The Domatic Number of Regular Graphs

Peter Dankelmann [*]
University of Natal
Durban, South Africa

Neil Calkin
Clemson University,
Clemson, SC, USA

April 29, 2003

Abstract

The domatic number of a graph G is the maximum number of dominating sets into which the vertex set of G can be partitioned.

We show that the domatic number of a random r-regular graph is almost surely at most r, and that for 3-regular random graphs, the domatic number is almost surely equal to 3.

We also give a lower bound on the domatic number of a graph in terms of order, minimum degree and maximum degree. As a corollary, we obtain the result that the domatic number of an r-regular graph is at least $(r+1)/(3\ln(r+1))$.

1 Introduction

A dominating set of a graph G is a subset S of the vertex set $V(G)$, such that every vertex of G is either in S or has a neighbour in S. It is well known that the complement of a dominating set of minimum cardinality of a graph G without isolated vertices is also a dominating set. Hence one can partition the vertex set of G into at least two disjoint dominating sets. The maximum number of dominating sets into which the vertex set of a graph G can be partitioned is called the *domatic number* of G, and denoted by $dom(G)$. This graph invariant was introduced by Cockayne and Hedetniemi [3]. The word domatic, an amalgamation of the words 'domination' and 'chromatic', refers to an analogy between the chromatic number (partitioning of the vertex set into independent sets) and the domatic number (partitioning into dominating sets). For a survey of results on the domatic number of graphs we refer the reader to [8].

---

[*] Supported by the South African National Research Foundation

It was first observed by Cockayne and Hedetniemi [3] that for every graph without isolated vertices $2 \leq dom(G) \leq \delta + 1$, where δ is the minimum degree of G. The upper bound is attained for interval graphs [5], for example.

Intuitively, it seems reasonable to expect that a graph with large minimum degree will have a large domatic number. Zelinka [7] showed that this is not necessarily the case. He gave examples for graphs of arbitrarily large minimum degree with domatic number 2. In this paper we study the domatic number of regular graphs. We focus on two aspects of the domatic number of regular graphs: the domatic number of random regular graphs and bounds on the domatic number of regular graphs in terms of degree only.

In the first part of the paper we show that the domatic number of a random 3-regular graph is almost surely equal to 3, and we prove that the upper bound $r + 1$ on the domatic number of an r-regular graph is almost never attained.

In the second part of the paper we prove the somewhat surprising fact that, for regular graphs, a large minimum degree does guarantee a large domatic number. More precisely, we show that the domatic number of every r-regular graph is at least $\frac{r+1}{3\ln(r+1)}$.

The notation we use is as follows. If G is a graph we denote its vertex set by $V(G)$ and its edge set by $E(G)$, respectively. For the set of vertices adjacent to a vertex v of G, the *neighbourhood* of v in G, we write $N_G(v)$ and for the set $N_G(v) \cup \{v\}$, the *closed neighbourhood* of v in G, we write $N_G[v]$. If the graph is understood we drop the subscript G. The order, minimum degree, and maximum degree of G are denoted by n, δ, and Δ, respectively. If $C = v_1, v_2, \ldots, v_n, v_1$ is a cycle and v_i, v_k are distinct vertices of C, then the *segment* $[v_i, v_k]$ of C is defined as the set $\{v_i, v_{i+1}, v_{i+2}, \ldots, v_k\}$, where the subscripts are taken modulo n.

If $f(n)$ and $g(n)$ are real valued functions of an integer variable n, then we write $f(n) = O(g(n))$ (or $f(n) = \Omega(g(n))$) if there exist constants $C > 0$ and n_0 such that $f(n) \leq Cg(n)$ (or $f(n) \geq Cg(n)$) for $n \geq n_0$. We also write $f(n) \sim g(n)$ if $lim f(n)/g(n) = 1$.

2 Random r-regular graphs

We use the following standard model $\mathcal{G}_{n,r}$ to generate r-regular graphs on n vertices uniformly: to construct a random r-regular graph on the vertex set $\{v_1, v_2, \ldots v_n\}$, take a random matching on the vertex set $\{v_{1,1}, v_{1,2}, \ldots v_{1,r}, v_{2,1}, \ldots, v_{2,r}, \ldots, v_{n,r}\}$ and collapse each set $\{v_{i,1}, v_{i,2}, \ldots v_{i,r}\}$ into a single vertex v_i. If the resulting graph contains any loops or multiple edges, discard it. All r-regular graphs are generated uniformly with this method.

Wormald et al have shown that 3-regular graphs are almost surely Hamiltonian, and that the model $\mathcal{G}_{n,r}$ and $\mathcal{H}_n \oplus \mathcal{G}_{n,r-2}$ are contiguous, meaning roughly that events that are almost sure in one model are almost sure in the other. Thus if an event is almost surely true in a random graph constructed from a random Hamilton cycle plus a random matching, then it is almost surely true in a random 3-regular graph. For more details the reader is referred to [6].

3 A lower bound for the domatic number

In this section we will show that for fixed $r \geq 3$, the domatic number of a random r-regular graph is at least 3.

Definition 1 *Let G be a 3-regular graph obtained from a cycle $C = v_1, v_2, \ldots v_n, v_1$ by adding a perfect matching M. An edge $v_i v_{i+1}$ of C (indices mod n) is a* 3-edge *if v_i and v_{i+1} have matching partners v_j and v_k respectively, such that the cycle segments $[v_j, v_i]$ and $[v_{i+1}, v_k]$ are disjoint and have cardinality* $0 \pmod 3$.

Theorem 1 *Let G be a random 3-regular graph. Then*

$$dom(G) \geq 3 \qquad a.a.$$

The theorem follows from the following lemmas.

Lemma 1 *Let $G = C \cup M$ as above. If C has a 3-edge then $dom(G) \geq 3$.*

Proof: Let $v_i v_{i+1}$ be a 3-edge of C, and let v_j, v_k respectively be their matching partners. Without loss of generality, we may assume that $i = 1$.
Case 1: $n \equiv 0 \pmod 3$
For $l = 1, 2, 3$, let

$$V_l = \{v_m \mid m \equiv l \pmod 3\}.$$

Then each V_l is a dominating set of G and hence $\mathrm{dom}(G) \geq 3$.

Case 2: $n \equiv 1 \pmod 3$.

The cycle $C' = v_1, v_j, v_{j+1} \ldots v_n, v_1$ has length $0 \pmod 3$, since $v_1 v_2$ is a 3-edge. Since $n \equiv 1 \pmod 3$, the cycle $C'' = v_1, v_j, v_{j-1}, v_{j-2}, \ldots, v_1$ has length $n - |C'| + 2 \equiv 0 \pmod 3$. As above, we obtain three disjoint dominating sets of G by selecting every third vertex from each cycle, C' and C''. More precisely, we let

$$\begin{aligned} V_1 &= \{v_1, v_4, v_7, \ldots v_{j-2}\} \cup \{v_1, v_{n-2}, v_{n-5}, \ldots v_{j+2}\} \\ V_2 &= \{v_2, v_5, v_8, \ldots v_{j-1}\} \cup \{v_n, v_{n-3}, v_{n-6}, \ldots v_{j+1}\} \\ V_3 &= \{v_3, v_6, v_9, \ldots v_j\} \cup \{v_{n-1}, v_{n-4}, v_{n-7}, \ldots v_j\} \end{aligned}$$

It is easy to verify that each of the sets V_1, V_2, V_3 is a dominating set of G.

Case 3: $n \equiv 2 \pmod 3$.

Then $k \equiv j \equiv 1 \pmod 3$. As above, we choose three disjoint dominating sets of G by selecting every third vertex of the cycle C for the same set, with the exception of v_1 and v_2. More precisely, let

$$\begin{aligned} V_1 &= \{v_3, v_6, v_9, \ldots, v_{n-2}\} \cup \{v_1\} \\ V_2 &= \{v_2, v_5, v_8, \ldots, v_{n-3}, v_n\} \\ V_3 &= \{v_4, v_7, v_{10}, \ldots, v_{n-1}\}. \end{aligned}$$

It is easy to verify that each of the sets V_1, V_2, V_3 is a dominating set of G. □

Lemma 2 *Let G be a graph obtained from a cycle $C = v_1, v_2, \ldots, v_n, v_1$ of even order by adding a random matching M. Then G has a 3-edge a.a.*

Proof: Define random variables X_i, $i = 1, \ldots, n$ by

$$X_i = \begin{cases} 1 & \text{if } v_i v_{i+1} \in E(C) \text{ is a 3-edge} \\ 0 & \text{otherwise} \end{cases}$$

and let $X = \sum_{i=1}^{n} X_i$. Then each X_i has expectation $E(X_i) = 1/18 + O(1/n)$ and variance $\mathrm{var}(X_i) = E(X_i^2) - E(X_i)^2 = E(X_i) - E(X_i)^2 = 17/324 + O(1/n)$. The covariance of X_i and X_j for $i < j$ equals

$$\begin{aligned} \mathrm{cov}(X_i, X_j) &= E(X_i X_j) - E(X_i)E(X_j) \\ &= \begin{cases} 1/324 - (1/18)^2 + O(1/n) & \text{if } i < j-1, \\ 2/324 - (1/18)^2 + O(1/n) & \text{if } i = j-1 \text{ and } n \equiv 1 \pmod 3, \\ 0 - (1/18)^2 + O(1/n) & \text{if } i = j-1 \text{ and } n \equiv 0, 2 \pmod 3, \end{cases} \\ &= \begin{cases} O(1/n) & \text{if } i < j-1, \\ O(1) & \text{if } i = j-1. \end{cases} \end{aligned}$$

Note that $X_i X_{i+1} = 1$ implies $n \equiv 1 \pmod 3$. To see this let v_k be the matching partner of v_{i+1}. If $X_i X_{i+1} = 1$ then $v_i v_{i+1}$ and $v_{i+1} v_{i+2}$ are 3-edges and thus $n + 2 \equiv |[v_{i+1}, v_k]| + |[v_k, v_{i+1}]| \equiv 0 + 0 \equiv 0 \pmod 3$, i.e., $n \equiv 1 \pmod 3$.

Hence the random variable X has expectation

$$E(X) = \sum_{i=1}^{n} E(X_i) = n/18 + O(1) = O(n)$$

and variance

$$\begin{aligned}
\mathrm{var}(X) &= \sum_{i=1}^{n}\mathrm{var}(X_i)+2\sum_{i<j-1}\mathrm{cov}(X_i,X_j)+2\sum_{i=1}^{n}\mathrm{cov}(X_i,X_{i+1})\\
&= \frac{17}{324}n+2\sum_{i<j-1}O(1/n)+2\sum_{i=1}^{n}O(1)\\
&= O(n).
\end{aligned}$$

By Chebyschev's inequality, we have

$$\mathrm{prob}(X=0)\le\frac{\mathrm{var}(X)}{E(X)^2}=\frac{O(n)}{(O(n))^2}=O(1/n).$$

Hence $X>0$ a.a., i.e., G has a 3-edge. □

Lemma 3 *If G is a 3-regular random graph, then a.a. G consists of a hamilton cycle plus a random matching.*

4 An upper bound for the domatic number

Theorem 2 *Let G be a random r-regular graph. Then dom(G) $\le r$ a.a.*

Proof: We first give an upper bound on the number of r-regular, $(r+1)$-domatic graphs. If G is an r-regular graph with domatic partition $V_1, V_2, \ldots V_{r+1}$, then each vertex is either in a given V_i, or has a neighbour in V_i. Hence

$$|N[v]\cap V_i|=1 \qquad \text{for all } v\in V(G) \text{ and } i\in\{1,2,\ldots,r+1\}$$

implying that for $i\neq j$,

$$E_{ij}:=\{uv\in E(G)\mid u\in V_i,\ v\in V_j\} \text{ is a perfect } V_i-V_j \text{ matching} \quad (1)$$

and thus

$$|V_1|=|V_2|=\ldots=|V_{r+1}|. \quad (2)$$

From the above it follows that every r-regular, $(r+1)$-domatic graph on the vertex set V can be obtained by first partitioning V into $r+1$ sets, all of equal cardinality, and then adding perfect matchings between all pairs of partition sets. If n is a multiple of $r+1$, the former can be done in

$$\binom{n}{n/(r+1),n/(r+1),\ldots,n/(r+1)}\frac{1}{(r+1)!}$$

ways, since the sets are not distinguishable; the latter can be done in

$$\left(\left(\frac{n}{r+1}\right)!\right)^{\binom{r+1}{2}}$$

ways, since there are $\binom{r+1}{2}$ different pairs of sets V_i, V_j, and between each pair a matching can be added in $\left(\frac{n}{r+1}\right)!$ ways.

Hence an upper bound on the number of labelled, r-regular, $(r+1)$-domatic graphs of order n is

$$\binom{n}{n/(r+1), n/(r+1), \ldots, n/(r+1)} \cdot \frac{1}{(r+1)!} \cdot \left(\left(\frac{n}{r+1}\right)!\right)^{\binom{r+1}{2}}$$

$$= \frac{n!}{\left(\left(\frac{n}{r+1}\right)!\right)^{r+1}} \cdot \frac{1}{(r+1)!} \cdot \left(\left(\frac{n}{r+1}\right)!\right)^{r(r-1)/2}$$

$$= \frac{n!}{(r+1)!} \cdot \left(\left(\frac{n}{r+1}\right)!\right)^{(r+1)(r-2)/2}$$

and hence, by Stirling's formula ($n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}(1 + \frac{1}{12n} + O(\frac{1}{n^2}))$ the upper bound is, for large n and constant r,

$$\left(\frac{n}{e}\right)^n \cdot \frac{1}{(r+1)!} \cdot \left(\frac{n}{e(r+1)}\right)^{\frac{n}{r+1} \cdot \frac{1}{2} \cdot (r+1)(r-2)} \sqrt{2\pi n}$$

$$\cdot \left(1 + \frac{1}{12n} + O\left(\frac{1}{n^2}\right)\right) \left(\sqrt{\frac{2\pi n}{r+1}} \left(1 + \frac{r+1}{12n} + O\left(\frac{1}{n^2}\right)\right)\right)^{\frac{1}{2}(r+1)(r-2)}$$

$$= \left(\frac{n}{e}\right)^{\frac{1}{2}nr} \cdot \frac{1}{(r+1)!(r+1)^{\frac{1}{2}n(r-2)}} . O\left(n^{\frac{1}{4}r(r-1)}\right).$$

Denote this last expression by DOM(r, n).

The total number of r-regular graphs, as given in [2] is asymptotic to

$$e^{-(r^2-1)/4} \frac{(rn)!}{(rn/2)! 2^{rn/2} (r!)^n} \sim \sqrt{2} e^{-(r^2-1)/4} \left(\frac{r^{r/2}}{e^{r/2} r!}\right)^n n^{rn/2}.$$

Denote this last expression by TOTAL(r, n). Then the proportion of r-regular graphs that are $(r+1)$-domatic, $DOM(r, n)/TOTAL(r, n)$, is at most

The fraction in brackets is less than 1, so the limit $DOM(r, n)/TOTAL(r, n)$ tends to 0, as desired. □

5 Domatic number and minimum degree

Zelinka [7] gave the following lower bound on the domatic number,

$$dom(G) \geq \left\lfloor \frac{n}{n - \delta(G)} \right\rfloor.$$

This bound is clearly not best possible. In order to guarantee domatic number at least 3, Zelinka's bound requires roughly $\delta(G) \geq 2n/3$.

Zelinka [7] also exhibited graphs with domatic number equal to 2 and arbitrarily large minimum degree, thus demonstrating that there is no non-trivial lower bound on the domatic number in terms of minimum degree only. His graphs have, however, very large maximum degree,

$$\Delta(G) > \binom{3\delta(G) - 1}{\delta(G) - 1},$$

i.e., the maximum degree is exponential in the minimum degree. If the maximum degree of the graph is not too big relative to the minimum degree, then the following, much stronger, bound holds.

Theorem 3 *Let G be a graph of order n with minimum degree δ and maximum degree Δ, and let k be a nonnegative integer. If*

$$e(\Delta^2 + 1)k(1 - \frac{1}{k})^{\delta+1} < 1,$$

then $dom(G)$ is at least k.

Proof: Let $f : V(G) \to \{1, 2, \ldots, k\}$ be a random colouring of the vertices of G. For $1 \leq i \leq k$ let $V_i = \{v \in V(G) | f(v) = i\}$. The partition $(V_1, V_2, \ldots, V_k)$ is a domatic partition of G if

$$f(N[v]) = \{1, 2, \ldots, k\}, \quad \text{for all } v \in V(G). \tag{3}$$

It suffices to show that the probability for a partition to satisfy

(3) is positive. For a vertex v let A_v be the event that $f(N[v])$ does not equal $\{1, 2, \ldots, k\}$. Then

$$prob(A_v) \leq \sum_{i=1}^{k} prob(i \notin f(N[v])) = k(1 - \frac{1}{k})^{deg v+1} \leq k(1 - \frac{1}{k})^{\delta+1}.$$

If vertices u and v of G have no neighbours in common, then the events A_u and A_v are independent. Thus the event A_v is dependent from at most Δ^2 other events. By the hypothesis we have

$$e(\Delta^2 + 1)k(1 - \frac{1}{k})^{\delta+1} < 1.$$

Therefore, by the Lovász Local Lemma, the probability that none of the events A_v occurs is positive. Hence there exists a colouring $f : V(G) \to \{1, 2, \ldots, k\}$ satisfying (3), which implies $dom(G) \geq k$. □

For the special case of a regular graph, we obtain a significant improvement of Zelinka's bound.

Corollary 1 *Let G be an r-regular graph. Then*

$$dom(G) \geq \frac{r+1}{3\ln(r+1)}.$$

Proof: With $\Delta = \delta = r$ and $k = \frac{r+1}{3\ln(r+1)}$ we have

$$\begin{aligned} e(\Delta^2+1)k(1-\frac{1}{k})^{\delta+1} &= e(r^2+1)k(1-\frac{1}{k})^{r+1} \\ &\leq e(r^2+1)\frac{r+1}{3\ln(r+1)}\exp(-(r+1)\frac{3\ln(r+1)}{r+1}) \\ &= \frac{e(r^2+1)(r+1)}{3(r+1)^3\ln(r+1)} \\ &< 1. \end{aligned}$$

By Theorem 3, $dom(G) \geq k$. □

A question that arises naturally is whether the bound in Corollary 1 is best possible. For a positive integer r let $f(r)$ be the maximum domatic number of all r-regular graphs. By Corollary 1 we have $f(r) \leq \frac{r+1}{3\ln(r+1)}$.

On the other hand, Alon [1] proved that there exist r-regular graphs of order n with domination number $(1+o(1))\frac{r+1}{n\ln(r+1)}$.

The domatic number of those graphs is at most $n/\gamma = (1+o(1))\frac{r+1}{\ln(r+1)}$. This proves

$$f(r) = \Omega\Big(\frac{r+1}{\ln(r+1)}\Big),$$

and the order of magnitude of the bound in Corollary 1 is best possible.

Note added in proof: A bound slightly stronger than Theorem 3 was independently proved by Feige, Halldórsson and Kortsarz [4].

References

[1] N. Alon, Transversal numbers of uniform hypergraphs. Graphs and Combinatorics **6** (1990), 1-4.

[2] E.A. Bender and E.R. Canfield, The asymptotic number of labeled graphs with given degree sequences. J. Combinatorial Theory Ser. A **24** (1978), 296-307.

[3] E.J. Cockayne and S. Hedetniemi, Towards a theory of domination in graphs. Networks **7** (1977), 247-261.

[4] U. Feige, M.M. Halldórsson and G. Kortsarz [4], Approximating the domatic number. Proceedings of the thirty-second Annual ACM Symposium on Theory of Computing, Portland, Oregon, 2000. ACM Press (New York), 134-143.

[5] Tung-Lin Lu, Pei-Hsin Ho, and Gerard J Chang, The domatic number problem in interval graphs. SIAM J. Discrete Math. **4** (1990), 531-536.

[6] N.C. Wormald, Models of Random Regular Graphs. In: Surveys in Combinatorics 1999 (J.D. Lamb and D.A. Preece, eds., 239-298), London Mathematical Society Lecture Notes Series 267, Cambridge University Press, Cambridge (1999).

[7] B. Zelinka, Domatic number and degrees of vertices of a graph. Math. Slovaca **33** (1983), 145-147.

[8] B. Zelinka, Domatic numbers of graphs and their variants: a survey. In: Domination in graphs, advanced topics (T.W. Haynes, S.T. Hedetniemi, and P. Slater, eds., 351-377). Marcel Dekker, New York, (1998).