A variant of the discrete isoperimetric problem

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Abstract

We consider a variant of what is known as the discrete isoperimetric problem, namely the problem of minimising the size of the boundary of a family of subsets of a finite set. We use the technique of 'shifting' to provide an alternative proof of a result of Hart. This technique was introduced in the early 1980s by Frankl and Füredi and gave alternative proofs of previously known classical results like the discrete isoperimetric problem itself and the Kruskal-Katona theorem. Hence our purpose is to bring Hart's result into this general framework.

1. Introduction

All notation and terminology is consistent with [5], Ch. 24, except that we use the symbols $+, -$ instead of $\cup, \setminus$ to denote set union and difference respectively, once no confusion can arise. We denote the set $\{1, .., n\}$ of positive integers by $[n]$. Let $\mathcal{F}$ be a family of subsets of $[n]$. The number of sets in the family $\mathcal{F}$ is denoted by $|\mathcal{F}|$. The boundary $\partial\mathcal{F}$ of the family $\mathcal{F}$ is defined as

$$\partial\mathcal{F} = \{A \subseteq [n] : A \notin \mathcal{F},\ \exists\, B \in \mathcal{F} \text{ such that } |A \Delta B| = 1\}. \tag{1}$$

The classical discrete isoperimetric problem asks to minimise $|\partial\mathcal{F}|$ when $|\mathcal{F}|$ is fixed. This problem was first solved by Harper [3] and a simpler solution given later by Frankl and Füredi [2]. The answer is that the size of the boundary is minimised when the family $\mathcal{F}$ is a generalised ball, by which we mean the following : for each $A \subseteq [n]$ and $r \geq 0$ we define the ball $\mathcal{B}(A, r)$ of center A and radius r by

$$\mathcal{B}(A, r) = \{B \subseteq [n] : |A \Delta B| \leq r\}. \tag{2}$$

The family $\mathcal{F}$ is then called a generalised ball if there exists a pair (A, r) such that $\mathcal{B}(A, r) \subseteq \mathcal{F} \subseteq \mathcal{B}(A, r + 1)$.

The question we are interested in is a variant of the isoperimetric problem : instead of seeking to minimise the size of the boundary of a family, one asks how to minimise the 'number of ways to get there'. To be precise, we define

$$\mathcal{P}(\mathcal{F}) = \{(A, B) : A \in \mathcal{F},\ B \notin \mathcal{F},\ |A \Delta B| = 1\}. \tag{3}$$

Then the problem is :

Given $|\mathcal{F}|$, *minimise* $|\mathcal{P}(\mathcal{F})|$.

This problem was solved in 1976 by Hart [4], who was motivated by a problem in game theory. Perhaps surprisingly, the minimum is not achieved by generalised balls. This is easily seen by considering the following example. Consider an odd n and take $|\mathcal{F}| = 2^{n-1}$. The generlised ball around $\emptyset$ would consist of all subsets of $[n]$ of size at most $\frac{n-1}{2}$. Each set of size $\frac{n-1}{2}$ is 'adjacent' to exactly $\frac{n+1}{2}$ boundary sets, hence

$$\left|\mathcal{P}(\mathcal{B}(\emptyset, \frac{n-1}{2}))\right| = \frac{n+1}{2} \binom{n}{\frac{n-1}{2}} \sim \frac{c}{\sqrt{n}} 2^n, \tag{4}$$

for some $c > 0$, by Stirling's formula. On the other hand if we take $\mathcal{F}$ to be the family consisting of all subsets of $[n]$ not containing n, then $|\mathcal{F}| = |\mathcal{P}(\mathcal{F})| = 2^{n-1}$, since $\mathcal{P}(\mathcal{F}) = \{(A, A + \{n\}) : A \in \mathcal{F}\}$.

This simple example may alert one to what the answer to the problem might be. Recall that the reverse lexicographic ordering $<_L$ of the finite subsets of N is defined by

'$A <_L B$ if $A \subset B$ or $\max\{x \in A - B\} < \max\{x \in B - A\}$'.

We denote by $\mathcal{R}(s)$ the family consisting of the s smallest finite subsets of N in the reverse lexicographic order. Then Hart proved

Theorem 1.1 [4]. *For any* $n > 0$ *and any* $\mathcal{F} \subseteq 2^{[n]}$, $|\mathcal{P}(\mathcal{F})| \geq |\mathcal{P}(\mathcal{R}(|\mathcal{F}|))|$.

2. Alternative proof of Hart's theorem

Our purpose is to show how Theorem 1.1 can be proved using the well-known technqique of 'shifting' (the definitions will be recalled below). This technique was introduced by Frankl and Füredi in 1981, in the paper [2] referred to earlier. Soon after, Frankl [1] used the same technique to give a simple proof of the Kruskal-Katona theorem, and a weaker version of it due to Lovasz. By applying the technique to Hart's theorem, we hope we offer a proof which is conceptually clearer than the original, though we admit that it is not any shorter.

The proof is established in two main steps. The first of these involves 'shifting' the family $\mathcal{F}$ and showing that $|\mathcal{P}(\mathcal{F})|$ is not increased in this way. The family $\mathcal{F}$ is thereby brought to a sort of canonical form which in the second step is compared to the family $\mathcal{R}(|\mathcal{F}|)$, with the help of an induction argument. Hence our whole method closely parallels that in [1]. The proof will be punctuated by a sequence of lemmas.

We begin by recalling for the reader the definitions of the shifting operations. Let $\mathcal{F} \subseteq 2^{[n]}$ and $i \in [n]$. The *down-shift* D_i is defined by $D_i(\mathcal{F}) = \{D_i(A) : A \in \mathcal{F}\}$ where

$$D_i(A) = \begin{cases} A - \{i\}, & \text{if } i \in A \text{ and } A - \{i\} \notin \mathcal{F}, \\ A, & \text{otherwise.} \end{cases}$$

There is another type of shifting, where one shifts sideways instead of downwards. For $1 \le i < j \le n$ we define the *(i,j)-shift* S_{ij} by $S_{ij}(\mathcal{F}) = \{S_{ij}(A) : A \in \mathcal{F}\}$ where

$$S_{ij}(A) = \begin{cases} A - \{j\} + \{i\} =: \bar{A}, & \text{if } j \in A, i \notin A \text{ and } \bar{A} \notin \mathcal{F}, \\ A, & \text{otherwise.} \end{cases}$$

Lemma 2.1. *For any family $\mathcal{F} \subseteq 2^{[n]}$ and $i \in [n]$, $|\mathcal{P}(D_i(\mathcal{F}))| \le |\mathcal{P}(\mathcal{F})|$.*

PROOF. We construct an explicit injection $i : \mathcal{P}(D_i(\mathcal{F})) \hookrightarrow \mathcal{P}(\mathcal{F})$. So let $(A, B) \in \mathcal{P}(D_i(\mathcal{F}))$. We consider four possible cases :

CASE I : $A \in \mathcal{F}, B \notin \mathcal{F}$.
CASE II : $A \notin \mathcal{F}, B \in \mathcal{F}$.
CASE III : $A \in \mathcal{F}, B \in \mathcal{F}$.
CASE IV : $A \notin \mathcal{F}, B \notin \mathcal{F}$.

CASE I : Set $i(A, B) = (A, B)$.

CASE II : Set $i(A, B) = (B, A)$. Note that the definition of D_i is easily checked to imply in this case that $B = A + \{i\}$.

CASE III : Since $B \in \mathcal{F}$ and $B \notin D_i(\mathcal{F})$ it follows that $i \in B$ and $B - \{i\} \notin \mathcal{F}$. Since $A \in \mathcal{F}$ we cannot therefore have $A = B - \{i\}$. Thus the only possibilities are that $A = B - \{j\}$ or $A = B + \{j\}$ for some $j \neq i$.

In either case one checks that it is feasible to take $i(A, B) = (A - \{i\}, B - \{i\})$.

CASE IV : Since $A \notin \mathcal{F}$ but $A \in D_i(\mathcal{F})$ it follows that $A = A_1 - \{i\}$ for some $A_1 \in \mathcal{F}$. Since $B \notin \mathcal{F}$ we cannot therefore have $B = A_1$. Thus the only possibilities are that $B = A - \{j\}$ or $B = A + \{j\}$ for some $j \neq i$.

In either case one checks that it is feasible to take $i(A, B) = (A + \{i\}, B + \{i\})$.

It remains to check that there can be no repititions in the images under i of the pairs arising in CASES I-IV. This is straightforward and left to the reader.

Recall that a family $\mathcal{F}$ is called a *lower ideal* if it is closed under subsets, i.e.: $A \in \mathcal{F}$ and $B \subset A \Rightarrow B \in \mathcal{F}$. It follows from Lemma 2.1 that $|\mathcal{P}(\mathcal{F})|$ is minimised, for a fixed $|\mathcal{F}|$, when $\mathcal{F}$ is some lower ideal.

The following fact is presumably well-known, though a proof does not seem to be given in [5]. Hence we will give a proof here for the sake of completeness.

Lemma 2.2. *If $\mathcal{F}$ is a lower ideal, then so is $S_{ij}(\mathcal{F})$ for $1 \leq i < j \leq n$.*

PROOF. Let $\mathcal{F}$ be a lower ideal, $X \in S_{ij}(\mathcal{F})$ and $Y \subset X$. We must show that $Y \in S_{ij}(\mathcal{F})$. Let $X = S_{ij}(A)$ for some $A \in \mathcal{F}$.

CASE I : $X = A$. Then $Y \in \mathcal{F}$ since $\mathcal{F}$ is a lower ideal. If $S_{ij}(Y) = Y$ there is nothing more to prove. Also, $S_{ij}(A) = A$ so there are three possibilities

(i) $j \notin A$.
(ii) $j \in A, i \in A$.
(iii) $j \in A, i \notin A$, but $A - j + i \in \mathcal{F}$.

If (i) holds then $j \notin Y$ either so $S_{ij}(Y) = Y$.
If (ii) holds and $S_{ij}(Y) \neq Y$ then $i \notin Y$ and $Y - j + i \notin \mathcal{F}$. But $Y - j + i \subseteq A - j \in \mathcal{F}$, contradicting the fact that $\mathcal{F}$ is a lower ideal.
If (iii) holds and $S_{ij}(Y) \neq Y$ then $j \in Y$ and $Y - j + i \notin \mathcal{F}$. But $Y - j + i \subseteq A - j + i \in \mathcal{F}$, so once again we have a contradiction to the fact that $\mathcal{F}$ is a lower ideal.

CASE II : $X \neq A$. Thus $X = A - j + i \notin \mathcal{F}$, for some $A \in \mathcal{F}$. There are two possibilities to consider :

(i) $Y \subseteq A - j$.
(ii) $i \in Y$.

If (i) holds then $Y \in \mathcal{F}$ since $\mathcal{F}$ is a lower ideal. Since $j \notin Y$ we have $S_{ij}(Y) = Y \in S_{ij}(\mathcal{F})$.
If (ii) holds and $Y \in \mathcal{F}$ then $S_{ij}(Y) = Y \in S_{ij}(\mathcal{F})$, since $j \notin Y$, and we are done. Thus we may suppose that $Y \notin \mathcal{F}$. Let $A_1 = Y - i + j$. Since $A_1 \subseteq A$ we have $A_1 \in \mathcal{F}$. But $Y \notin \mathcal{F}$ so $S_{ij}(A) = Y \in S_{ij}(\mathcal{F})$. This completes the proof of Lemma 2.2.

Lemma 2.3. *If $\mathcal{F}$ is a lower ideal then, for $1 \leq i < j \leq n$, we have*

$|\mathcal{P}(S_{ij}(\mathcal{F}))| \leq |\mathcal{P}(\mathcal{F})|$[1].

PROOF. We will construct an explicit injection $i : \mathcal{P}(S_{ij}(\mathcal{F})) \hookrightarrow \mathcal{P}(\mathcal{F})$. So let $(A, B) \in \mathcal{P}(S_{ij}(\mathcal{F}))$. Notice that, by Lemma 2.2, we must have $A \subset B$. As in the proof of Lemma 2.1, we consider four possible cases :

CASE I : $A \in \mathcal{F}, B \notin \mathcal{F}$.
CASE II : $A \notin \mathcal{F}, B \in \mathcal{F}$.
CASE III : $A \in \mathcal{F}, B \in \mathcal{F}$.
CASE IV : $A \notin \mathcal{F}, B \notin \mathcal{F}$.

We note immediately that CASE II cannot even arise, however, since $A \subset B$ and $\mathcal{F}$ is a lower ideal.

CASE I : Set $i(A, B) = (A, B)$.

CASE III : Since $B \in \mathcal{F}$ and $B \notin S_{ij}(\mathcal{F})$, it follows that $j \in B$, $i \notin B$ and $B - j + i \notin \mathcal{F}$. Since $A \subset B$ we may consider two possibilities :

(i) $A = B - j$.
(ii) $A = B - k$ for some $k \neq i$ or j.

If (i) holds we set $i(A, B) = (A, B - j + i)$.
If (ii) holds then $j \in A$ and $i \notin A$. But $A \in S_{ij}(\mathcal{F})$ so we must have $A = S_{ij}(A)$, which is only possible if $A - j + i \in \mathcal{F}$. We therefore take $i(A, B) = (A - j + i, B - j + i)$ in this case.

CASE IV : Since $A \notin \mathcal{F}$ but $A \in S_{ij}(\mathcal{F})$ it follows that $A = A_1 - j + i$ for some $A_1 \in \mathcal{F}$. Since $A \subset B$ we may again consider two possibilities :

(i) $B = A + j$.
(ii) $B = A + k$ for some $k \neq i$ or j.

If (i) holds we set $i(A, B) = (A_1, B) = (A + j - i, B)$.
If (ii) holds we wish to take $i(A, B) = (A + j - i, B + j - i)$. We must check that $B + j - i := B_1 \notin \mathcal{F}$. But if $B_1 \in \mathcal{F}$ then $j \in B_1$, $i \notin B_1$ and $B_1 - j + i = B \notin \mathcal{F}$ so that $S_{ij}(B_1) = B$, contradicting the fact that $B \notin S_{ij}(\mathcal{F})$.

So it remains to check that there can be no repititions in the images under i of the pairs arising in CASES I,III and IV. Once again this is straightforward and left to the reader.

Before proceeding any further we need to make some more definitions. For

[1]This result still holds when $\mathcal{F}$ is not a lower ideal, but the proof is a bit longer.

a family $\mathcal{F}$ we define $a(\mathcal{F})$ to be the average size of a set in $\mathcal{F}$, i.e.:

$$a(\mathcal{F}) := \frac{1}{|\mathcal{F}|} \sum_{A \in \mathcal{F}} |A|. \tag{5}$$

We also introduce the families

$$\mathcal{F}^1 := \{A \in \mathcal{F} : 1 \notin A\}, \tag{6}$$
$$\mathcal{F}_1 := \{A - 1 : A \in \mathcal{F} - \mathcal{F}^1\}, \tag{7}$$
$$\mathcal{F}^* := \mathcal{F}^1 - \mathcal{F}_1. \tag{8}$$

Note that if $\mathcal{F}$ is a lower ideal then $\mathcal{F}_1 \subseteq \mathcal{F}$ and

$$|\mathcal{F}^1| = \frac{1}{2}\left(|\mathcal{F}| + |\mathcal{F}^*|\right), \tag{9}$$
$$|\mathcal{F}_1| = \frac{1}{2}\left(|\mathcal{F}| - |\mathcal{F}^*|\right). \tag{10}$$

Finally we introduce the notations

$$\mathcal{P}^+(\mathcal{F}) := \{(A, B) \in \mathcal{P}(\mathcal{F}) : A \subset B\}, \tag{11}$$
$$\mathcal{P}^-(\mathcal{F}) := \{(A, B) \in \mathcal{P}(\mathcal{F}) : B \subset A\}. \tag{12}$$

Once again note that if $\mathcal{F}$ is a lower ideal then $\mathcal{P}(\mathcal{F}) = \mathcal{P}^+(\mathcal{F})$.

Now back to the proof. So we are considering families $\mathcal{F} \subseteq 2^{[n]}$ for a fixed n and $|\mathcal{F}|$. After a suitable renumbering of $1, ..., n$ and iteration of the shifts S_{1j} for $j = 2, ..., n$, Lemmas 2.1-2.3 imply that we may assume that the family $\mathcal{F}$ which minimises $|\mathcal{P}(\mathcal{F})|$ has the following canonical form :

(A) $\mathcal{F} \neq \mathcal{F}^1$.
(B) $\mathcal{F}$ is a lower ideal.
(C) Every member of $\mathcal{F}^*$ is a maximal set in $\mathcal{F}^1$.

From now on we assume that our family $\mathcal{F}$ has properties (A) - (C). For such a family we first prove

Lemma 2.4.

$$|\mathcal{P}(\mathcal{F})| = 2|\mathcal{P}(\mathcal{F}^1)| - |\mathcal{F}| - [n - 1 - 2a(\mathcal{F}^*)]\,|\mathcal{F}^*| \tag{13}$$

and $$|\mathcal{P}(\mathcal{F})| = 2|\mathcal{P}(\mathcal{F}_1)| - |\mathcal{F}| + [n + 1 - 2a(\mathcal{F}^*)]\,|\mathcal{F}^*|. \tag{14}$$

PROOF. We first establish that

$$|\mathcal{P}(\mathcal{F}^1)| = |\mathcal{P}(\mathcal{F}_1)| + |\mathcal{P}^+(\mathcal{F}^*)| - |\mathcal{P}^-(\mathcal{F}^*)|. \tag{15}$$

For let $(A, B) \in \mathcal{P}(\mathcal{F}^1)$. We consider two possibilities :

(i) $A \in \mathcal{F}_1$.
(ii) $A \in \mathcal{F}^*$.

Because of property (C) there are exactly $|\mathcal{P}(\mathcal{F}_1)| - |\mathcal{P}^-(\mathcal{F}^*)|$ pairs of type (i). And there are exactly $|\mathcal{P}^+(\mathcal{F}^*)|$ pairs of type (ii). This establishes (15). Note further that because of property (C) we have indeed that

$$|\mathcal{P}^+(\mathcal{F}^*)| = \sum_{A \in \mathcal{F}^*} (n - |A|) = [n - a(\mathcal{F}^*)]\,|\mathcal{F}^*|, \tag{16}$$

$$|\mathcal{P}^-(\mathcal{F}^*)| = \sum_{A \in \mathcal{F}^*} |A| = a(\mathcal{F}^*)|\mathcal{F}^*|, \tag{17}$$

so that (15) can be written in the form

$$|\mathcal{P}(\mathcal{F}^1)| = |\mathcal{P}(\mathcal{F}_1)| + [n - 2a(\mathcal{F}^*)]\,|\mathcal{F}^*|. \tag{18}$$

To verify the lemma it thus remains to establish (13), because (14) follows from (13) and (18).

So let $(A, B) \in \mathcal{P}(\mathcal{F})$. Because of property (B) we know that $A \subset B$. We consider three possibilities :

(i) $1 \notin A$, $1 \notin B$.
(ii) $1 \in A$, $1 \in B$.
(iii) $1 \notin A$, $1 \in B$.

A pair of type (i) lies in $\mathcal{P}(\mathcal{F}^1)$. Moreover any member of $\mathcal{P}(\mathcal{F}^1)$ is either of type (i) or of the form $(A, A+1)$ for some $A \in \mathcal{F}^1$. Hence the number of pairs of type (i) is exactly $|\mathcal{P}(\mathcal{F}^1)| - |\mathcal{F}^1|$.
For each pair (A, B) of type (ii) consider the pair $(A-1, B-1)$. This lies in $\mathcal{P}(\mathcal{F}_1)$ because of property (B). Moreover the only members of $\mathcal{P}(\mathcal{F}_1)$ which do not arise in this way are pairs of the form $(A, A+1)$ for some $A \in \mathcal{F}_1$. Hence the number of pairs of type (ii) is precisely $|\mathcal{P}(\mathcal{F}_1)| - |\mathcal{F}_1|$.
Finally for each pair of type (iii) we must have $A \in \mathcal{F}^*$ and $B = A + 1$. Hence there are exactly $|\mathcal{F}^*|$ pairs of this type.

Adding and using (18), one checks that (13) is obtained. This completes the proof of Lemma 2.4.

Our eventual aim is to prove Theorem 1.1 by induction on $|\mathcal{F}|$. In order to be able to do this we need some information about the behaviour of the function $|\mathcal{P}(\mathcal{R}(s))|$. First, we must make more precise our notation. For any family $\mathcal{F}$, if $\mathcal{F} \subseteq 2^{[n_0]}$ then $\mathcal{F} \subseteq 2^{[n]}$ for all $n \geq n_0$. Thus we shall use the notation $\mathcal{P}_n(\mathcal{F})$ in order to specify the value of n under consideration. We will also employ the less cumbersome notation

$$p_n(s) := |\mathcal{P}_n(\mathcal{R}(s))| \quad (0 \leq s \leq 2^n).$$

We let X_s denote the s^{th}-smallest finite subset of N in $<_L$ - that is, the largest member of $\mathcal{R}(s)$ - and set $x_s := |X_s|$. First off, we have the recursion formula

$$p_n(s+1) = p_n(s) + n - 2x_{s+1}. \tag{19}$$

To see this observe that the definition of the family $\mathcal{R}(s)$ implies that $\mathcal{P}_n(\mathcal{R}(s+1)) = \mathcal{P}_n(\mathcal{R}(s)) + \Psi - \Phi$ where $\Psi = \{(X_{s+1}, X_{s+1} + i) : i \notin X_{s+1}\}$ and $\Phi = \{(X_{s+1} - i, X_{s+1}) : i \in X_{s+1}\}$, so that $|\Psi| = n - x_{s+1}$ and $|\Phi| = x_{s+1}$.

Iteration of (19) gives the useful formulas

$$p_n(s+t) = p_n(s) + tn - 2\sum_{i=s+1}^{s+t} x_i, \tag{20}$$

$$p_n(s-t) = p_n(s) - tn + 2\sum_{i=s-t+1}^{s} x_i. \tag{21}$$

The crucial information is contained in the next two lemmas :

Lemma 2.5. *For any* $0 \le t \le s$ *we have*

$$\sum_{i=s+1}^{s+t} x_i - \sum_{i=s-t+1}^{s} x_i \le t. \tag{22}$$

PROOF. Let's look at the sequence of numbers x_s. It begins as follows :

$0,1,1,2,1,2,2,3,1,2,2,3,2,3,3,4,1,2,2,3,2,3,3,4,2,3,3,4,3,4,4,5,\ldots.$

The pattern here is obvious. There are two features of this sequence that we wish to make note of :

(I) Something we call *divide-by-two replication.* If we define the 2-element vectors $\mathbf{i} := (i, i+1)$ then, dividing the terms of the sequence into pairs, we get the sequence of vectors

$\mathbf{0,1,1,2,1,2,2,3,1,2,2,3,2,3,3,4,1,2,2,3,2,3,3,4,2,3,3,4,3,4,4,5,\ldots}$

In other words we get the 'same' sequence as before.

(II) Two special consequences of (I), namely that the sequence consists of groups of 2 terms, each group of the form $\{c, c+1\}$, and of groups of 4 terms, each group of the form $\{c, c+1, c+1, c+2\}$.

We now prove (22) by induction on t. For $t = 0$ the inequality holds trivially for all s. Now consider $t > 0$ fixed and s arbitrary. We denote the left-hand side of (22) in what follows by $\Delta(s,t)$. We consider four possibilities :

(i) s, t both even.
(ii) s even, t odd.
(iii) s odd, t even.
(iv) s, t both odd.

CASE (i) : Property (I) immediately implies that $\Delta(s,t) = 2\Delta(\frac{s}{2}, \frac{t}{2})$ and (22) holds by the induction on t.

CASE (ii) : Let $x_{s-t+1} = a+1$, $x_{s+t} = b$. Property (II) implies that $x_{s-t} = a$, $x_{s+t+1} = b+1$. Property (I) and induction on t give the inequalities

$$\Delta(s,t+1) = 2\Delta(\frac{s}{2}, \frac{t+1}{2}) = \Delta(s,t) + (b+1-a) \leq t+1,$$
$$\Delta(s,t-1) = \Delta(s,t) - (b-a-1) \leq t-1.$$

Adding these inequalities gives $\Delta(s,t) \leq t-1$.

CASE (iii) : As above let $x_{s-t+1} = a+1$, $x_{s+t} = b$ so that property (II) implies that $x_{s-t} = a$, $x_{s+t+1} = b+1$. Also, since s is odd, we have $x_{s+1} = x_s + 1$. Property (I) and the induction on t thus give the pair of inequalities

$$\Delta(s+1,t) = 2\Delta(\frac{s+1}{2}, \frac{t}{2}) = \Delta(s,t) - 2x_s + (a+b) \leq t,$$
$$\Delta(s-1,t) = 2\Delta(\frac{s-1}{2}, \frac{t}{2}) = \Delta(s,t) + 2x_s - (a+b) \leq t.$$

Adding these gives $\Delta(s,t) \leq t$ as required.

CASE (iv) : This time let $x_{s-t+1} = a$, $x_{s+t} = b+1$. Then $x_{s-t+2} = a+1$ and $x_{s+t-1} = b$. Also $x_{s+1} = x_s + 1$. Induction on t gives the pair of inequalities

$$\Delta(s-1,t-1) = \Delta(s,t) + 2x_s - (2b+1) \leq t-1, \tag{23}$$
$$\Delta(s+1,t-1) = \Delta(s,t) - (2x_s+2) + (2a+1) \leq t-1. \tag{24}$$

Next, since s and t are both odd, property (II) implies that one of the following two possibilities must occur :

(i) $x_{s+t+1} = b+1$, $x_{s+t+2} = b+2$,
(ii) $s \geq t+2$, $x_{s-t-1} = a-1$, $x_{s-t} = a$.

If (i) holds, then property (I) and induction on t give

$$\Delta(s+1,t+1) = 2\Delta(\frac{s+1}{2}, \frac{t+1}{2}) = \Delta(s,t) - (2x_s+2) + (2b+3) \leq t+1. \tag{25}$$

Adding (23) and (25) give $\Delta(s,t) \leq t$, as required.

If (ii) holds then property (I) and the induction give

$$\Delta(s-1,t+1) = 2\Delta(\frac{s-1}{2},\frac{t+1}{2}) = \Delta(s,t) + 2x_s - (2a-1) \le t+1. \quad (26)$$

This time adding (24) and (26) gives $\Delta(s,t) \le t$. This completes the proof of Lemma 2.5.

Lemma 2.6. *For any n and $0 \le s \le 2^{n-1}$ we have*

$$p_n(2s) = 2p_n(s) - 2s. \quad (27)$$

PROOF. Putting $s = 0$ in (20) we get (since $p_n(0) = 0$)

$$p_n(t) = nt - 2\sum_{i=1}^{t} x_i. \quad (28)$$

The family $\mathcal{R}(t)$ can be described as follows. Let

$$t = 2^{k_1} + 2^{k_2} + \ \dots \ + 2^{k_p}, \quad k_1 > k_2 > \ \dots \ k_p \ge 0,$$

be the binary decomposition of t. Then $\mathcal{R}(t) = \bigsqcup_{j=1}^{p} \Psi_j$ where

$$\Psi_j = \{A \cup B_j : A \in 2^{[k_j]},\ B_j = \{k_l + 1 : 1 \le l \le j-1\}\}.$$

With this notation, (28) becomes

$$p_n(t) = nt - \sum_{j=1}^{p}\left(j - 1 + \frac{k_j}{2}\right)2^{k_j+1}.$$

From this (27) is easily seen to follow.

We are now in a position to complete the proof of the main theorem. We proceed by induction on $|\mathcal{F}|$. The theorem is trivially true if $|\mathcal{F}| = 0$ or 1. Now fix $|\mathcal{F}| > 1$. To simplify the notation, let $s := |\mathcal{F}|$, $t := |\mathcal{F}^*|$, $a := a(\mathcal{F}^*)$.

Let n_0 be any integer such that $\mathcal{F} \subseteq 2^{[n_0]}$. We must show that $|\mathcal{P}_{n_0}(\mathcal{F})| \le p_{n_0}(|\mathcal{F}|)$. Notice that, for any $n \ge n_0$,

$$|\mathcal{P}_n(\mathcal{F})| - |\mathcal{P}_{n_0}(\mathcal{F})| = p_n(|\mathcal{F}|) - p_{n_0}(|\mathcal{F}|) = (n - n_0)|\mathcal{F}|, \quad (29)$$

so that it suffices to prove that $|\mathcal{P}_n(\mathcal{F})| \le p_n(|\mathcal{F}|)$ for any single $n \ge n_0$. In the following argument we shall need the quantity $p_n(s+t)$ to be defined (see equation (31)), so we fix a choice of $n \ge n_0$ such that $|\mathcal{F}| \le 2^{n-1}$.

By (13) and induction on $|\mathcal{F}|$ we have

$$|\mathcal{P}_n(\mathcal{F})| \ge 2p_n(|\mathcal{F}^1|) - s - (n - 1 - 2a)t. \quad (30)$$

Eqs. (9) and (27) give

$$2p_n(|\mathcal{F}^1|) = p_n(s+t) + (s+t) \tag{31}$$

which, substituted into (30), gives

$$|\mathcal{P}_n(\mathcal{F})| \geq p_n(s+t) - (n-2-2a)t. \tag{32}$$

Hence we are done unless

$$p_n(s+t) - (n-2-2a)t < p_n(s). \tag{33}$$

By (20), this inequality is equivalent to

$$\sum_{i=s+1}^{s+t} x_i > (1+a)t. \tag{34}$$

We now repeat this kind of procedure, but starting instead from (14). Induction on $|\mathcal{F}|$ gives

$$|\mathcal{P}_n(\mathcal{F})| \geq 2p_n(|\mathcal{F}_1|) - s + (n+1-2a)t. \tag{35}$$

Eqs. (10) and (27) give

$$2p_n(|\mathcal{F}_1|) = p_n(s-t) + (s-t) \tag{36}$$

which, substituted into (35), gives

$$|\mathcal{P}_n(\mathcal{F})| \geq p_n(s-t) + (n-2a)t. \tag{37}$$

Hence we are done unless

$$p_n(s-t) + (n-2a)t < p_n(s). \tag{38}$$

By (21), this inequality is equivalent to

$$\sum_{i=s-t+1}^{s} x_i < at. \tag{39}$$

To complete the proof we just have to note that (34) and (39) cannot both hold as that would give a contradiction to (22).

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