The Edge-Coloring of Graphs with Small Genus

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Abstract In this note, we prove that a graph is of class one if G can be embedded in a surface with positive characteristic and satisfies one of the following conditions: (i) $\Delta(G) \geq 3$ and $g(G)$ (the girth of G) ≥ 8 (ii) $\Delta(G) \geq 4$ and $g(G) \geq 5$; and (iii) $\Delta(G) \geq 5$ and $g(G) \geq 4$.

An edge-coloring of a graph G is a mapping from $E(G)$ into Z^+ such that incident edges receive distinct values. The chromatic index of $G, \chi'(G)$, is defined to be the smallest positive integer k such that the edge coloring only uses colors in $\{1, 2, 3, \cdots, k\}$. It is easy to see that $\chi'(G) \geq \Delta(G)$ and $\chi'(G) \leq \Delta(G)+1$ for a simple graph was obtained by Vizing [5]. Therefore, for a simple graph (no multiple edges), $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$. The graph G with $\chi'(G) = \Delta(G)$ is said to be of class one, and of class two if $\chi'(G) = \Delta(G) + 1$. A critical graph is a connected graph of class two and $G - e$ is of class one for each edge e of G.

The girth of a graph G, $g(G)$, is defined to be the smallest length of a cycle in G and the girth of an acyclic graph is zero. The surface we consider in this paper are compact, connected 2-manifolds without boundary. All

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embeddings are 2-cell embeddings throughout.

Given an embedded graph G, let $V(G)$, $E(G)$, and $F(G)$ be the vertex set, edge set and face set of G, respectively. A k-vertex is a vertex of degree k and a k-face is a face with k edges. We shall use n_i to denote the number of i-vertices in G and n_Δ to denote the number of vertices with maximum degree of G, $\Delta(G)$.

We define the Euler characteristic $\chi(S)$ of a surface S by $\chi(S_h) = 2-2h$, for the orientable surface S_h, and $\chi(N_k) = 2 - k$, for the non-orientable surface N_k. The following results are well-known.

Theorem 1. *For a 2-cell embedding of a connected graph with p vertices, q edges, and r regions (faces) in a surface S, we have $p - q + r = \chi(S)$.*

Theorem 2. [1, 2, 6, 7] *If G is a critical graph with maximum degree Δ, then*

(i) *for each vertex x, the number of Δ-vertices adjacent to x, $d_\Delta(x) \geq \Delta - k + 1$, provided that $d_k(x) \geq 1$;*

(ii) *every vertex is adjacent to at least two vertices of maximum degree Δ;*

(iii) *the sum of the degrees of any two adjacent vertices is at least $\Delta + 2$;*

(iv) *for each $k, 2 \leq k \leq \Delta - 1$, we have $n_\Delta \geq 2\Sigma_{j=2}^{k} n_j/(j-1)$.*

The following result was obtained by Kronk et, al. and its proof can be found in [1].

Theorem 3. *Let G be a planar graph. Then G is of class one if one of the following conditions holds:*

(i) $\Delta(G) \geq 3$ *and* $g \geq 8$; *(ii)* $\Delta(G) \geq 8$ *and* $g \geq 3$; *(iii)* $\Delta(G) \geq 4$ *and* $g \geq 5$; *(iv)* $\Delta(G) \geq 5$ *and* $g \geq 4$.

The second part of Theorem 3 was improved later.

Theorem 4. [4] *Let G be a graph which can be 2-cell embedded in a projective plane. Then G is of class one provided that* $\Delta(G) \geq 8$.

Theorem 5. [3] *Let G be a graph with* $\chi(G) \geq 0$. *Then G is of class one provided that* $\Delta(G) \geq 8$.

In this note, we mainly improve the other parts in Theorem 3 and prove the following result.

Theorem 6. *Let G be a graph with* $\chi(G) > 0$. *Then G is of class one if one of the following conditions holds:*

(i) $\Delta(G) \geq 3$ *and* $g \geq 8$; *(ii)* $\Delta(G) \geq 4$ *and* $g \geq 5$; *(iii)* $\Delta(G) \geq 5$ *and* $g \geq 4$.

Proof. We shall apply the well-known discharging method to prove (i), (ii), and (iii) is obtained by direct counting.

(i) Let $d(x)$ be the degree of x if x is a vertex in $V(G)$ and the number of edges in the boundary of x if x is a face in $F(G)$. Then, by Theorem 1,

$$\sum_{x \in V(G) \cup F(G)} (2 - \frac{1}{2}d(x)) = 2\chi(G) > 0. \quad (1)$$

For convenience, we shall call $2 - \frac{1}{2}d(x)$ the initial charge of x. Clearly, for vertices, only 2-vertices and 3-vertices have positive initial charges. Now, we rearrange the charges by the following discharging rules:

R1. For each 2-vertex v, send $\frac{1}{2}$ from v to each face which is incident to v.

R2. For each 3-vertex v, send $\frac{1}{6}$ from v to each face which is incident to v.

Let the new charge of x obtained by R1 and R2 be $M(x)$. It is obvious that $\sum_{x \in V(G) \cup F(G)} M(x) = 2\chi(G)$.

Now, it is easy to check that $M(x) \leq 0$ for each vertex x. As to the face x, by Theorem 2, since every vertex is adjacent to at least two major vertices (vertices of maximum degree), there are at most two vertices of degree 2 on the boundary of an 8-face and at most three vertices of degree 2 on the boundary of a 9-face. Therefore, $M(x) \leq (-2) + 2 \times \frac{1}{2} + 6 \times \frac{1}{6}$ and $M(x) \leq -2.5 + 3 \times \frac{1}{2} + 6 \times \frac{1}{6}$, respectively. By a similar argument, $M(x) \leq 0$ for each k-face x, $k \geq 10$. Since $M(x) \leq 0$ for each $x \in V(G) \cup F(G)$, $\sum_{x \in V(G) \cup F(G)} M(x) \leq 0$; this contradicts (1). This completes the proof of part (i).

(ii) Following the same discharging rules, we have $M(x) \leq 0$ for each vertex x. Now, consider the face x. By Theorem 2, the adjacency relation between 2-vertices, 3-vertices and the major vertices are as follows:

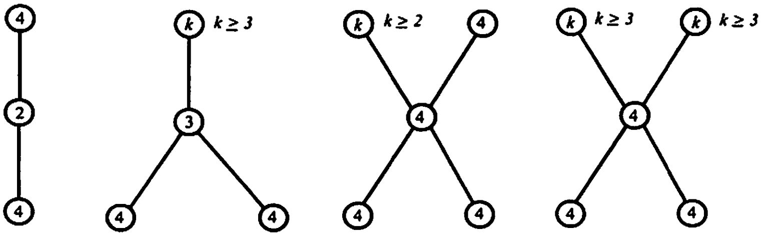

Figure 1: ⓚ denote a vertex with degree k

This implies that if x is a 5-face on the boundary of x, the degree sequences of the vertices are either $\{2,4,4,4,4\}$ or $\{3,3,3,4,4\}$. In either case, $M(x) = 0$. Similarly, on the boundary of a 6-face x we may have degree sequences $\{2,2,4,4,4,4\}$ or $\{3,3,3,3,4,4\}$; both lead to $M(x) \le 0$. Now, as the size of face increases, the initial charge $2 - \frac{1}{2}d(x)$ gets smaller, decreasing by $\frac{1}{2}$, but we can have at worst one more 2-vertices which sends $\frac{1}{2}$ to the face to increase $M(x)$. This shows that $M(x) \le 0$ for each k-face x, $x \ge 5$. Hence, we again have a contradiction and the proof of part(ii) is complete.

Case(iii) Since $g \ge 4$, the initial charge $2 - \frac{1}{2}d(x)$ for each face x is not greater than zero. For vertices, the total charge is equal to $n_2 + \frac{1}{2}n_3 \cdot n_4 + (-\frac{1}{2})n_5 + (-1)n_6 + \cdots + (2 - \frac{1}{2}\Delta)n_\Delta$, where n_i is the number of i-vertices in G. By Theorem 2 (iv), $n_5 \ge 2(n_2 + \frac{1}{2}n_3)$; hence $\sum_{x\in V(G)} M(x) \le 0$. Combining this with the charge on the faces, we have a contradiction to (1). This concludes the proof.

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References

[1] S. Fiorini, The chromatic index of a simple graph, Ph.D. thesis, The Open University, 1974.

[2] S. Fiorini and R. J. Wilson, Edge-colorings of graphs, Research Notes in Matheatics 16, Pitman Publishing, London, 1977.

[3] H. Hind and Yue Zhao, Edge colorings of graphs embeddable in a surface of low genus, Discrete Math. 190(1998), 107-114.

[4] L .S. Melnikov, The chromatic class and location of a graph on a closed surface, Mat. Zametk 7(1970), 671-681.

[5] V. G. Vizing, On an estimate of the chromatic class of a p-graph, Diskret. Analiz 3(1964), 25-30.

[6] V. G. Vizing, Critical graphs with given chromatic class, Diskret. Analiz 5(1965), 9-17.

[7] V. G. Vizing, The chromatic class of a multigraph, Kibernetika 3(1965), 29-39.