A Counterexample to a Conjecture on Packing Two Copies of a Tree into its Third Power

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Abstract

A graph H of order n is said to be embeddable in a graph G of order n, if G contains a spanning subgraph isomorphic to H. It is well known that any non-star tree T of order n is embeddable in its complement (i.e. in $K_n - E(T)$). In the paper "*Packing two copies of a tree into its fourth power*" by Hamamache Kheddouci, Jean-Francois Saclé, and Mariusz Woźniak, *Discrete Mathematics* **213** (2000), 169-178, it is proved that any non-star tree T is embeddable in $T^4 - E(T)$. They asked whether every non-star tree T is embeddable in $T^3 - E(T)$. In this paper, answering their question negatively, we show that there exist trees T such that T is not embeddable in $T^3 - E(T)$.

1 Introduction

We consider finite graphs without loops or multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. An edge joining a vertex x to a vertex y is denoted by xy or yx. Let $d_G(v)$ be the degree of a vertex v in G. For a graph G and two vertices x, y, denote by $d_G(x,y)$ the length of a shortest $x - y$ path in G. For a positive integer i, we denote by G^i the graph such that $V(G^i) = V(G)$ and $E(G^i) = \{xy \mid x, y \in V(G), x \neq y, d_G(x,y) \leq i\}$. For a vertex x, the set of all vertices adjacent to x in G is denoted by $N_G(x)$. In this paper, a graph S is called *star* if S is a complete bipartite graph $K_{1,n}$

or an isolated vertex, and a tree T is called *non-star tree* if T is neither a complete bipartite graph $K_{1,n}$ nor an isolated vertex. (Thus every non-star tree has at least four vertices.)

Suppose $G_1, \ldots, G_k$ and G are graphs of order n. We say that there is *a packing of* $G_1, \ldots, G_k$ *into* G if G contains pairwise edge-disjoint copies of $G_1, \ldots, G_k$, i.e. if for $i = 1, \ldots, k$, there exist bijections $\alpha_i : V(G_i) \to V(G)$ and injections $\alpha_i^* : E(G_i) \to E(G)$ (let $E^*(G_i) = \{\alpha_i^*(e) | \forall e \in E(G_i)\}$), such that for any i if an edge x_iy_i belongs to $E(G_i)$ then $\alpha(x_i)\alpha(y_i)$ belongs to $E^*(G_i)$, and for all i and j, $1 \leq i < j \leq k$, $E^*(G_i) \cap E^*(G_j) = \emptyset$.

A graph H of order n is said to be *embeddable in a graph* G of order n, if there is a packing of H into G. Then a bijection $\alpha : V(H) \to V(G)$ is called *an embedding in* G.

It is well known that any non-star tree T of order n is embeddable in its complement. Hamamache et al. [1] proved that any non-star tree T is embeddable in $T^4 - E(T)$.

They asked whether every non-star tree T is embeddable in $T^3 - E(T)$. In this paper, answering their question negatively, we show that there exist trees T such that T is not embeddable in $T^3 - E(T)$.

Suppose $P, P_1, P_2, \ldots, P_k$, are pairwise vertex-disjoint paths of order 7, and x, y_i are the centers of P, P_i $(i = 1, 2, \ldots, k)$, respectively. Define T_k to be the tree obtained from these paths by joining x to y_i, for all i, $1 \leq i \leq k$, (see Figure 1).

Theorem 1. *For any integer $k \geq 1$, T_k is not embeddable in $T_k^3 - E(T_k)$.*

2 Proof of Theorem 1

Suppose for some integer $k \geq 1$, T_k is embeddable in $T_k^3 - E(T_k)$. Then there exists the embedding $\alpha : V(T_k) \to V(T_k^3 - E(T_k))$. We denote $\alpha(v)$, $E^*(T_k)$, $T_k^3 - E(T_k)$ by v', E', H, respectively. Moreover, we denote by T_k' a graph isomorphic to T_k induced by E'. Notice that $V(T_k) = V(H) = V(T_k')$. (i.e. α is the permutation on $V(T_k)$.)

Set
$V(P) = \{x, z_1, z_2, z_3, w_1, w_2, w_3\}$, $E(P) = \{z_3z_2, z_2z_1, z_1x, xw_1, w_1w_2, w_2w_3\}$
$V(P_i) = \{y_i, a_i, b_i, c_i, d_i, e_i, f_i\}$, $E(P_i) = \{c_ib_i, b_ia_i, a_iy_i, y_id_i, d_ie_i, e_if_i\}$,
for all i, $1 \leq i \leq k$,
and $U = \{x, y_1, y_2, \ldots, y_k\}$, $U' = \{x', y_1', y_2', \ldots, y_k'\}$
(see Figure 1)

Let T be a tree and let u and v be two distinct vertices in T. We say that $u - v$ path R is *good in* T if R satisfies the following conditions;

(i) either $d_T(u) = 1$ and $d_T(v) \geq 3$, or $d_T(u) \geq 3$ and $d_T(v) = 1$.

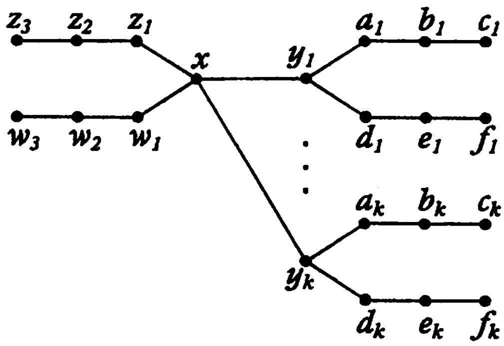

Figure 1: T_k

(ii) $d_T(w) = 2$ for every vertex w in $V(R) - \{u, v\}$.

Remark 1. *If R is a good path in T'_k, then $|E(R)| = 3$.*

Claim 1. $U' \cap (\{z_2, z_3, w_2, w_3\} \cup \{b_i, c_i, e_i, f_i \mid 1 \le i \le k\}) = \emptyset$.

Proof. For $v \in \{z_3, w_3, c_i, f_i\}$, $d_H(v) = 2$, so $U' \cap \{z_3, w_3, c_i, f_i \mid 1 \le i \le k\} = \emptyset$.

If $U' \cap \{b_i, e_i \mid 1 \le i \le k\} \ne \emptyset$, say $b_1 \in U'$, then $N_{T'_k}(b_1) = \{x, y_1, d_1\}$. Hence $\{f_1y_1, f_1d_1\} \not\subseteq E'$, namely $d_{T'_k}(f_1) = 1$. This implies that T'_k has the good $b_1 - f_1$ path R with $|E(R)| \le 2$, contradicting Remark 1.

Thus assume z_2 or $w_2 \in U'$, say $z_2 \in U'$, If $z_2x, z_2w_1 \in E'$, then similarly considering the vertex w_3, we have a contradiction. Hence we have $\{z_2x, z_2w_1\} \not\subseteq E'$. If $z = x'$, then $d_{T'_k}(z_2) = k + 2$, implying that $\{z_2x, z_2w_1\} \subseteq E'$, a contradiction. Thus assume that $z_2 \in U' - \{x'\}$. Since $d_{T'_k}(z_2) = 3$, we have $z_2y_i, z_2y_j \in E'$, for some i and j, $1 \le i < j \le k$. We may assume that $d_{T'_k}(y_i) = 2$. Since every vertex in $U' - \{x'\}$ has two neighbors of degree two in T'_k. We consider two cases.

Case 1. $b_ix \notin E'$ or $e_ix \notin E'$, say $b_ix \notin E'$.

If $b_id_i \notin E'$, then $b_iy_i \in E'$ and $d_{T'_k}(b_i) = 1$. Hence the path $z_2y_ib_i$ in T'_k is a good path with two edges, contradicting Remark 1. If $f_id_i \notin E'$, then $f_iy_i \in E'$ and $d_{T'_k}(f_i) = 1$. Similarly the path $z_2y_if_i$ in T'_k is a good path with two edges, contradicting Remark 1. Thus $b_id_i, f_id_i \in E'$. Since $b_ix \notin E'$, we see that either $b_iy_i \in E'$ or $f_iy_i \in E'$ holds. If b_iy_i is in E', then $d_{T'_k}(f_i) = 1$, $d_{T'_k}(y_i) = 2$ and $d_{T'_k}(b_i) = 2$. Clearly either f_id_i is a good path in T'_k or $f_id_ib_iy_iz_2$ is a good path in T'_k. In either case, we have a contradiction to Remark 1. Thus assume that f_iy_i is in E'. Clearly we have $d_{T'_k}(y_i) = 2$, $d_{T'_k}(f_i) = 2$ and $d_{T'_k}(b_i) = 1$. Similarly either b_id_i is a good path in T'_k or $b_i d_i f_i y_i z_2$ is a good path in T'_k. In either case, we have a contradiction to Remark 1.

Case 2. $b_i x$ and $e_i x \in E'$.

If $b_j x \in E'$ or $e_j x \in E'$, say $b_j x \in E'$, then since $U' \cap \{b_i, e_i, b_j\} = \emptyset$, x is adjacent to at least three vertices of degree at most 2 in T'_k. However, every vertex is adjacent to at most 2 vertices of degree at most 2 in T_k, a contradiction. Thus assume that $b_j x, e_j x \notin E'$.

Assume first that $y_j \neq x'$. Now if $b_j d_j \notin E'$, then $b_j y_j \in E'$ and $d_{T'_k}(b_j) = 1$. Hence the path $z_2 y_j b_j$ in T'_k is a good path with two edges, contradicting Remark 1. If $f_j d_j \notin E'$, then $f_j y_j \in E'$ and $d_{T'_k}(f_j) = 1$. Similarly the path $z_2 y_j f_j$ in T'_k is a good path with two edges, contradicting Remark 1. Thus $b_j d_j, f_j d_j \in E'$. Since $b_j x \notin E'$, we see that either $b_j y_j \in E'$ or $f_j y_j \in E'$ holds. If $b_j y_j$ is in E', then $d_{T'_k}(f_j) = 1$, $d_{T'_k}(y_j) = 2$ and $d_{T'_k}(b_j) = 2$. Clearly either $f_j d_j$ is a good path in T'_k or $f_j d_j b_j y_j z_2$ is a good path in T'_k. In either case, we have a contradiction to Remark 1. Thus assume that $f_j y_j$ is in E'. Clearly we have $d_{T'_k}(y_j) = 2$, $d_{T'_k}(f_j) = 2$ and $d_{T'_k}(b_j) = 1$. Similarly either $b_j d_j$ is a good path in T'_k or $b_j d_j f_j y_j z_2$ is a good path in T'_k. In either case, we have a contradiction to Remark 1.

Next, assume that $y_j = x'$. Now if $x \in U'$, then since $x \neq y_j = x'$, $xx' \in E'$, implying that $xy_j \in E$ and $xy_j = xx' \in E'$, a contradiction. Thus assume that $d_{T'_k}(x) = 2$. If $f_i d_i \notin E'$ or $c_i a_i \notin E'$, say $f_i d_i \notin E'$, then $f_i y_i \in E'$ and $d_{T'_k}(f_i) = 1$. The path $z_2 y_i f_i$ in T'_k is a good path with two edges, contradicting Remark 1. Thus $f_i d_i, c_i a_i \in E'$.

If $d_i \in U'$ or $a_i \in U'$, say $d_i \in U'$, then $d_i x' = d_i y_j \in E'$. Thus $f_i y_i \notin E'$, otherwise $f_i y_i z_2 y_j d_i f_i$ is a cycle in T'_k. Clearly $f_i d_i$ is a good path in T'_k, a contradiction to Remark 1. Thus $d_{T'_k}(d_i) \leq 2, d_{T'_k}(a_i) \leq 2$.

If $b_i d_i, e_i a_i \notin E'$, then $b_i y_i \in E'$ or $e_i y_i \in E'$, say $b_i y_i \in E'$. Clearly we have $d_{T'_k}(y_i) = 2$, $d_{T'_k}(b_i) = 2$ and $d_{T'_k}(e_i) = 1$, implying that $e_i x b_i y_i z_2$ is a good path in T'_k, a contradiction to Remark 1. Thus assume that $b_i d_i \in E'$ or $e_i a_i \in E'$, say $b_i d_i \in E'$.

If $f_i y_i \in E'$, then we have $d_{T'_k}(y_i) = 2$, $d_{T'_k}(f_i) = 2$, $d_{T'_k}(d_i) = 2$, $d_{T'_k}(b_i) = 2$, $d_{T'_k}(x) = 2$, and $d_{T'_k}(c_i) = 1$. Clearly either $z_2 y_i f_i d_i b_i x e_i$ is a good path in T'_k, or $z_2 y_i f_i d_i b_i x e_i a_i c_i$ is a good path in T'_k. In either case, we have a contradiction to Remark 1. Thus assume that $f_i y_i \in E'$. Clearly we have $d_{T'_k}(y_i) = 2$, $d_{T'_k}(f_i) = 1$, $d_{T'_k}(d_i) = 2$, $d_{T'_k}(b_i) = 2$, $d_{T'_k}(x) = 2$, and $d_{T'_k}(c_i) = 1$. Similarly either $f_i d_i b_i x e_i y_i z_2$ is a good path in T'_k or $f_i d_i b_i x e_i a_i c_i y_i z_2$ is a good path in T'_k. In either case, we have a contradiction to Remark 1. ∎

Claim 2.
(1) $\{z_1, x\} \not\subseteq U'$ *and* $\{w_1, x\} \not\subseteq U'$.
(2) $\{a_i, y_i\} \not\subseteq U'$ *and* $\{d_i, y_i\} \not\subseteq U'$ *for all* i, $1 \leq i \leq k$.

Proof. If $\{z_1, x\} \subseteq U'$ or $\{w_1, x\} \subseteq U'$, say $\{z_1, x\} \subseteq U'$, then since $z_3 \notin U'$ by Claim 1, $\{z_3z_1, z_3x\} \not\subseteq E'$, namely $d_{T'_k}(z_3) = 1$, Therefore either z_3z_1 is a good path in T'_k or z_3x is a good path in T'_k. In either case, we have a contradiction to Remark 1. Thus $\{z_1, x\} \not\subseteq U'$ and $\{w_1, x\} \not\subseteq U'$.

Similarly If $\{a_i, y_i\} \subseteq U'$ or $\{d_i, y_i\} \subseteq U'$ for some i, say $\{a_i, y_i\} \subseteq U'$, then since $c_i \notin U'$ by Claim 1, $\{c_ia_i, c_iy_i\} \not\subseteq E'$, namely $d_{T'_k}(c_i) = 1$, Therefore either c_ia_i is a good path in T'_k or c_iy_i is a good path in T'_k. In either case, we have a contradiction to Remark 1. Thus $\{a_i, y_i\} \not\subseteq U'$ and $\{d_i, y_i\} \not\subseteq U'$ for all i, $1 \le i \le k$. ∎

Claim 3.
(1) $z_1 \in U' \Rightarrow z_3x \in E'$
(2) $w_1 \in U' \Rightarrow w_3x \in E'$
(3) $x \in U' \Rightarrow z_1z_3, w_1w_3 \in E'$
(4) $a_i \in U' \Rightarrow c_iy_i \in E'$ *for all* i, $1 \le i \le k$
(5) $d_i \in U' \Rightarrow f_iy_i \in E'$ *for all* i, $1 \le i \le k$
(6) $y_i \in U' \Rightarrow a_ic_i, d_if_i \in E'$ *for all* i, $1 \le i \le k$

Proof. If $z_1 \in U'$ and $z_3x \notin E'$, then $z_3z_1 \in E'$ and $d_{T'_k}(z_3) = 1$, implying that z_3z_1 is a good path in T'_k, a contradiction to Remark 1. Thus statement (1) is true.

If $w_1 \in U'$ and $w_3x \notin E'$, then $w_3w_1 \in E'$ and $d_{T'_k}(w_3) = 1$, implying that w_3w_1 is a good path in T'_k, a contradiction to Remark 1. Thus statement (2) is true.

If $x \in U'$ and $z_1z_3 \notin E'$ or $w_1w_3 \notin E'$, say $z_1z_3 \notin E'$, then $z_3x \in E'$ and $d_{T'_k}(z_3) = 1$, implying that z_3x is a good path in T'_k, a contradiction to Remark 1. Thus statement (3) is true.

Similarly If $a_i \in U'$ and $c_iy_i \notin E'$ for some i, then $c_ia_i \in E'$ and $d_{T'_k}(c_i) = 1$, implying that c_ia_i is a good path in T'_k, a contradiction to Remark 1. Thus statement (4) is true.

If $d_i \in U'$ and $f_iy_i \notin E'$ for some i, then $f_id_i \in E'$ and $d_{T'_k}(f_i) = 1$, implying that f_id_i is a good path in T'_k, a contradiction to Remark 1. Thus statement (5) is true.

If $y_i \in U'$ and $a_ic_i \notin E'$ or $d_if_i \notin E'$ for some i, say $a_ic_i \notin E'$, then $c_iy_i \in E'$ and $d_{T'_k}(c_i) = 1$, implying that c_iy_i is a good path in T'_k, a contradiction to Remark 1. Thus statement (6) is true. ∎

Claim 4. $\{a_i, d_i\} \cap U' \neq \emptyset \Rightarrow \{b_ix, e_ix\} \cap E' \neq \emptyset$, *for all* i, $1 \le i \le k$.

Proof. Assume that a_i or $d_i \in U'$, say $a_i \in U'$, and $\{b_ix, e_ix\} \cap E' = \emptyset$. By Claim 2-(2), $y_i \notin U'$. By Claim 3-(4), $c_iy_i \in E'$. If $e_iy_i \notin E'$, then $e_ia_i \in E'$ and $d_{T'_k}(e_i) = 1$, implying that e_ia_i is a good path in T'_k, a contradiction to Remark 1. Hence $e_iy_i \in E'$ and $d_{T'_k}(y_i) = 2$. Therefore $b_iy_i, f_iy_i \notin E'$. Then $b_id_i, f_id_i \in E'$ and $d_{T'_k}(b_i) = d_{T'_k}(f_i) = 1$. Since T'_k is connected, $d_{T'_k}(d_i) \geq 3$, implying that b_id_i is a good path in T'_k, a contradiction to Remark 1. ∎

Claim 5. *$\{a_i, d_i, y_i\} \cap U' = \emptyset \Rightarrow \{b_ix, e_ix\} \cap E' \neq \emptyset$, for all i, $1 \leq i \leq k$.*

Proof. Assume $\{b_ix, e_ix\} \cap E' = \emptyset$. If $y_ib_i, y_if_i \notin E'$ or $y_ie_i, y_ic_i \notin E'$, say $y_ib_i, y_if_i \notin E'$, then $b_id_i, f_id_i \in E'$ and $d_{T'_k}(b_i) = d_{T'_k}(f_i) = 1$. Since T'_k is connected, $d_{T'_k}(d_i) \geq 3$, implying that b_id_i is a good path in T'_k, a contradiction to Remark 1. Hence $y_ib_i, y_ie_i \in E'$ or $y_ib_i, y_ic_i \in E'$ or $y_if_i, y_ie_i \in E'$ or $y_if_i, y_ic_i \in E'$.

If $y_ib_i, y_ie_i \in E'$, then $b_id_i \in E'$ or $e_ia_i \in E'$, say $b_id_i \in E'$. Since $y_i \notin U'$, $f_iy_i \notin E'$, so $f_id_i \in E'$. Since $d_i \notin U'$, $d_{T'_k}(d_i) = 2$. Thus $d_{T'_k}(f_i) = 1$, $d_{T'_k}(b_i) = 2$, and $d_{T'_k}(y_i) = 2$. Since T'_k is connected, $e_ia_i \in E'$. However, $c_iy_i \notin E'$ and $c_ia_i \in E'$. Hence the path $c_ia_ie_iy_ib_id_if_i$ is a component of T'_k, a contradiction.

If $y_ib_i, y_ic_i \in E'$ or $y_if_i, y_ie_i \in E'$, say $y_ib_i, y_ic_i \in E'$, then $b_id_i \in E'$ or $c_ia_i \in E'$. Since $f_iy_i, e_iy_i \notin E'$, $f_id_i, e_ia_i \in E'$. If $b_id_i \in E'$ then since T'_k is connected, $c_ia_i \in E'$. However, the path $f_id_ib_iy_ic_ia_ie_i$ is a component of T'_k, a contradiction.

If $y_if_i, y_ic_i \in E'$, then $a_ic_i \in E'$ or $d_if_i \in E'$, say $a_ic_i \in E'$. Since $y_i \notin U'$, $b_iy_i, e_iy_i \notin E'$, so $b_id_i, e_ia_i \in E'$. Since T'_k is connected, $f_id_i \in E'$. However, the path $bid_if_iy_ic_ia_ie_i$ is a component of T'_k, a contradiction. ∎

Now every vertex is adjacent to at most 2 vertices of degree at most 2 in T_k. Hence x is adjacent to at most 2 vertices of degree at most 2 in T'_k, so x is not adjacent to at least $2k-2$ vertices of $\{b_i, e_i | 1 \leq i \leq k\}$. By Claim 4 and Claim 5, If for some i $\{b_ix, e_ix\} \cap E' = \emptyset$, then $a_i, d_i \notin U'$ and $\{a_i, d_i, y_i\} \cap U' \neq \emptyset$, namely $y_i \in U'$. Therefore at least $k-2$ vertices of $\{y_i | 1 \leq i \leq k\}$ are contained in U'.

Claim 6. *For all i, $1 \leq i \leq k$, $\{a_i, d_i\} \not\subseteq U'$.*

Proof. Assume for some i, $a_i, d_i \in U'$. By Claim 3-(4) and Claim 3-(5), $c_iy_i, f_iy_i \in E'$. By Claim 2-(2), $d_{T'_k}(y_i) = 2$. If $b_ix \notin E'$, then $b_id_i \in E'$ and $d_{T'_k}(b_i) = 1$, implying that b_id_i is a good path in T'_k, a contradiction to Remark 1. Hence $b_ix \in E'$. Similarly, $e_ix \in E'$.

Since x is adjacent to at most 2 vertices of degree 2 in T'_k, for all $j (\neq i)$, $1 \leq j \leq k$, $\{b_jx, e_jx\} \cap E' = \emptyset$. Thus by Claim 4 and Claim 5, $a_j, d_j \notin U'$ and $\{a_j, d_j, y_j\} \cap U' \neq \emptyset\}$, namely $y_j \in U'$. Since x is adjacent to at most 2 vertices of degree 2 in T'_k, $z_3x, z_2x, w_3x, w_3x \notin E'$, so $z_3z_1, w_3w_1 \in E'$.

If $z_2w_1 \notin E'$, then for some $j(\neq i)$, $1 \leq j \leq k$, $z_2y_j \in E'$ and $d_{T'_k}(z_2) = 1$, implying that z_2y_j is a good path in T'_k, a contradiction to Remark 1. Hence $z_2w_1 \in E'$. Similarly, $w_2z_1 \in E'$.

Now $U' = \{a_i, d_i, y_j | j \neq i, 1 \leq j \leq k\}$. Hence $z_1, w_1 \notin U'$ and $d_{T'_k}(z_1) = d_{T'_k}(w_1) = 2$. Then $a_iz_1, a_iw_1, a_ix, d_iz_1, d_iw_1, d_ix \notin E'$. Since all vertex in U is adjacent to two vertices of degree 2 in T_k, all vertex in U' is adjacent to two vertices of degree 2 in T'_k. Hence $a_ic_i, a_ie_i, d_ib_i, d_if_i \in E'$, implying T'_k has a cycle $a_ic_iy_if_id_ib_ixe_ia_i$, a contradiction. ∎

We consider six cases.

Case 1. $z_1 \in U'$ and $w_1 \in U'$.

By Claim 3-(1) and Claim 3-(2), $z_3x, w_3x \in E'$. By Claim 2-(1), $d_{T'_k}(x) = 2$. Thus for all i, $1 \leq i \leq k$, $b_ix, e_ix \notin E'$. Hence by Claim 4 and Claim 5, for all i, $1 \leq i \leq k$, $y_i \in U'$. Therefore $|U'| = k + 2$, a contradiction.

Case 2. $z_1 \in U'$, $w_1 \notin U'$, and for some i, $a_i \in U'$ or $d_i \in U'$, say $a_i \in U'$.

By Claim 3-(1) and Claim 3-(4), $z_3x, c_iy_i \in E'$. By Claim 4 and Claim 6, b_ix or $e_ix \in E'$. Hence by Claim 2-(1) $d_{T'_k}(x) = 2$, so for all $j(\neq i)$, $1 \leq j \leq k$, $b_jx, e_jx \notin E'$. By Claim 4 and Claim 5, $y_j \in U'$. Then $U' = \{z_1, a_i, y_j | j \neq i, 1 \leq j \leq k\}$. If $w_2y_i \notin E'$, then clearly w_2z_1 or w_2y_j for some $j(\neq i)$ is a good path in T'_k, a contradiction to Remark 1. Thus $w_2y_i \in E'$, and $d_{T'_k}(y_i) = 2$, so $f_id_i \in E'$. If $e_ix \notin E'$ then e_i is adjacent to one and only one vertex a_i, implying that a_ie_i is a good path in T'_k, a contradiction to Remark 1. Hence $e_ix \in E'$ and $b_ix \notin E'$.

Then b_i is adjacent to one and only one vertex d_i. Now $d_{T'_k}(b_i) = d_{T'_k}(f_i) = 1$, so clearly b_id_i or f_id_i is a good path in T'_k, a contradiction to Remark 1. (See Figure 2)

Case 3. $z_1 \in U'$, $w_1 \notin U'$, and for any i, $a_i, d_i \notin U'$.

By our assumption and Claim 2-(1), $U' = \{z_1, y_i | 1 \leq i \leq k\}$. If $w_2x \notin E'$, then clearly w_2z_1 or w_2y_i for some i is a good path in T'_k, a contradiction to Remark 1. Thus $w_2x \in E'$. If $z_3x \notin E'$, then clearly z_3z_1 is a good path in T'_k, a contradiction to Remark 1. Thus $z_3x \in E'$. By Claim 2-(1), $d_{T'_k}(x) = 2$, so w_3w_1. If $z_2w_1 \notin E'$, then clearly z_2y_i for some i is a good path in T'_k, a contradiction to Remark 1. Thus $z_2w_1 \in E'$. Moreover, for some i, $z_2y_i \in E'$, but then by $d_{T'_k}(x) = 2$ and Claim 3-(6) $b_id_i, d_if_i, e_ia_i, a_ic_i \in E'$. Therefore b_iy_i or $f_iy_i \in E'$ and e_iy_i or $c_iy_i \in E'$, implying that y_i is adjacent to three vertices of degree 2 at T'_k, a contradiction.

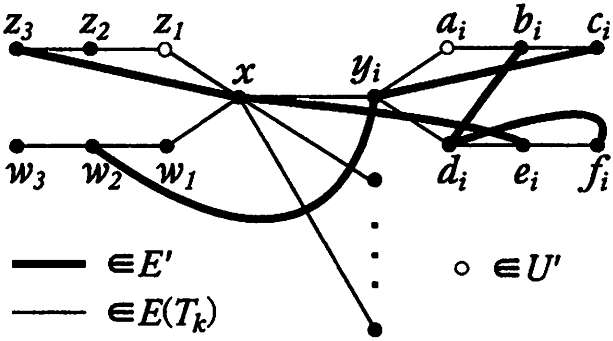

Figure 2: Case 2

Now, we can assume $z_1, w_1 \notin U'$ by proofs in three cases above.

Case 4. $x \in U'$ and for some i, $a_i \in U'$.

By Claim 3-(3), $z_3z_1 \in E'$. If $z_1a_i \in E'$ then $z_3x \notin E'$, implying that $z_3z_1a_i$ is a good path in T'_k, a contradiction to Remark 1. Hence $z_1a_i \notin E'$. Similarly $w_1a_i \notin E'$. By Claim 3-(4), $c_iy_i \in E'$. If $e_iy_i \notin E'$, then clearly e_ix or e_ia_i is a good path in T'_k, a contradiction to Remark 1. Thus $e_iy_i \in E'$. By Claim 2-(2), $d_{T'_k}(y_i) = 2$. Hence $f_id_i \in E'$. If $b_id_i \notin E'$, then clearly b_ix is a good path in T'_k, a contradiction to Remark 1. Thus $b_id_i \in E'$. By Claim 6, $d_{T'_k}(d_i) = 2$, so $a_id_i \notin E'$.

Now, a_i is adjacent exactly two vertices of degree 2 at T'_k, and a_id_i, a_iz_1, a_iw_1 E'. Thus for some $j (\neq i)$, $a_iy_j \in E'$ and $d_{T'_k}(y_j) = 2$. If $a_j, d_j \notin U'$ then by Claim 2 and claim 6, $|U'| \leq k$. Thus $a_j \in U'$ or $d_j \in U'$, say $a_j \in U'$.

By Claim 3-(4), $c_jy_j \in E'$. If $e_jy_j \notin E'$, then clearly e_jx or e_ja_j is a good path in T'_k, a contradiction to Remark 1. Thus $e_jy_j \in E'$, implying that $d_{T'_k}(y_j) \geq 3$, a contradiction. (See Figure 3)

Case 5. $x \in U'$ and for any i, $a_i, d_i \notin U'$.

In this case, $x \in U'$ and for any i, $y_i \in U'$. Thus for some i, $x'y'_i \in E(T_k)$, which is a contradiction.

Case 6. $V(P) \cap U' = \emptyset$.

In this case, by Claim 2-(2) and Claim 6, $|U'| \leq k$, which is a contradiction. ∎

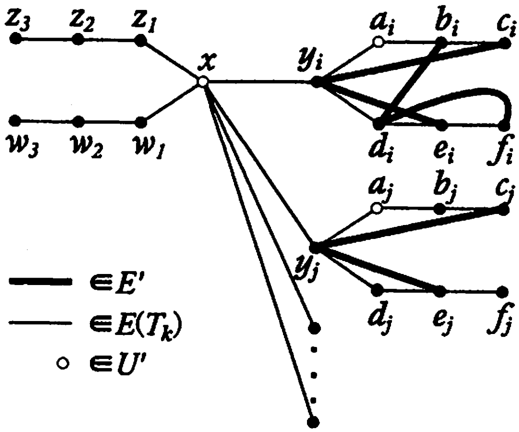

Figure 3: Case 4

References

[1] Hamamache Kheddouci, Jean-Francois Saclé, and Mariusz Woźniak, Packing two copies of a tree into its fourth power, *Discrete Mathematics* **213** (2000), 169-178.