COMBINATORICS OF GEOMETRICALLY DISTRIBUTED RANDOM VARIABLES: WORDS AND PERMUTATIONS AVOIDING TWO OR THREE ADJACENT PATTERNS

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ABSTRACT. A word $w = w_1w_2\ldots w_n$ avoids an adjacent pattern τ iff w has no subsequence of adjacent letters having all the same pairwise comparisons as τ. In [12] and [13] the concept of words and permutations avoiding single adjacent pattern was introduced. We investigate the probability that words and permutations of length n avoid two or three adjacent patterns.

1. INTRODUCTION

We consider words $w = a_1a_2\ldots a_n$, where the letters a_i are taken from the set $\mathbb{Z}$ or a finite subset thereof. We assume that the letters are obtained from the geometric distribution, i.e., $\mathbb{P}\{a = k\} = pq^{k-1}$ for $k \in \mathbb{N}$ and $p = 1 - q$. The interest in the geometric distribution comes from computer science applications, namely a data structure called skip lists [4], and permutation counting. A permutation $\sigma_1\sigma_2\ldots\sigma_n$ does not enjoy the independence property of letters in a word; a letter σ_i can only occur if it was not already used in $\sigma_1\sigma_2\ldots\sigma_{n-1}$. This is often cumbersome to model. However, with the present approach, we can (often) consider the limit $q \to 1$. Then, the probability that a letter appears more than once, goes to 0, and each relative ordering of the letters is equiprobable. We are interested in the probability that a random word of length n avoid adjacent pattern of length three. In [12] and [13], the concept of words and permutations avoiding single adjacent 3-letter pattern was introduced. In this paper we derive formulae for the probabilities of words of length n avoiding two or three adjacent 3-letter patterns.

As discussed already in [10], the limit $q \to 1$ yields the model of permutations (in one line notation). Thus we get the generating functions (or formulae) of permutations avoiding two or three adjacent three letter patterns as corollaries. Permutations with these restrictions were studied independently by Kitaev in [5] and Tshifhumulo in [12] and [13].

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Definition 1. *A word is a sequence of characters or letters drawn from a fixed alphabet. That is, an ordered n-tuple of symbols is an n-word. The empty word is denoted by ε.*

Definition 2. *An n-word w, say $w = a_1a_2\cdots a_n$, contains an adjacent* 123 *pattern if and only if there exists $1 \le i \le n-2$ such that $a_i < a_{i+1} < a_{i+2}$. That is, if there is a 3-letter block $a_ia_{i+1}a_{i+2}$ satisfying $a_i < a_{i+1} < a_{i+2}$. Otherwise w is said to avoid a adjacent* 123 *pattern.*

The other five adjacent 3-letter patterns, namely 132, 231, 213, 321, and 312, are defined in the same manner.

Definition 3. *A 3-letter word $a_ia_{i+1}a_{i+2}$ is said to be*
(i) an up-down pattern if $a_i < a_{i+1} > a_{i+2}$;
(ii) a down-up pattern if $a_i > a_{i+1} < a_{i+2}$.

The other cases, namely up-up and down-down are defined similarly.

Definition 4. *A 3-letter word $a_ia_{i+1}a_{i+2}$ is said to be an up-down pattern in the sense of* 132 (231) *if $a_i < a_{i+2} < a_{i+1}$ $(a_{i+2} < a_i < a_{i+1})$.*

We can also define down-up patterns in the sense of 312 or 213 in the same manner.
Since we can use $<$ or $\le$ for "up" and $>$ or $\ge$ for "down", four possibilities can be considered, namely $\{<,>\}$, $\{<,\ge\}$, $\{\le,>\}$ and $\{\le,\ge\}$. In this paper, we consider only one possibility, namely $\{<,\ge\}$. That is, we will always denote an "up pattern" by $<$ and a "down pattern" by $\ge$.
The generating function related to an up, given that the previous element was i is

$$\sum_{i<j} pq^{j-1}x^j = \frac{px}{1-qx}(qx)^i \tag{1}$$

where the indices $i < j$ show that the pattern of the last two letters is an up. The factor $(qx)^i$ of the term on the right side of (1) means that we substitute x by qx. The generating function related to a down, given that the last element was i is

$$\sum_{i\ge j} pq^{j-1}x^j = \frac{px}{1-qx} - \frac{px}{1-qx}(qx)^i, \tag{2}$$

where the indices $i \ge j$ indicate that the pattern of the last two letters is a down. The first term means that we forget the labelling of the last part $(x := 1)$ and the second term means that we replace x by (qx).
In order to find probabilities of words avoiding 3-letter patterns, we will use a method called *adding-a-new-slice*. This method was used successfully by Flajolet and Prodinger in [2] and Knopfmacher and Prodinger in [7] and more recently by Prodinger in [10] and Tshifhumulo in [12] and [13].

The probability that words of length n avoid an adjacent 3-letter pattern, say $\alpha \in S_3$, will be denoted by $\omega_n^{(\alpha)}(q)$.
In order to obtain the recursions for the probabilities we introduce a variable u, which marks the last letter.

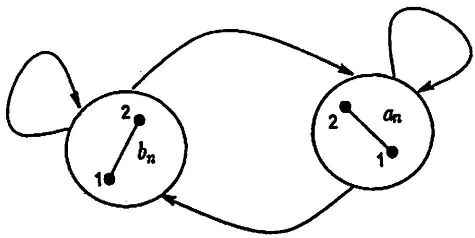

Figure 1. Automaton for the construction of a word of length n.

In order to illustrate our approach we use the automaton in Figure 1, where the two states a_n and b_n are the n^{th} down and up step, respectively, in the construction of a word of length n.
Let $a_n(u)$ ($b_n(u)$) be the counting function of words of length n when the last step is a down (up). Figure 1 above gives rise to the following counting functions for $n > 2$

$$a_n(u) = \frac{pu}{1-qu}a_{n-1}(1) - \frac{pu}{1-qu}a_{n-1}(qu) + \frac{pu}{1-qu}b_{n-1}(1) - \frac{pu}{1-qu}b_{n-1}(qu)$$

and

$$b_n(u) = \frac{pu}{1-qu}a_{n-1}(qu) + \frac{pu}{1-qu}b_{n-1}(qu).$$

Note carefully that there are no restrictions involved and every possible word is taken care of.
We define $a_2(u)$ and $b_2(u)$ as follows:

$$\begin{aligned} a_2(u) &= \sum_{i\geq 1}\sum_{j\leq i} p^2 q^{i+j-2}u^j \\ &= \frac{p^2u}{(1-qu)(1-q)} - \frac{p^2qu^2}{(1-qu)(1-q^2u)} \end{aligned} \tag{3}$$

meaning that the pattern of the first two letters is a down and

$$\begin{aligned} b_2(u) &= \sum_{i\geq 1}\sum_{j> i} p^2 q^{i+j-2}u^j \\ &= \frac{p^2qu^2}{(1-qu)(1-q^2u)}, \end{aligned} \tag{4}$$

meaning that the pattern of the first two letters is an up. Throughout this paper, a_2 and b_2 will be given by (3) and (4), unless otherwise stated.

Note that in all cases $a_2(1) + b_2(1) = 1$ and $\omega_1^{\langle\alpha\rangle}(q) = 1$ for all $\alpha \in S_3$. In particular, the initial conditions $a_0(u)$, $a_1(u)$, $b_0(u)$ and $b_1(u)$ will depend on the problem concerned.

The discussion above shows that the probability that words of length n admit every 3-letter pattern is given by

$$\omega_n(q) = a_n(1) + b_n(1)$$

for $n > 1$ and $\omega_1(q) = 1$. Since there are no restrictions, $\omega_n(q) = 1$ for each $q \in [0,1]$. With the help of Mathematica, we obtain the exponential generating function for the number of permutations in S_n by taking the limit $q \to 1$ in our recursion formulae.

2. Results

In this section we derive formulae for the probabilities that words of length n avoid two and three different adjacent 3-letter patterns. Although there are $\binom{6}{2} = 15$ and $\binom{6}{3} = 20$ pairs of restrictions, we shall only need to consider 6 cases for two different patterns and 6 cases for three different patterns, as shown in the table below. In each case, we will consider only one restriction.

Since the method for deriving the formulae for the probabilities is similar for all cases, we shall prove case 1, 2 and 5 and only give formulae for the remaining cases. As a consequence, we give (where possible) without proof results for permutations avoiding adjacent patterns as corollaries.

Table 1

Case	Restrictions	Formula / Numbers
1	$\{123,321\}$	$e.g.f. = 2(\sec z + \tan z)$, [13]
2	$\{123,231\},\{123,312\}$ $\{321,132\},\{321,213\}$	$1,2,4,11,39,161,784,4368,27260,$ $189540,1448860,12076408,109102564,\ldots$ [13]
3	$\{123,132\},\{123,213\}$ $\{321,231\},\{321,312\}$	$a_n = a_{n-1} + (n-1)a_{n-2}$ [13]
4	$\{213,312\},\{132,231\}$	$a_n = 2^{n-1}$ [13, 5]
5	$\{132,312\},\{213,231\}$	$1,2,4,10,30,108,454,2186,11840,71254,\ldots$ [13]
6	$\{132,213\},\{231,312\}$	
7	$\{123,321,231\}$ $\{123,321,213\}$ $\{123,321,132\}$ $\{123,321,312\}$	$a_n = (n-2)a_{n-2} + (n-3)!!$, [13] or $a_n = (n-1)!! + (n-2)!!$ [5]
8	$\{123,312,213\}$ $\{132,231,321\}$ $\{123,132,231\}$ $\{321,213,312\}$	$a_n = n$, [13, 5]
9	$\{132,231,312\}$ $\{132,213,312\}$ $\{132,213,231\}$ $\{213,231,312\}$	$a_n = 2^{n-2} + 1$, [13, 5]
10	$\{123,132,312\}$ $\{132,312,321\}$ $\{123,213,231\}$ $\{213,231,321\}$	$a_0 = 1,\ a_1 = 1$ [5] $a_n = \sum_i \binom{n-i-1}{i} a_{n-2i-1} + ((n+1) \bmod 2)$
11	$\{123,312,231\}$ $\{321,213,132\}$	$e.g.f = 1 + z(\sec z + \tan z)$, [6]
12	$\{123,213,132\}$ $\{321,231,312\}$	$a_n = \binom{n}{\lfloor n/2 \rfloor}$, [13, 5]

Case 1. We look at the probability that words of length n avoid adjacent $(123,321)$ patterns. That is, words avoiding two different adjacent 3-letter patterns, namely 123 and 321 patterns. These are precisely alternating words.

Theorem 1. *The probability that words of length n avoid adjacent* $(123,321)$ *patterns is given by*

$$\omega_n^{(123,321)}(q) = a_n(1) + b_n(1),$$

where for $n > 2$ we have

$$a_n(u) = \frac{pu}{1-qu} b_{n-1}(1) - \frac{pu}{1-qu} b_{n-1}(qu),$$
$$b_n(u) = \frac{pu}{1-qu} a_{n-1}(qu).$$

Proof. In order to prove this theorem, we make use of the automaton in Figure 2. The n^{th} step can either be an up or down.

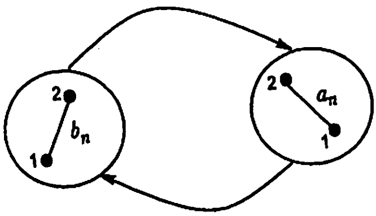

Figure 2. Automaton for adjacent (123, 321)-avoiding patterns

(a) If the n^{th} step is a down, then the $(n-1)^{st}$ step can only be a down, otherwise the word constructed will not avoid an up-down pattern. Therefore

$$a_n(u) = \frac{pu}{1-qu} a_{n-1}(1) - \frac{pu}{1-qu} a_{n-1}(qu).$$

(b) If the n^{th} step is an up, then the $(n-1)^{st}$ step can be a down or an up. Therefore

$$b_n(u) = \frac{pu}{1-qu} a_{n-1}(qu) + \frac{pu}{1-qu} b_{n-1}(qu).$$

Adding a_n and b_n and then setting $u := 1$ we obtain the desired results. □

Since permutation avoiding adjacent (123, 321) patterns are precisely alternating permutations, it follows that

Corollary 1. *The exponential generating function for the number of permutations in S_n avoiding adjacent* (123, 321) *patterns is*

$$f(z) = 2(\sec z + \tan z).$$

Case 2. In this case we consider words avoiding adjacent (321, 132) patterns. That is, words with no two adjacent down patterns and no up-down pattern in the sense of 132. Let $c_n(u)$ denote an up-down pattern in the sense of 231. Then $c_0(u) = c_1(u) = c_2(u) = 0$.

Theorem 2. *The probability that words of length n avoid adjacent* (321, 132) *is given by*

$$\omega_n^{(321,132)}(q) = a_n(1) + b_n(1) + c_n(1),$$

where for $n > 2$ we have

$$\begin{aligned} a_n(u) &= 0, \\ b_n(u) &= \frac{pv}{1-qv}a_{n-1}(qu) + \frac{pv}{1-qv}b_{n-1}(qu) + \frac{pv}{1-qv}c_{n-1}(qu), \\ c_n(u) &= \frac{p^2u}{(1-q)(1-qu)}a_{n-2}(q) - \frac{p^2u}{(1-q)(1-qu)}a_{n-2}(q^2u) \\ &+ \frac{p^2u}{(1-q)(1-qu)}b_{n-2}(q) - \frac{p^2u}{(1-q)(1-qu)}b_{n-2}(q^2u) \\ &+ \frac{p^2u}{(1-q)(1-qu)}c_{n-2}(q) - \frac{p^2u}{(1-q)(1-qu)}c_{n-2}(q^2u). \end{aligned}$$

Proof. In order to prove this theorem we use the automaton in Figure 3. In this case, the n^{th} step can be a down or an up-down in the sense of 231.

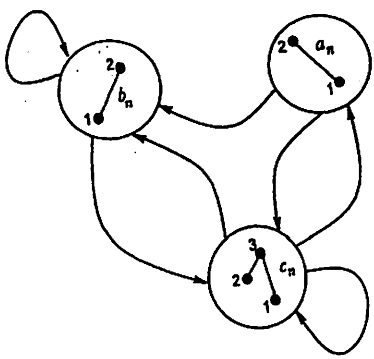

Figure 3. Automaton for adjacent (321, 132)-avoiding patterns

As seen in Figure 3, the n^{th} step cannot be a down since that is taken care of by an up-down in the sense of 231 and hence

$$a_n(u) = 0.$$

If the n^{th} step is an up, Figure 3 shows that the $(n-1)^{st}$ step can either be a down or an up-down in the sense of 231. Therefore

$$b_n(u) = \frac{pu}{1-qu}a_{n-1}(qu) + \frac{pu}{1-qu}c_{n-1}(qu).$$

If the n^{th} step is an up-down in the sense of 231, then Figure 3 shows that the $(n-1)^{st}$ step can either be a down or an up or an up-down in the sense of 213. Therefore

$$\begin{aligned} c_n(u) &= \frac{p^2u}{(1-q)(1-qu)}a_{n-2}(q) - \frac{p^2u}{(1-q)(1-qu)}a_{n-2}(q^2u) \\ &+ \frac{p^2u}{(1-q)(1-qu)}b_{n-2}(q) - \frac{p^2u}{(1-q)(1-qu)}b_{n-2}(q^2u) \\ &+ \frac{p^2u}{(1-q)(1-qu)}c_{n-2}(q) - \frac{p^2u}{(1-q)(1-qu)}c_{n-2}(q^2u). \end{aligned}$$

Adding a_n, b_n and c_n and setting $u := 1$ we obtain the desired result. □

Case 3. We consider the probability that words of length n avoid adjacent $(123, 132)$ patterns. As before, $c_n(u)$ will denote the up-down pattern in the sense 231. This case leads to

Theorem 3. *The probability that words of length n avoid adjacent* $(123, 132)$ *patterns is given by*

$$\omega_n^{(123,132)}(q) = a_n(1) + b_n(1) + c_n(1),$$

where for $n > 2$ *we have*

$$\begin{aligned} a_n(u) &= \frac{pu}{1-qu}a_{n-1}(1) - \frac{pu}{1-qu}a_{n-1}(qu) + \frac{pu}{1-qu}c_{n-1}(1) - \frac{pu}{1-qu}c_{n-1}(q \\ b_n(u) &= \frac{pv}{1-qv}a_{n-1}(qu) + \frac{pv}{1-qv}c_{n-1}(qu), \\ c_n(u) &= \frac{p^2u}{(1-q)(1-qu)}a_{n-2}(q) - \frac{p^2u}{(1-q)(1-qu)}a_{n-2}(q^2u) \\ &+ \frac{p^2u}{(1-q)(1-qu)}c_{n-2}(q) - \frac{p^2u}{(1-q)(1-qu)}c_{n-2}(q^2u). \end{aligned}$$

Corollary 2. *The number of permutations in* S_n *avoiding adjacent* $(123, 132)$ *patterns satisfy the following recurrence relation*

$$a_n = a_{n-1} + (n-1)a_{n-2},$$

for $n > 2$ *with* $a_1 = 1$ *and* $a_2 = 2$.

This recurrence relation also enumerates permutations consisting only of singleton cycles and doubleton cycles (involutions) (see [3]). Therefore we can conclude that there is a bijection between permutations of length n avoiding adjacent $(123, 132)$ patterns and involutions as we show in the proposition below.

Let us recall that every permutation can be written as a product of cycles and that the cyclic notation for a permutation is not unique. Permutations whose cycles have length one or two are called involutions. Let us consider a canonical form for a permutation. To get a canonical form, we proceed as follows (see Knuth [8], p. 148):

(*a*) Write a permutation in cycle form, with all singleton cycles written explicitly.
(*b*) Within each cycle, choose the smallest number (the cycle leader) first.
(*c*) Order the cycles in decreasing order of the cycle leaders in the cycle.

Example 1. *Starting with a* 6 *element permutation* (2 1 6)(4 3) *we would get*
(*a*) : (2 1 6)(4 3)(5); (*b*) : (1 6 2)(3 4)(5); (*c*) : (5)(3 4)(1 6 2).

The important property of this canonical form is that the parentheses may be dropped and uniquely reconstructed again. The left parentheses are inserted just before each left-to-right minimum (i.e. just before each element that is preceded by no smaller elements). Corresponding to each permutation there exists exactly one canonical representation.

Proposition 1. *There is a one-to-one correspondence between* $S_n(\langle 123, 132\rangle)$ *and the set of all involutions in* S_n.

Proof. To prove this theorem, we shall show that every canonical form for an involution is a adjacent (123, 132)-avoiding permutation and vice versa. That is, $\sigma \in S_n(\langle 123, 132\rangle)$ if and only if σ is a canonical form of an involution.
Let σ^* be a canonical form for an involution σ in S_n. If σ consists only of cycles of length one, then there is nothing to prove as σ^* will be a decreasing sequence of length n. Therefore assume that σ consists of cycles of length one or two. Then the cycle leaders of σ form a decreasing sequence in σ^*. Since σ is an involution, then there exists at most one element between any two adjacent cycle leaders in σ^*. If such an element exists, it will be greater than both the preceding and the next element in σ^*. Hence every 3-letter string in σ^* will either be a 321, 231, 213 or 312 pattern. Therefore $\sigma^* \in S_n(\langle 123, 132\rangle)$.
On the other hand, let $\sigma \in S_n(\langle 123, 132\rangle)$. Then every 3-letter block in σ is either a 321, 231, 213 or 312 pattern. As it has been shown above, permutations with these properties are canonical forms of involutions. □

Case 4. We now consider the probability that words of length n avoid adjacent (132, 231) patterns. That is, words with no up-down patterns. In this case we obtain

Theorem 4. *The probability that words of length* n *avoid* (132, 231) *patterns is given by*

$$\omega_n^{\langle 132,231\rangle}(q) = a_n(1) + b_n(1),$$

where for $n > 2$ we have

$$\begin{aligned} a_n(u) &= \frac{pu}{1-qu}a_{n-1}(1) - \frac{pu}{1-qu}a_{n-1}(qu), \\ b_n(u) &= \frac{pu}{1-qu}a_{n-1}(qu) + \frac{pu}{1-qu}b_{n-1}(qu). \end{aligned}$$

We give the results without proof for permutations avoiding adjacent (231, 132) patterns. These are permutations without peaks and the proof for the results can be found in [11].

Corollary 3. *The number of adjacent* (231, 132) *avoiding permutations in* S_n, $n \geq 1$, *is*

$$A_n(\langle 231, 132\rangle) = 2^{n-1}.$$

Case 5. Next we look at words avoiding adjacent (132, 312) patterns. In this case we introduce the second variable v. Now the variable u labels the second to last letter while v labels the last letter.

Theorem 5. *The probability that words of length n avoid adjacent* (132, 312) *patterns is given by*

$$\omega_n^{\langle 132,312\rangle}(q) = a_n(1,1) + b_n(1,1),$$

where for $n > 2$ we have

$$\begin{aligned} a_n(u,v) &= \frac{pv}{1-qv}a_{n-1}(1,u) - \frac{pv}{1-qv}a_{n-1}(1,quv) \\ &\quad + \frac{pv}{1-qv}b_{n-1}(1,u) - \frac{pv}{1-qv}b_{n-1}(qv,u), \\ b_n(u,v) &= \frac{pv}{1-qv}a_{n-1}(qv,u) + \frac{pv}{1-qv}b_{n-1}(1,quv). \end{aligned}$$

Proof. In order to prove this theorem we use the automaton in Figure 4. We define $a_2(u,v)$ and $b_2(u,v)$ as follows:

$$\begin{aligned} a_2(u,v) &= \sum_{i\geq 1}\sum_{j\leq i} p^2 q^{i+j-2} u^i v^j \\ &= \frac{p^2uv}{(1-qu)(1-qv)} - \frac{p^2quv^2}{(1-qv)(1-q^2uv)} \end{aligned}$$

meaning that the pattern of the first two letters is a down and

$$\begin{aligned} b_2(u,v) &= \sum_{i\geq 1}\sum_{j> i} p^2 q^{i+j-2} u^i v^j \\ &= \frac{p^2quv^2}{(1-qv)(1-q^2uv)}, \end{aligned}$$

meaning that the pattern of the first two letters is an up.

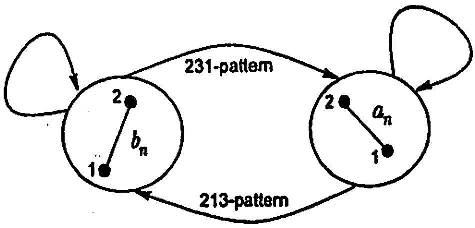

Figure 4. Automaton for adjacent (132, 312)-avoiding patterns

As seen in Figure 4, the n^{th} step can either be a down or an up.
(a) If the n^{th} step is a down, then the $(n-1)^{st}$ step can either be a down or an up (see Figure 4). If the $(n-1)^{st}$ step is a down, then adding a new slice means adding a pair (j,k) with $j \leq i,\ \ k \leq j$. If the $(n-1)^{st}$ step is an up, then adding a new slice means adding a pair (j,k) with $1 \leq j \leq i$, $k > i$. Therefore

$$\begin{aligned} a_n(u,v) &= \frac{pv}{1-qv}a_{n-1}(1,u) - \frac{pv}{1-qv}a_{n-1}(1,quv) \\ &\quad + \frac{pv}{1-qv}b_{n-1}(1,u) - \frac{pv}{1-qv}b_{n-1}(qv,u). \end{aligned}$$

(b) If the n^{th} step is an up, then the $(n-1)^{st}$ step can either be a down or an up (see Figure 4). If the $(n-1)^{st}$ step is a down, then adding a new slice means adding a pair (j,k) with $j > i,\ \ 1 \leq k \leq i$. If the $(n-1)^{st}$ step is an up, then adding a new slice means adding a pair (j,k) with $j > i$, $k > j$. Therefore

$$b_n(u,v) = \frac{pv}{1-qv}a_{n-1}(qv,u) + \frac{pv}{1-qv}b_{n-1}(1,quv).$$

Adding $a_n(u,v)$ and $b_n(u,v)$ and setting $u := 1$ and $v := 1$ we obtain the desired results. □

Case 7. We consider words avoiding adjacent (321, 123, 132) patterns. These are up-down and down-up words in which the up-down patterns are only in the sense of 231.

Theorem 6. *The probability that words of length n avoid* (123, 321, 132) *patterns is given by*

$$\omega_n^{(123,321,132)}(q) = a_n(1) + b_n(1) + c_n(1),$$

where for $n > 2$ *we have*

$$\begin{aligned} a_n(u) &= 0, \\ b_n(u) &= \frac{pu}{(1-qu)}a_{n-1}(qu) + \frac{pu}{(1-qu)}c_{n-1}(qu), \\ c_n(u) &= \frac{p^2u}{(1-q)(1-qu)}a_{n-2}(q) - \frac{p^2u}{(1-q)(1-qu)}a_{n-2}(q^2u) \\ &\quad + \frac{p^2u}{(1-q)(1-qu)}c_{n-2}(q) - \frac{p^2u}{(1-q)(1-qu)}c_{n-2}(q^2u). \end{aligned}$$

Conjecture 1. *The number of adjacent* (123, 321, 132)*-avoiding permutations is*

$$a_n = (n-2)a_{n-2} + (n-3)!! \tag{5}$$

for $n > 3$ *with* $a_1 = 1$, $a_2 = 2$ *and* $a_3 = 3$.

If we iterate (5) we obtain

Proposition 2. *For* $n \geq 4$, *we have*

$$A_n(\langle 123, 321, 132\rangle) = \begin{cases} (\frac{n-2}{2})!2^{\frac{n}{2}} + (\frac{n-2}{2})! \sum_{j=1}^{\frac{n-2}{2}} \frac{2^{(\frac{n-2}{2})-j}}{j!}(2j-1)!! & , \quad n \text{ even} \\ \frac{3(n-2)!}{2^{\frac{n-3}{2}}(\frac{n-3}{2})!} + \frac{(n-2)!}{2^{\frac{n-3}{2}}(\frac{n-3}{2})!} \sum_{j=1}^{\frac{n-3}{2}} \frac{2^j j!}{(2j+1)!}(2j)!! & , \quad n \text{ odd} \end{cases}$$

Case 8. We consider the probability that words of length n avoid adjacent (123, 132, 231) patterns. These are words with no up-down patterns and in which an up pattern cannot be followed by another up pattern.

Theorem 7. *The probability that words of length* n *avoid* (123, 132, 231) *patterns is given by*

$$\omega_n^{(123,132,231)}(q) = a_n(1) + b_n(1),$$

where for $n > 2$ *we have*

$$\begin{aligned} a_n(u) &= \frac{pu}{1-qu}a_{n-1}(1) - \frac{pu}{1-qu}a_{n-1}(qu), \\ b_n(u) &= \frac{pu}{1-qu}a_{n-1}(qu). \end{aligned}$$

We give without proof the formula for permutations avoiding adjacent (123, 231, 132) patterns.

Corollary 4. *The number of adjacent* (123, 132, 231)*-avoiding permutation is given by*

$$a_n = n$$

Case 9. We consider words avoiding adjacent (132, 231, 312) patterns. That is, words with no up-down patterns and whose down-up patterns are only in the sense of 213.

Theorem 8. *The probability that words of length n avoid* (132, 231, 312) *patterns is given by*

$$\omega_n^{(132,231,312)}(q) = a_n(1) + b_n(1) + d_n(1),$$

where for $n > 2$ we have

$$\begin{aligned} a_n(u) &= \frac{pu}{1-qu}a_{n-1}(1) - \frac{pu}{1-qu}a_{n-1}(qu), \\ b_n(u) &= \frac{pu}{(1-qu)}b_{n-1}(qu) + \frac{pu}{(1-qu)}d_{n-1}(qu), \\ d_n(u) &= \frac{p^2u}{(1-q)(1-qu)}a_{n-2}(qu) - \frac{p^2u}{(1-q)(1-qu)}a_{n-2}(q^2u). \end{aligned}$$

Case 10. Next we look at words avoiding adjacent (123, 132, 312) patterns. These are words in which an up pattern cannot be followed by another up pattern and in which an up-down pattern and down-up patterns are of the sense of 213 and 231, respectively.

Theorem 9. *The probability that words of length n avoid adjacent* (123, 132, 31 *patterns is given by*

$$\omega_n^{(123,132,312)}(q) = a_n(1,1) + b_n(1,1),$$

where for $n > 2$ we have

$$\begin{aligned} a_n(u,v) &= \frac{pv}{1-qv}a_{n-1}(1,u) - \frac{pv}{1-qv}a_{n-1}(1,quv) \\ &\quad + \frac{pv}{1-qv}b_{n-1}(1,u) - \frac{pv}{1-qv}b_{n-1}(qv,u), \\ b_n(u,v) &= \frac{pv}{1-qv}a_{n-1}(qv,u). \end{aligned}$$

Remark 1. *In cases* 6, 11 *and* 12, *we failed to obtain recursive formulae for the probability that words of length n avoid the given patterns. Although we did not obtain formulae for the number of permutations avoiding adjacent patterns in cases* 2 *and* 5, *we managed to recursively compute the initial terms of these sequences. This has been achieved by considering the probability that a word of length n avoid adjacent given patterns and taking the limit $q \to 1$ as discussed in the introduction.*

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