Plane-Line Collinearity Graph of the M_{24} Minimal Parabolic Geometry

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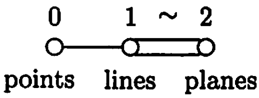
The purpose of this short note is to record the collapsed adjacency graph for the plane-line collinearity graph of the minimal parabolic geometry associated with M_{24}, the Mathieu group of degree 24. This geometry, Γ, owes much of its notoriety to it also appearing as a residue in geometries for the sporadic groups Co_1, Conway's largest simple group, and $\mathbb{M}$, the Monster simple group ([4],[5]). For a sample of related results see [2],[3],[6],[7],[11],[10]. The associated diagram of Γ is



where the numbers on the top give the type of the geometry objects and underneath their 'usual' names. Letting Γ_i denote the objects in Γ of type i we have $\Gamma = \Gamma_0 \dot{\cup} \Gamma_1 \dot{\cup} \Gamma_2$. Put $G = M_{24}$. The plane-line collinearity graph, $\mathcal{G}$, is the graph with vertex set Γ_2 where, for $x, y \in \Gamma_2$, x and y are adjacent if there exists $l \in \Gamma_1$ with which both x and y are incident. The point-line collinearity graph, $\mathcal{G}_0$, of Γ frequently steals the limelight from $\mathcal{G}$. The structure of $\mathcal{G}_0$ is fairly transparent – $\mathcal{G}_0$ has diameter 2 and G, in its action on the vertices of $\mathcal{G}_0$, has permutation rank 4 while, as we shall see below, the structure of $\mathcal{G}$ is far more intricate. We remark that $\mathcal{G}$ is encoded in the Co_1-maximal parabolic geometry, and hence in the maximal parabolic geometry for $\mathbb{M}$ (see [8]).

Though $\mathcal{G}$ has 11,385 vertices, and valency 14, a relatively small number of hand calculations, aided by Curtis's (amazing) MOG [1], will reveal many features of $\mathcal{G}$. To set the scene, let Ω be a 24-element set with Steiner system $S(24, 8, 5)$ as depicted in [1], and preserved by G. The blocks of this system are called octads. So we have, following [1],

$\Omega =$

O_1	O_2	O_3

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where O_1, O_2 and O_3 are the heavy bricks of the MOG. By a tetrad we understand a 4-element subset of Ω. A sextet is a partition of Ω into six tetrads with the property that the union of any two is an octad while a trio is a partition of Ω into three 8-element subsets each of which is an octad.

From [5], where Γ was first introduced, we extract the following concrete description of Γ:- Γ_2 consists of all involutions of G of cycle type $1^8 2^8$ (on Ω), Γ_1 all fours groups of G the involutions of which have cycle type $1^8 2^8$ and, pairwise, their fixed point sets have empty intersection and Γ_0 all sextets of Ω. A line l and a plane x are incident if the involution x is an element of the fours group l – for the other incidences see [5]. From now on we shall view Γ as just described. For $x \in \Gamma_2$, the fixed points of x on Ω is an octad which we shall denote by O_x.

Table 1 summarizes information about the G_a-orbits of Γ_2 where a is a fixed plane of Γ. The first column gives the orbit a name, $\Delta_i^j(a)$ indicating that the orbit is a subset of $\Delta_i(a)$, the set of planes distance i from a in $\mathcal{G}$. The second, third and fourth columns give, respectively, the size of the orbit, a representative involution x for the orbit and the size of the intersection of O_1 and O_x. The final column gives the cycle type of the permutation ax on Ω.

A character calculation (aided, perhaps, by GAP [9]) reveals that G has permutation rank 14 on Γ_2. The list of representatives are distinguished by $|O_1 \cap O_x|$ and the cycle type of ax with the exception of $\Delta_4^2(a)$ and $\Delta_4^3(a)$, and $\Delta_5^1(a)$ and $\Delta_5^2(a)$. However, noting that the representative for $\Delta_4^2(a)$ has two 2-cycles contained in O_1 and the representative for $\Delta_5^1(a)$ has four 2-cycles in O_1 while the representatives for $\Delta_4^3(a)$ and $\Delta_5^2(a)$ have no 2-cycles contained in O_1, we conclude the our list of G_a-orbit representatives is complete. We return to verifying the sizes of the G_a-orbits later.

Let $x \in \Gamma_2$. We now give a method, using the MOG, for calculating $\Delta_1(x)$, the neighbours of x in $\mathcal{G}$. Let $D_1, \ldots, D_8$ be the orbits of $\langle x \rangle$ on Ω of size 2 (so $\Omega = O_x \cup D_1 \cup \ldots \cup D_8$). First choose any D_i with $i \neq 1$ and then, using the MOG, find the (unique) sextet S_i for which $D_1 \cup D_i$ is a tetrad. Next select D_j ($j \neq 1$) so as $D_1 \cup D_j$ is not a tetrad of S_i. Let S_j be the unique sextet which has $D_1 \cup D_j$ as a tetrad. Finally, select D_k ($k \neq 1$) so as $D_1 \cup D_k$ is not a tetrad of either S_i or S_j, and let S_k be the sextet containing this tetrad. By pairing off tetrads in $\Omega \backslash O_x$ we obtain for each of S_i, S_j, S_k three trios which have O_x as one of its octads. One trio will, in fact, arise from each of the three sextets and so we obtain seven trios in total. (Observe that we only need to find the four tetrads for S_i, S_j, S_k in $\Omega \backslash O_x$.) Let T be one of the seven trios, and let O_T be an octad of T with $O_T \neq O_x$. Choose any D_l with $D_l \subseteq \Omega \backslash (O_T \cup O_x)$ and, using the MOG yet again (page 34, [1]), determine the (unique) involution y which fixes O_T point-wise and interchanges the two elements in D_l. Then x, y, xy are the three involutions (planes) incident with the line corresponding to

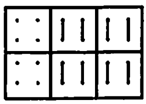
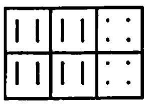
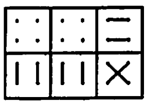
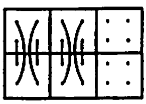
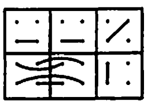
T. Repeating this procedure for the other trios yields the 14 involutions (planes) in $\Delta_1(x)$. To check the validity of the above method it suffices to see that it works for a and then observe that the action of G preserves all relevant parts of the recipe. So, for each representative x listed in Table 1 we may find $\Delta_1(x)$ and hence the number of edges from x to the various G_a-orbits of Γ_2. (Note that, often, we do not need to compute all of y – the involution with $O_y = O_T$ – in order to determine to which G_a-orbit y belongs.) This, as we may now calculate the sizes of $\Delta_j^i(a)$, gives all the data in Figure 1.

Remarks

(*i*) G is transitive on paths in $\mathcal{G}$ of length 2.

(*ii*) $\Delta_6(a)$ is a coclique in $\mathcal{G}$.

(*iii*) Inspecting Table 1 we see two instances of a lack of 'symmetry' in $\mathcal{G}$, namely if $x \in \Delta_4^2(a)$, then $a \in \Delta_4^3(x)$ and if $x \in \Delta_5^1(a)$, then $a \in \Delta_5^2(x)$.

Orbit	Size	Representative x	$\lvert O_1 \cap O_x \rvert$	Cycle type of ax
$\{a\}$	1		8	1^{24}
$\Delta_1(a)$	14		0	$1^8 2^8$
$\Delta_2(a)$	168		4	$1^8 2^8$
$\Delta_3^1(a)$	84		0	2^{12}
$\Delta_3^2(a)$	1344		2	$1^4 2^2 4^4$

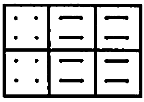
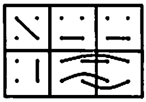
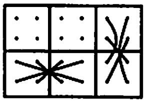
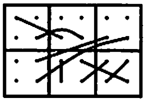
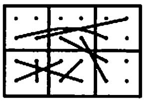
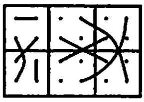
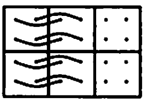
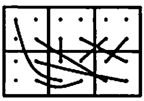
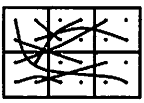
Orbit	Size	Representative x	$\lvert O_1 \cap O_x \rvert$	Cycle type of ax
$\Delta_4^1(a)$	14		8	$1^8 2^8$
$\Delta_4^2(a)$	672		4	$1^4 2^2 4^4$
$\Delta_4^3(a)$	672		4	$1^4 2^4 4^4$
$\Delta_4^4(a)$	896		4	$1^6 3^6$
$\Delta_4^5(a)$	5376		2	$1^2 2^2 3^2 6^2$
$\Delta_5^1(a)$	112		0	$2^4 4^4$
$\Delta_5^2(a)$	112		0	$2^4 4^4$
$\Delta_5^3(a)$	1792		4	$1^4 5^4$
$\Delta_6(a)$	128		0	3^6

Table 1: The G_a orbits of Γ_2

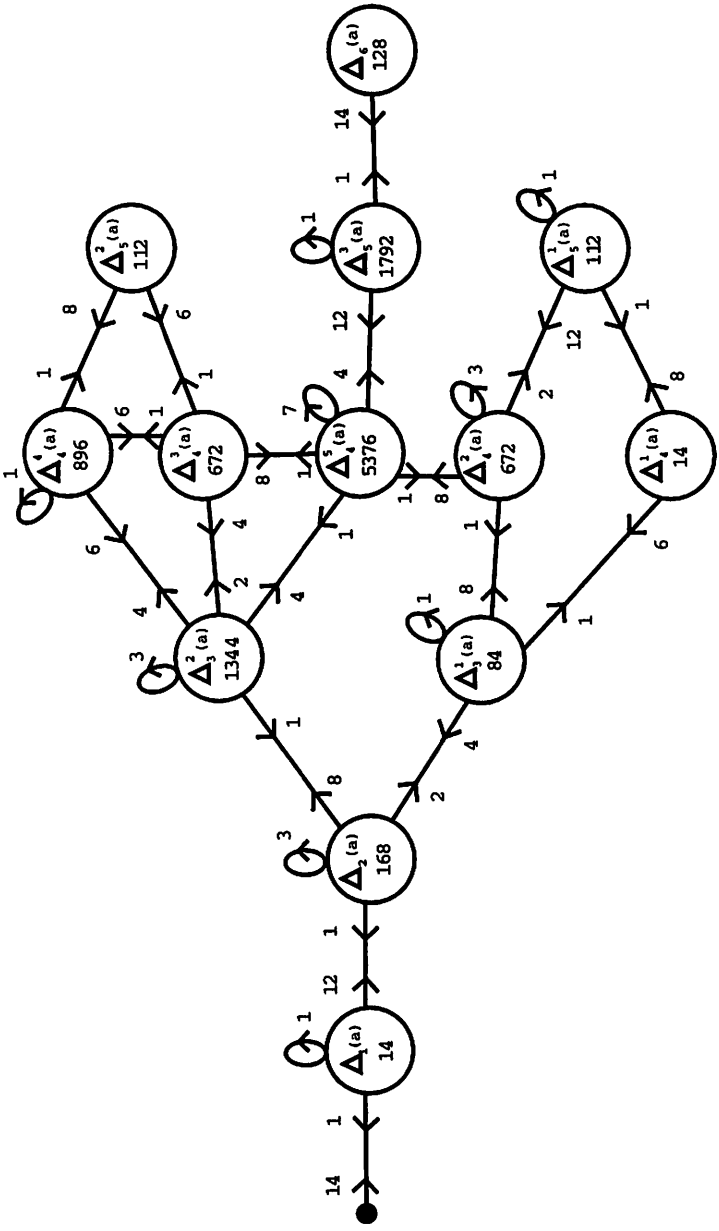


Figure 1: The Plane-Line Collinearity Graph of Γ

References

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