{n-1}, v_n\}$. Since $\langle S \rangle$ has at least two vertices of degree 1, let $S_1 = \{v_1, v_2, \cdots, v_m | 2 \leq m < n-3\}$ be the set of vertices with degree 1 in $\langle S \rangle$. We now claim every vertex $v_i \in S_1$ is adjacent to only one or two vertices of $V(G) - S$ for $1 \leq i \leq m$. Otherwise, if there exists a vertex v_i such that v_i is not adjacent to $V(G) - S$, then $S - \{v_i\}$ is a tree dominating set of G with cardinality $n - 4$, which is a contradiction. If v_i is adjacent to all vertices of $V(G) - S$, then there exists a vertex $v_j \in S_1$ and $i \neq j$ such that $S - v_j$ is a tree dominating set of G with cardinality $n - 4$, which is a contradiction.

Case 1 Every vertex of S_1 is adjacent to exactly one vertex of $V(G) - S$ for $i = 1, 2, \cdots, m$. Then any two vertices of S_1 must be adjacent to different vertices of $V(G) - S$. Since $|V(G) - S| = 3$, it follows that $2 \leq m = |S_1| \leq 3$.

Case 1.1 $|S_1| = 3$. Without loss of generality, we can assume $v_1v_{n-2} \in E(G)$, $v_2v_{n-1} \in E(G)$ and $v_3v_n \in E(G)$. It is obvious that every vertex $v_i \in V(G)-S$ must be not adjacent to any vertex of $S-S_1$ for $i = n-2, n-1, n$. If there exists v_i such that $d_{\langle V(G)-S\rangle}(v_i) = 2$ for $i = n-2, n-1, n$, say $d_{\langle V(G)-S\rangle}(v_n) = 2$, then $S \cup \{v_n\} \backslash \{v_1, v_2\}$ is a tree dominating set of G with cardinality less than $n-3$, which is a contradiction. So, $d_{\langle V(G)-S\rangle}(v_i) \leq 1$ for $i = n-2, n-1, n$. Hence, G is isomorphic to one graph of H.

Case 1.2 $|S_1| = 2$. It is obvious that $\langle S\rangle$ is a path of order $n-3$, say $\langle S\rangle = v_1v_3 \cdots v_{n-3}v_2$. We may assume without loss of generality that v_1 is adjacent to v_{n-2} and v_2 is adjacent to v_{n-1}. It is obvious that both v_{n-2} and v_{n-1} are not adjacent to any vertices of $S - S_1$. Let $N_{S-S_1}(v_n) = \{v_{t_1}, v_{t_2}, \cdots, v_{t_s}\}$ denote the neighborhood set of v_n in $S - S_1$. Without loss of generality, we assume that $3 \leq t_1 < t_2 < \cdots < t_s \leq n-3$. So, $|N_{S-S_1}(v_n)| = s$.

Case 1.2.1 $s = 1$. If $d_{\langle V(G)-S\rangle}(v_n) = 2$, then $S \cup \{v_n\} \backslash \{v_1, v_2\}$ is a tree dominating set of G with cardinality less than $n-3$, which is a contradiction. So, $d_{\langle V(G)-S\rangle}(v_n) \leq 1$. If $d_{\langle V(G)-S\rangle}(v_n) = 0$, then G is isomorphic to one graph of ξ_1. Without loss of generality, we can assume that $d_{\langle V(G)-S\rangle}(v_n) = 1$ and $v_nv_{n-2} \in E(G)$. If $d_{\langle V(G)-S\rangle}(v_{n-2}) \leq 1$, then G is isomorphic to one graph of ξ_1. If $d_{\langle V(G)-S\rangle}(v_{n-2}) = 2$, then $v_{t_1} = v_3$. Otherwise, $S \cup \{v_n, v_{n-2}\} \backslash \{v_1, v_2, v_3\}$ is a tree dominating set of G with cardinality less than $n-3$, which is a contradiction. Hence, G is isomorphic to G_1.

Case 1.2.2 $s = 2$. Then we claim that $t_2 - t_1 \leq 2$. Otherwise, if $t_2 - t_1 \geq 3$, then $S \cup \{v_n\} \backslash \{v_{t_1+1}, v_{t_1+2}\}$ is a tree dominating set of G with cardinality less than $n-3$, which is a contradiction. If $t_2 - t_1 = 2$, then $d_{\langle V(G)-S\rangle}(v_n) = 0$. Otherwise, say $v_nv_{n-2} \in E(G)$, $S \cup \{v_n\} \backslash \{v_1, v_{t_1+1}\}$ is a tree dominating set of G with cardinality less than $n-3$, which is a contradiction. So G is isomorphic to one graph of ξ_2. If $t_2 - t_1 = 1$, then $d_{\langle V(G)-S\rangle}(v_n) \leq 1$. Otherwise, if $v_{t_2} \neq v_{n-3}$, then $S \cup \{v_n, v_{n-1}\} \backslash \{v_1, v_{t_2}, v_{t_2+1}\}$ is a tree dominating set of G with cardinality less than $n-3$, which is a contradiction; if $v_{t_2} = v_{n-3}$, then $S \cup \{v_n, v_{n-1}\} \backslash \{v_1, v_{t_2}, v_2\}$ is a tree dominating set of G with cardinality less than $n-3$, which is a contradiction. In a similar way, $d_{\langle V(G)-S\rangle}(v_{n-2}) \leq 1$ and $d_{\langle V(G)-S\rangle}(v_{n-1}) \leq 1$. So, G is isomorphic to one graph of ξ_2.

Case 1.2.3 $s = 3$. Then we claim that $t_2 - t_1 = t_3 - t_2 = 1$. Otherwise, we assume without loss of generality that $t_2 - t_1 \geq 2$, then $S \cup \{v_n\} \backslash \{v_{t_2-1}, v_{t_2}\}$ is a tree dominating set of G with cardinality less than $n-3$, which is a contradiction. If $d_{\langle V(G)-S\rangle}(v_n) \geq 1$, say $v_nv_{n-2} \in E(G)$, then $S \cup \{v_n\} \backslash \{v_{t_2}, v_1\}$ is a tree dominating set of G with cardinality less than $n-3$, which is a contradiction. So $d_{\langle V(G)-S\rangle}(v_n) = 0$. Hence, G is isomorphic to one graph of ξ_3.

Case 1.2.4 $s \geq 4$. Then we claim that s is an even integer number.

Otherwise, if s is an even integer, then $S \cup \{v_n\} \backslash \{v_{t_i} | 1 \leq i \leq s$ and i is an even$\}$ is a tree dominating set of G with cardinality less than $n-3$, which is a contradiction. Now, we can prove that $v_{t_{2k-1}}$ and $v_{t_{2k}}$ are adjacent vertices in $S - S_1$ for $k = 1, 2, \cdots, \frac{s}{2}$. Suppose to the contrary, there exists k such that $v_{t_{2k-1}}$ and $v_{t_{2k}}$ are not adjacent, so $S \cup \{v_n\} \backslash \{\{v_{t_{2i}} | 1 \leq i \leq k-1\} \cup \{v_{t_{2i+1}} | k \leq i \leq \frac{s-2}{2}\} \cup \{v_{t_{2k-1}+1}\}\}$ is a tree dominating set of G with cardinality less than $n-3$, which is a contradiction. Finally, we can prove that $v_{t_{2k}}$ and $v_{t_{2k+1}}$ are not adjacent vertices in $S - S_1$ for $k = 1, 2, \cdots, \frac{s-2}{2}$. Suppose to the contrary, there exists k such that $v_{t_{2k}}$ and $v_{t_{2k+1}}$ are adjacent. So, $S \cup \{v_n\} \backslash \{\{v_{t_{2i}} | 1 \leq i \leq k\} \cup \{v_{t_{2i+1}} | k \leq i \leq \frac{s-2}{2}\}\}$ is a tree dominating set of G with cardinality less than $n-3$, which is a contradiction.

If $d_{\langle V(G)-S \rangle}(v_n) \geq 1$, then assume $v_n v_{n-2} \in E(G)$. If $v_{t_1} \neq v_3$, then $S \cup \{v_n, v_{n-2}\} \backslash \{\{v_{t_{2i-1}} | 1 \leq i \leq k\} \cup \{v_{t_1 - 1}\}\}$ is a tree dominating set of G with cardinality less than $n-3$, which is a contradiction. If $v_{t_1} = v_3$, then $S \cup \{v_n, v_{n-2}\} \backslash \{\{v_{t_{2i-1}} | 1 \leq i \leq k\} \cup \{v_1\}\}$ is a tree dominating set of G with cardinality less than $n-3$, which is a contradiction. So $d_{\langle V(G)-S \rangle}(v_n) = 0$. Hence, G is isomorphic to one graph of ξ_4.

Case 2 There exists a vertex of S_1 is adjacent to two vertices of $V(G) - S$. We may assume without loss of generality that v_1 is adjacent to v_{n-2} and v_n. So, other vertex $v_i \in S_1$ must be adjacent to at most one vertex of v_{n-2} and v_n for $i = 2, 3, \cdots, m$.

Case 2.1 Every vertex $v_i \in S_1$ is not adjacent to v_{n-2} and v_n for $i = 2, 3, \cdots, m$. Then they must be adjacent to v_{n-1}. So, $m = |S_1| = 2$ and v_2 is adjacent to v_{n-1}. Hence, $\langle S \rangle$ is a path, i.e., $\langle S \rangle = v_1 v_3 \cdots v_{n-3} v_2$. It is obvious that v_{n-1} is not adjacent to any vertex of $S - S_1$. If v_{n-2} and v_n are not adjacent to any vertex of $S - S_1$, then G is isomorphic to one graph of ξ_1.

If there exists a vertex of $\{v_{n-2}, v_n\}$ such that is adjacent to some vertices of $S - S_1$, we may assume without loss of generality that v_n is adjacent to some vertices of $S - S_1$. Let $N_{S-S_1}(v_n) = \{v_{t_2}, v_{t_3}, \cdots, v_{t_s}\}$ denote the neighborhood set of v_n in $S - S_1$. Without loss of generality, we assume that $v_{t_1} = v_1$ and $1 = t_1 < 3 \leq t_2 < \cdots < t_s \leq n-3$ so, $|N_{S-S_1}(v_n)| = s - 1 \geq 1$.

Case 2.1.1 $s = 2$. In a similar way as case 1.2.2, it follows that $v_{t_2} = v_3$ or $v_{t_2} = v_4$. If $v_{t_2} = v_4$, in a similar way as case 1.2.2, then it follows that $d_F(v_n) = 0$. So G is isomorphic to one graph of ξ_2. If $v_{t_2} = v_3$, then it is obvious that $d_F(v_n) \leq 1$ and $d_F(v_{n-1}) \leq 1$. If $d_F(v_{n-2}) = 2$, then G is isomorphic to G_2; if $d_F(v_{n-2}) \leq 1$, then G is isomorphic to one graph of ξ_2.

Case 2.1.2 $s = 3$. In a similar way as case 1.2.3, it follows that G is isomorphic to one graph of ξ_3.

Case 2.1.3 $s \geq 4$. In a similar way as case 1.2.4, it follows that G is isomorphic to one graph of ξ_4.

Case 2.2 There exists a vertex $v_i \in S_1$ such that is adjacent to at most one vertex of v_{n-2} and v_n for $i = 1, 2, \cdots, m$. Assume v_2 is adjacent to v_n, then v_{n-2} must be not adjacent to other any vertex of $S_1 - \{v_1, v_2\}$ and v_2 must be adjacent to v_{n-1}. It is obvious that v_{n-1} is not adjacent to other any vertex of $S_1 - \{v_1, v_2\}$. We claim that $|S_1| = m \leq 2$. Otherwise, if $|S_1| = m \geq 3$, then there exist a vertex $v_3 \in S_1$ such that it is adjacent to v_n. So $S - v_3$ a tree dominating set of G with cardinality less than $n - 4$, which is a contradiction. Hence, $\langle S \rangle$ is a path, i.e., $\langle S \rangle = v_1 v_3 \cdots v_{n-3} v_2$.

Let $N_{S-S_1}(v_n) = \{v_{t_3}, v_{t_4}, \cdots, v_{t_s}\}$ denote the neighborhood set of v_n in $S - S_1$. Without loss of generality, we assume that $v_{t_1} = v_1$, $v_{t_2} = v_2$ and $3 \leq t_3 < \cdots < t_s \leq n-3$. If $s = 2$, then let $N_{S-S_1}(v_n) = \emptyset$, otherwise, $|N_{S-S_1}(v_n)| = s - 2$.

Case 2.2.1 $s = 2$. In a similar way as case 1.2.2, it follows that $n = 5, 6$. So G is isomorphic to one graph of $\xi_2 \cup \{G_2\}$.

Case 2.2.2 $s = 3$. In a similar way as case 1.2.3, it follows that $n = 6$ and $d_F(v_n) = 0$. So, G is isomorphic to one graph of ξ_3.

Case 2.2.3 $s \geq 4$. In a similar way as case 1.2.4, it follows that G is isomorphic to one graph of ξ_4.

4 Relationships between tree domination number and other domination parameters

In this section, we study the relationships between the tree domination number and the connected and acyclic domination numbers. We know that every connected graph has both a connected dominating set and an acyclic dominating set, but not necessarily a tree dominating set. But if a graph G has a tree dominating set, then clearly $\gamma_a(G) \leq \gamma_{tr}(G)$ and $\gamma_c(G) \leq \gamma_{tr}(G)$.

Theorem 10. *Every connected graph G is a spanning subgraph of a connected graph H such that $\gamma_a(H) \leq \gamma_{tr}(H) \leq \gamma_a(G)$.*

Proof. Let S be a minimum cardinality acyclic dominating set having a minimum number of connected components among all γ_a-sets of G. Let $\omega(\langle S \rangle)$ denote the connected components number of $\langle S \rangle$. If $\omega(\langle S \rangle) = 1$, then $\langle S \rangle$ is connected. Let $H = G$ and H has a tree dominating set with cardinality $|S|$. Hence,

$$\gamma_a(G) \leq \gamma_{tr}(G) = \gamma_{tr}(H) \leq \gamma_a(G)$$

that is,

$$\gamma_{tr}(H) = \gamma_a(G).$$

If $\omega(\langle S\rangle) > 1$, then let H be obtained from G by joining the components of $\omega(\langle S\rangle)$ with a path. So H has a tree dominating set with cardinality $|S|$, $\gamma_{tr}(H) \leq |S|$. We can obtain the following inequality

$$\gamma_a(H) \leq \gamma_{tr}(H) \leq |S| = \gamma_a(G).$$

Theorem 11. *Every connected graph G contains a spanning connected subgraph H such that $\gamma_{tr}(H) = \gamma_c(G)$.*

Proof. Let S be a minimum connected dominating set having the least cyclic number among all γ_c-sets of G. Let $\omega(\langle S\rangle)$ denote the cyclic number of $\langle S\rangle$. If $\omega(\langle S\rangle) = 0$, then $\langle S\rangle$ is a tree. Let $H = G$ and H has a tree dominating set with cardinality $|S|$. Hence,

$$\gamma_c(G) \leq \gamma_{tr}(G) = \gamma_{tr}(H) \leq \gamma_c(G)$$

that is,

$$\gamma_{tr}(H) = \gamma_c(G).$$

If $\omega(\langle S\rangle) > 0$, then let H be obtained from G by removing an edge from each cycle of $\langle S\rangle$. So, H has a tree dominating set with cardinality $|S|$, $\gamma_{tr}(H) \leq |S|$. We can obtain the following inequality

$$\gamma_c(G) \leq \gamma_c(H) \leq \gamma_{tr}(H) \leq |S| = \gamma_c(G)$$

So,

$$\gamma_{tr}(H) = \gamma_c(G).$$

5 Open questions

Now we give a few open questions

(1) *Characterize the graphs G with $\gamma_{tr}(G) = 0$.*

(2) *Characterize the regular graphs G having $\gamma_{tr}(G) = 0$.*

(3) *Does there exist a family of 2-connected graphs F, for which a polynomial algorithm exists for finding a tree dominating set (of any size) in any graph in F, if it exists?*

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