Total Coloring of Series-Parallel Graphs

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Abstract

It is proved that the total chromatic number of any series-parallel graph of degree at least 3 is $\Delta(G) + 1$.

Let $G = (V, E)$ be a graph. A k-vertex is a vertex of degree k. A k-cycle is a cycle of length k. $N_G(v)$ denotes the set of vertices adjacent to v. A *total k-coloring* of G is a mapping $\psi : E(G) \bigcup V(G) \to \{1, 2, ..., k\}$ such that no two incident or adjacent elements receive the same color. The *total chromatic number* of G, $\chi_T(G)$, is the smallest integer k such that G has a total k-coloring. Behzad and Vizing posed independently the famous total coloring conjecture (**TCC**): For any graph G, $\Delta(G)+1 \leq \chi_T(G) \leq \Delta(G)+2$. Clearly, the lower bound is trivial. The upper bound has been unsolved completely, but a lot of beautiful results have been obtained (cf. [2]). Zhang *etc* [3] (also see Chapter 7 in [2]) determined completely the total chromatic number of all outerplanar graphs. In the paper, we generalize the result to series-parallel graphs and our proof is very simple. A graph is a *series-parallel* graph if it contains no subgraphs homeomorphic to K_4, the complete graph of order 4. It is well-known [1] that any series-parallel graph has a vertex of degree at most 2.

Lemma 1. *Let G be a 2-connected series-parallel graph of order at least 4. Then*

(1) *G has two adjacent 2-vertices u and v, or*

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(2) *G has a 3-cycle $uwxu$ such that $d(u) = 2$ and $d(w) = 3$, or*

(3) *G has a 4-cycle $uxvyu$ such that $d(u) = d(v) = 2$, or*

(4) *G has a 4-vertex w, $N(w) = \{u, v, x, y\}$, such that $d(u) = d(v) = 2$, $N(u) = \{w, x\}$ and $N(v) = \{w, y\}$.*

Proof. The proof is by contradiction. Let G be a counterexample such that its order is as small as possible and its edge number is as many as possible. Let z be a 2-vertex of G and $N(z) = \{z_1, z_2\}$. It follows from (1) that $\min\{d(z_1), d(z_2)\} \geq 3$. Suppose that $z_1z_2 \notin E(G)$. Let $G' = G + z_1z_2$. Then G' is also a series-parallel graph and satisfies the lemma. Let $u, v, w, x, y \in V(G')$ be vertices as in the lemma which G' satisfies. Since u, v are 2-vertices and G does not satisfy the lemma, z_1z_2 must be wx of (2), or wx, wy of (4). But if z_1z_2 is wx of (2), then (1) holds for G. If z_1z_2 is wx or wy of (4), then (3) holds for G. Hence $z_1z_2 \in E(G)$.

Let $G^* = G\backslash z$. Then G^* is also a 2-connected series-parallel graph. If $|V(G^*)| \leq 3$, then the lemma is obvious true for G. So $|V(G^*)| \geq 4$. Thus G^* satisfies the lemma. Let $u, v, w, x, y \in V(G^*)$ be vertices as in the lemma which G^* satisfies. Since G does not satisfy the lemma, $\{z_1, z_2\} \cap \{u, v, w\} \neq \emptyset$. But if $\{z_1, z_2\} \cap \{u, v\} \neq \emptyset$, then (2) holds for G. If $w \in \{z_1, z_2\}$ but $\{z_1, z_2\} \cap \{u, v\} = \emptyset$, then (3) or (4) holds for G, a contradiction. □

Lemma 2. *Let G be a 2-connected series-parallel graph having $\Delta(G) = 3$. Then G has a cycle such that there are just two 3-vertices on it.*

The proof of Lemma 2 is similar to that of Lemma 1, we omit here. Now we prove the main result in the paper.

Theorem 3. *Let G be a connected series-parallel graph. If $G = K_2$ or $C_n(n \not\equiv 0 \,(mod\, 3))$, then $\chi_T(G) = \Delta(G) + 2$; otherwise, $\chi_T(G) = \Delta(G) + 1$.*

Proof. We shall prove the theorem by induction on $|V(G)|$. If $\Delta(G)| = 2$, the result is obvious. Let G be a series-parallel graph having $\Delta(G) \geq 3$, and assume that G is 2-connected.

Suppose that $\Delta(G) \geq 4$. By Lemma 1, there are four possibilities. The first three possibilities are easy to be settled, so assume that G has a 4-vertex w, $N(w) = \{u, v, x, y\}$ such that $d(u) = d(v) = 2$, $N(u) = \{w, x\}$ and $N(v) = \{w, y\}$. Let $G^* = G\backslash\{u, v\}$. Then G^* is also a 2-connected series-parallel graph. So by the induction hypothesis, G^* has a total $(\Delta(G) + 1)$-coloring φ. In the following, we extend φ to a total $(\Delta(G) + 1)$-coloring of G. Since $\max\{d_{G^*}(x), d_{G^*}(y)\} < \Delta(G)$, it is easy to color ux and vy. If $\varphi(xu) = \varphi(vy)$, then interchange colors of wy and vy, and recolor w. So assume that $\varphi(xu) \neq \varphi(vy)$. If $\varphi(ux) \in \{\varphi(w), \varphi(wy)\}$, then let $\varphi(wv) \in \{1, 2, 3, 4, 5\}\backslash\{\varphi(w), \varphi(wx), \varphi(wy), \varphi(vy)\}$; otherwise let $\varphi(wv) = \varphi(xu)$. Thus $\varphi(ux) \in \{\varphi(w), \varphi(wx), \varphi(wy)\}$. Now it is easy to color uw, u and v.

Suppose that $\Delta(G) = 3$. We shall give a total 4-coloring σ of G(called *neat*) such that for any two adjacent 2-vertices w_1 and w_2, $N(w_1) = \{w_1', w_2\}$ and $N(w_2) = \{w_2', w_1\}$, we have $\sigma(w_1w_1') \neq \sigma(w_2w_2')$ and $|\{\sigma(w_1), \sigma(w_1w_1'), \sigma(w_2), \sigma(w_2w_2')\}| = 3$. By Lemma 2, G has a cycle $u_1u_2\cdots u_nu_1$ $(n \geq 3)$ such that $d(u_1) = d(u_k) = 3$ for some $k(1 < k \leq n)$. Without loss of generality, assume $k \geq n/2 + 1 > 2$.

Suppose $5 \leq k < n$. Then $G^* = G\backslash\{u_2, u_3, u_4\} + u_1u_5$ is also a 2-connected series-parallel graph of maximum degree 3. So by the induction hypothesis, G^* has a neat total 4-coloring σ. Without loss of generality, assume that $\sigma(u_1u_5) = 1$, $\sigma(u_1) = 2$ and $\sigma(u_5) = 3$. First, let $\sigma(u_1u_2) = \sigma(u_4u_5) = \sigma(u_3) = 1$, $\sigma(u_2) = \sigma(u_4) = 4$. Then if some edge incident with u_5 is colored with color 2, then let $\sigma(u_2u_3) = 2$ and $\sigma(u_3u_4) = 3$, otherwise let $\sigma(u_2u_3) = 3$ and $\sigma(u_3u_4) = 2$. Thus σ is extended to a neat total 4-coloring of G.

Suppose $3 \leq k = n \leq 5$. Let $G^{**} = G\backslash\{u_1, u_2, \cdots, u_{n-1}\}$. Then G^{**} is also 2-connected series-parallel graph. If $\Delta(G^{**}) = 2$, that is, G^{**} is a cycle, then it is easy to give a neat total 4-coloring of G. So assume that $\Delta(G^{**}) = 3$. Thus by the induction hypothesis, G^{**} has a neat total 4-coloring σ. Since $d_{G^{**}}(u_1) = d_{G^{**}}(u_n) = 2$, it is easy to color $u_1u_2, u_{n-1}u_n$. Without loss of generality, assume that $\sigma(u_1u_2) = \sigma(u_n) = 1$, $\sigma(u_1) = 2$ and $\sigma(u_{n-1}u_n) = 3$. If $k = n = 3$, then let $\sigma(u_2) = 4$. If $k = n = 4$, then let $\sigma(u_2) = 3$, $\sigma(u_2u_3) = 2$ and $\sigma(u_3) = 4$. If $k = 5$, then let $\sigma(u_3) = 1$, $\sigma(u_2u_3) = \sigma(u_4) = 2$, $\sigma(u_2) = 3$ and $\sigma(u_3u_4) = 4$. So σ is extended to a neat total 4-coloring of G.

If $5 \leq k < n$ and $3 \leq k = n \leq 5$ do not hold, then $n = k + 1 = 4$, or $n = k + 1 = 5$, or $n = k + 2 = 6$. These can be settled similarly, we omit here. Hence we complete the proof. □

References

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