Unique total domination graphs

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Abstract

A set D of vertices in a graph G is a total dominating set if every vertex of G has at least one neighbor in D. The minimum cardinality of a total dominating set of G is called the total domination number of G, denoted by $\gamma_t(G)$. A total dominating set of G with cardinality $\gamma_t(G)$ is called a γ_t-set of G. We characterize trees with unique γ_t-sets. Further, we prove that $\gamma_t(G) \leq \frac{3}{5}n(G)$ for graphs with unique γ_t-sets, and we characterize all graphs with unique γ_t-sets where $\gamma_t(G) = \frac{3}{5}n(G)$.
Keywords: Total dominating set; Unique minimum total dominating set; Unique γ_t-set.

1 Terminology and Introduction

In this paper we consider only simple and finite graphs. For any graph G the vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively, and $n(G) = |V(G)|$ and $m(G) = |E(G)|$. For any subset $A \subseteq V(G)$ we define the induced subgraph $G[A]$ as the graph with vertex set A and edge set $\{ab \in E(G) \mid a, b \in A\}$. For any set $A \subseteq V(G)$ and any vertex $x \in V(G)$ we define $G - A = G[V(G) \setminus A]$ and $G - x = G - \{x\}$. For any vertex x the set of neighbors of x in G is denoted by $N_G(x)$, the closed neighborhood by $N_G[x] = N_G(x) \cup \{x\}$, and $d_G(x) = |N_G(x)|$ is called the degree of x. Further, for any subset $A \subseteq V(G)$ we define $N_G(A) = \bigcup_{x \in A} N_G(x)$ and $N_G[A] = N_G(A) \cup A$. In any graph G a vertex of degree 0 is called an isolated vertex and a vertex of degree 1 is called an endvertex of G. For any positive integer t we denote by $K_{1,t}$ a graph of order $t+1$ where one vertex has degree t and the t neighbors of this vertex are endvertices. For two vertices x and y in a connected graph G the *distance* $d_G(x, y)$ between x and y is the minimum number of edges of a path in G from x to y. The *diameter* of G is $\text{diam}(G) = \max_{v,w \in V(G)} d_G(v, w)$. We also use the shorter notations $N(x), N[x], d(x), N(A), N[A], d(x, y)$ if it is obvious to which graph the notations refer. In 1980 Cockayne, Dawes, and Hedetniemi [4] introduced the total domination, motivated by the Five Queens Problem. A set $D \subseteq V(G)$ is a *total dominating set* of G if every vertex in $V(G)$ has at least one neighbor in D. Note that a graph has a total dominating set if and only if it has no isolated vertices. A total dominating set D of G is called *minimal* if no subset $D' \subseteq D$ with $D' \neq D$ is a total dominating set of G. Further, a total dominating set D of G is called a *minimum total dominating set* if there is no total dominating set D' of G with $|D'| < |D|$. If a graph G has no isolated vertices, then we define the *total domination number* of G, denoted by $\gamma_t(G)$, as the cardinality of a minimum total dominating set of G. We call a minimum total dominating set of G also briefly a *γ_t-set.* If a graph has a unique γ_t-set, then we call this graph a ***unique total domination graph*** or a ***utd-graph.*** There are numerous publications on total domination as e.g. [1], [2], [3], [4], [5], [9], [10], [11], [12], and [13]. Of interest are for example upper bounds on the total domination number, characterizations of extremal graphs, and the complexity of the total domination problem. Pfaff [11] has demonstrated that the total domination problem is NP-complete. Cockayne, Dawes, and Hedetniemi [4] have shown that $\gamma_t(G) \leq \frac{2}{3}n(G)$ for any connected graph G of order $n(G) \geq 3$. There are many open questions on total domination, L. Volkmann [15] posed the following ones.

- Is it possible to characterize graphs, or in particular trees, with unique γ_t-sets.
- What is the upper bound on the total domination number for unique total domination graphs.
- Which unique total domination graphs have their total domination number equal this upper bound.

These questions will be answered here. In this context we need a few more definitions.
For a subset D of $V(G)$ and a vertex $x \in D$ we call the set $P_t(x, D) = N(x) \setminus N(D \setminus \{x\})$ the *total private neighborhood* of x with regard to D and we call a vertex $y \in P_t(x, D)$ a *total private neighbor* of x with regard to D. The set $P_t(D, D) = \bigcup_{x \in D} P_t(x, D) = \{v \in V(G) \mid |N_G(v) \cap D| = 1\}$ is called the *open boundary* of D. For other graph theory terminology we follow [7].

2 Unique minimum total domination

It is easy to see ([10]) that a total dominating set D of a graph G is minimal if and only if $P_t(D, D)$ dominates D which is equivalent to the property that every vertex $x \in D$ has at least one total private neighbor. Now, we present a necessary condition for a total dominating set being a unique γ_t-set.

Lemma 2.1 *Let G be a connected graph of order at least 3. If D is the unique γ_t-set of G, then for every vertex $x \in D$ we have $|P_t(x, D)| \geq 2$ or $P_t(x, D) = P_t(x, D) \setminus D = \{y\}$ for some endvertex y of G.*

Proof.
Let D be the unique γ_t-set of G and let $x \in D$ arbitrary. Since D is minimal, we have $P_t(x, D) \neq \emptyset$. If $|P_t(x, D)| \geq 2$, then there is nothing to prove. Now, let $P_t(x, D) = \{y\}$ for some vertex y in G. Suppose $y \in D$. If there is a vertex $z \in N(y) \setminus \{x\}$, then $D' = (D \setminus \{x\}) \cup \{z\} \neq D$ is also a γ_t-set of G, which is a contradiction. If $N(y) = \{x\}$, then there is a vertex $z \in N(x) \setminus \{y\}$, by $n(G) \geq 3$. This leads to the contradiction, that $D' = (D \setminus \{y\}) \cup \{z\} \neq D$ is a second γ_t-set of G. Hence, $P_t(x, D) = P_t(x, D) \setminus D = \{y\}$. Suppose, there is a vertex $z \in N(y) \setminus \{x\}$. Then, $z \in N(D) \setminus D$ and $D' = (D \setminus \{x\}) \cup \{z\}$ is a γ_t-set of G different from D, which again is a contradiction. □

The following theorem shows that for trees the necessary condition in Lemma 2.1 is also sufficient.

Theorem 2.2 [1] *Let T be a tree of order at least 3 and let D be a subset of $V(T)$. Then the following conditions are equivalent:*

(i) D is the unique γ_t-set of T.

(ii) D is a total dominating set of T such that for every vertex $x \in D$ we have $|P_t(x, D)| \geq 2$ or $P_t(x, D) = P_t(x, D) \setminus D = \{y\}$ for some endvertex y.

Proof.
(i) ⇒ (ii): Follows immediately from Lemma 2.1.
(ii) ⇒ (i): We prove this by induction on the order $n(T)$. For any tree T let $\Gamma(T)$ be the set of endvertices of T and let $W(T) = N(\Gamma(T))$. If a tree T has a total dominating set D as in (ii), then $\Gamma(T) \cap D = \emptyset$, $W(T) \subseteq D$, and $T[D]$ has no trivial component. Hence, the diameter of T is greater or equal 3 and $n(T) \geq 4$.
First, let T be a tree of order $n(T) = 4$ that has a total dominating set D as in (ii). Then, the tree T is isomorphic to the path $x_1x_2x_3x_4$ and $D = \{x_2, x_3\}$. Obviously, D is the unique γ_t-set of T. Assume, the claim holds for every tree T' of order $4 \leq n(T') < n$. Now, let T be a tree of order $n(T) = n$, and let D be a total dominating set of T as in (ii). Let d be the diameter of T and let $P = v_0v_1 \ldots v_d$ be a longest path in T such that the first index $i(P) > 0$ with $v_{i(P)} \notin D$ is as small as possible. If

[1] Theorem 2.2 has been independently obtained by T.W. Haynes and M.A. Henning (Discussiones Mathematicae Graph Theory 22(2) (2002), 233-246)

$3 \le d \le 4$, then $T - \Gamma(T) \cong K_{1,t}$ for some positive integer t which implies that $D = V(T) \setminus \Gamma(T)$ and D is the unique γ_t-set of T. Now, let $d \ge 5$. Suppose, there exists a γ_t-set F of T different from D. If there is a vertex $x \in F \cap \Gamma(T)$, then the vertex $y \in N(x)$ is in F. Let $z \in N(y) \setminus \{x\}$. Then, also $F' = (F \setminus \{x\}) \cup \{z\}$ is a γ_t-set of T. Since $P_t(z, F') = \{y\} \subseteq F'$, the set F' does not fulfill (ii) and $F' \neq D$. Thus, successively we can get a γ_t-set $D' \neq D$ of T such that $D' \cap \Gamma(T) = \emptyset$ which yields $W(T) \subseteq D'$. For the first three vertices of the path P we get that $v_0 \in \Gamma(T)$ and $v_1, v_2 \in D \cap D'$. Now, we choose the edge $ab \in E(T)$ as follows.
Case I: If $v_3 \notin P_t(v_2, D)$, then let $a = v_2 \in D$ and $b = v_3$.
Case II: If $v_3 \in P_t(v_2, D) \setminus D$, then let $a = v_3 \notin D$ and $b = v_4$.
Case III: If $v_3 \in P_t(v_2, D) \cap D$ and $v_4 \notin P_t(v_3, D)$, then let $a = v_3 \in D$ and $b = v_4$.
Case IV: If $v_3 \in P_t(v_2, D) \cap D$, $v_4 \in P_t(v_3, D)$, and $d(v_4) = 2$, then let $a = v_4 \notin D$ and $b = v_5$.
Case V: If $v_3 \in P_t(v_2, D) \cap D$, $v_4 \in P_t(v_3, D)$, and $d(v_4) > 2$, then there is at least one vertex $v_3' \in N(v_4) \setminus V(P)$ and, since $v_3', v_4 \notin D$, there is a second path $P' = v_0', v_1', v_2', v_3', v_4, \ldots, v_d$ of length d with $v_1', v_2' \in D$ and $v_3' \notin D$ which is a contradiction to the choice of P.

Now, let T_1 and T_2 be the two components of $T - ab$ such that $a \in V(T_1)$ and $b \in V(T_2)$. For $i = 1, 2$ let $D_i = D \cap V(T_i)$ and $D_i' = D' \cap V(T_i)$. By the choice of the edge ab, the set D_i and the tree T_i fulfill Condition (ii) for $i = 1, 2$. Since $n(T_i) < n(T)$, we get by the induction hypothesis that D_i is the unique γ_t-set of T_i for $i = 1, 2$. Further, since P is a longest path of T and D_1 fulfills Condition (ii) for T_1, we get that $T_1 - \Gamma(T_1) \cong K_{1,t}$ for some positive integer t, $D_1 = V(T_1) \setminus \Gamma(T_1)$, and $d_T(a) \ge 3$ if and only if $a \in D$. If $d_T(a) = 2$, then it is $\Gamma(T_1) = (\Gamma(T) \cap V(T_1)) \cup \{a\}$, and if $d_T(a) \ge 3$, then $\Gamma(T_1) = \Gamma(T) \cap V(T_1)$. This leads to

$$\begin{array}{lcll} D_1 & = & V(T_1) \setminus \Gamma(T) & \text{, if } a \in D \text{ (Case I, III), and} \\ D_1 & = & V(T_1) \setminus (\Gamma(T) \cup \{a\}) & \text{, if } a \notin D \text{ (Case II, IV).} \end{array}$$

In addition, the set $\tilde{W} = \{v_2\} \cup (W(T) \cap V(T_1)) \subseteq D_1' \subseteq V(T_1) \setminus \Gamma(T)$.
Case I: Since $a = v_2 \in D$, we have $\tilde{W} = V(T_1) \setminus \Gamma(T) = D_1$ and $D_1' = D_1$.
Case II: Since $a = v_3 \notin D$, we have $\tilde{W} = V(T_1) \setminus (\Gamma(T) \cup \{a\}) = D_1$ and $D_1 \subseteq D_1' \subseteq D_1 \cup \{a\}$.
Case III,IV: Since $v_3 \in D$, $v_2 \notin P_t(v_3, D)$, and $v_4 \notin \Gamma(T)$, there is at least one total private neighbor of v_3 in $N(v_3) \setminus \{v_2, v_4\} \neq \emptyset$. Since P is a longest path and $N(v_3) \cap D = \{v_2\}$, we get that $N(v_3) \setminus \{v_2, v_4\} \subseteq \Gamma(T)$. Thus, $v_3 \in W(T) \cap V(T_1) \subseteq \tilde{W}$.
In Case III where $a = v_3$ this leads to $\tilde{W} = V(T_1) \setminus \Gamma(T) = D_1$ and $D_1' = D_1$.

In Case IV where $a = v_4$ we get that $\tilde{W} = V(T_1) \setminus (\Gamma(T) \cup \{a\}) = D_1$ and $D_1 \subseteq D_1' \subseteq D_1 \cup \{a\}$.
Thus, in every case we obtain that $|D_2'| \leq |D_2|$ and $D_2' \neq D_2$ which implies that D_2' is not a total dominating set of T_2. Since b is the only vertex in T_2 that has a neighbor outside of T_2, we get that every vertex in $V(T_2) \setminus \{b\}$ has a neighbor in D_2', $a \in D'$, and $b \in P_t(a, D')$.
Suppose, $a \notin D$ (Case II or IV). Then, $D_1' = D_1 \cup \{a\} \neq D_1$ and $|D_2'| < |D_2|$. Let $x \in N(b) \cap V(T_2)$. Since every vertex in $V(T_2) \setminus \{b\}$ has a neighbor in D_2' and $b \in P_t(a, D')$, we obtain that $x \in N(D_2') \setminus D_2'$ and the set $F_2 = D_2' \cup \{x\}$ is a total dominating set of T_2 with $|F_2| = |D_2|$. This leads to $D_2 = D_2' \cup \{x\}$. Since b is no endvertex of T, there is a vertex $w \in P_t(x, D) \setminus \{b\}$ that has no neighbor in $D_2 \setminus \{x\} = D_2'$, which is a contradiction.
Hence, we have $a \in D$ (Case I or III) and $D_1 = D_1'$. Let $x \in (N(b) \cap D_2)$. Since $b \in P_t(a, D')$, we have $x \notin D_2'$, $N(x) \cap \Gamma(T) = \emptyset$, and $|P_t(x, D)| \geq 2$. By $b \in V(P) \setminus P_t(x, D)$, there is at least one vertex $y_1 \in P_t(x, D) \setminus V(P)$. This vertex y_1 has at least one neighbor $y_2 \in D_2'$. This implies that $y_2 \notin \Gamma(T)$, $y_2 \neq x$, and $y_2 \notin D$, by $y_1 \in P_t(x, D)$. Thus, there is a vertex $y_3 \in N(y_2) \setminus \{y_1\}$ and a vertex $y_4 \in N(y_3) \cap D$. Thus, $y_4 \neq y_2$, $y_4 \notin \Gamma(T)$, and there is a vertex $y_5 \in N(y_4) \setminus \{y_3\}$.
In Case I where $b = v_3$ the path P' in T from y_5 to v_d has length

$$d(y_5, v_d) = d(y_5, x) + d(x, v_4) + d(v_4, v_d) \geq 5 + 0 + (d - 4) > d,$$

which is a contradiction.
In Case III where $b = v_4$ the path P' in T from y_5 to v_d has length

$$d(y_5, v_d) = d(y_5, x) + d(x, v_5) + d(v_5, v_d) \geq 5 + 0 + (d - 5) = d.$$

Hence, $P' = v_0'v_1' \dots v_d'$ with $v_0' = y_5$ and $v_d' = v_d$ is a longest path of T which fulfills $v_3' = y_2 \notin D$ and $i(P') = 3$. But, since $a = v_3 \in D$, we have $i(P) > 3$, which is a contradiction to the choice of P. □

In the end of this section we want to characterize the class of trees that are utd-graphs. In order to do this, we first define two classes of trees.
Let $\mathcal{F}_3 = \{T(s_1, s_2) \mid s_1, s_2 \geq 1\}$ where $T(s_1, s_2)$ is the tree with vertex set

$$V(T(s_1, s_2)) = \{x_1, x_2\} \cup \{y_{1,i}, y_{2,j} \mid 1 \leq i \leq s_1, 1 \leq j \leq s_2\}$$

and edge set

$$E(T(s_1, s_2)) = \{x_1x_2\} \cup \{x_1y_{1,i}, x_2y_{2,j} \mid 1 \leq i \leq s_1, 1 \leq j \leq s_2\}.$$

Let $\mathcal{F}_4 = \{T(t; s_0, s_1, \dots, s_t) \mid t \geq 2, s_0 \geq 0, s_1, s_2, \dots, s_t \geq 1\}$ where

$T(t; s_0, s_1, \ldots, s_t)$ is the tree with vertex set

$$V(T(t; s_0, s_1, \ldots, s_t)) = \{x_0, x_1, \ldots, x_t\} \cup \{y_{i,j_i} \mid 1 \leq j_i \leq s_i, 0 \leq i \leq t\}$$

and edge set

$$E(T(t; s_0, s_1, \ldots, s_t)) = \{x_0 x_i \mid 1 \leq i \leq t\} \cup \{x_i y_{i,j_i} \mid 1 \leq j_i \leq s_i, 0 \leq i \leq t\}.$$

Observation 2.3 *Every tree $T(s_1, s_2) \in \mathcal{F}_3$ is of diameter 3 and has the total dominating set $\{x_1, x_2\}$, and every tree $T(t; s_0, s_1, \ldots, s_t) \in \mathcal{F}_4$ is of diameter 4 and has the total dominating set $\{x_0, x_1, \ldots, x_t\}$. By Theorem 2.2, it is straightforward to see that for $d = 3, 4$ the class $\mathcal{F}_d$ is equal the set of trees of diameter d that are utd-graphs. For these trees T the set $V(T) \setminus \Gamma(T)$ is the unique γ_t-set.*

Let $\mathcal{T}_1 = \mathcal{F}_3 \cup \mathcal{F}_4$ and for every $i \geq 1$ let $\mathcal{T}_{i+1} = \{op(T_1, T) \mid T_1 \in \mathcal{T}_1, T \in \mathcal{T}_i\}$ where op is one of the three operations listed below. We denote $\mathcal{T} = \bigcup_{i \geq 1} \mathcal{T}_i$.

For every non trivial tree T with γ_t-set D and for any tree $T_1 \in \mathcal{T}_1$ with γ_t-set D_1 we define $op^1(T_1, T)$, $op^2(T_1, T)$, and $op^3(T_1, T)$ to be the trees with vertex set

$$V(T_1) \cup V(T)$$

and with the following edge sets.

$op^1(T_1, T)$ has edge set $E(T_1) \cup E(T) \cup \{xv\}$ for some $x \in D_1$ and some $v \in V(T) \setminus D$ where either $v \notin P_t(D, D)$ or $v \in P_t(w, D)$ for a vertex $w \in D$ with $|P_t(w, D)| \geq 3$ or $P_t(w, D) = \{v, v'\}$ for some endvertex v' of T.

$op^2(T_1, T)$ has edge set $E(T_1) \cup E(T) \cup \{xv\}$ for some $x \in D_1$ if $T_1 \in \mathcal{F}_3$, and $x = x_0$ if $T_1 \in \mathcal{F}_4$, and for some $v \in D$ where either $v \notin P_t(D, D)$ or $v \in P_t(w, D)$ for a vertex $w \in D$ with $|P_t(w, D)| \geq 3$ or $P_t(w, D) = \{v, v'\}$ for some endvertex v' of T.

$op^3(T_1, T)$ has edge set $E(T_1) \cup E(T) \cup \{yv\}$ for some $y \in V(T_1) \setminus D_1$ where, if $T_1 \in \mathcal{F}_4$ and $y \in N(x_i)$ for some $0 < i \leq t$, then $s_i \geq 2$, and for some $v \in V(T) \setminus D$ where $v \in N(w)$ for a vertex $w \in D$ with $P_t(w, D) \neq \{v\}$.

Theorem 2.4 *Let T be a tree of order at least 3. Then, T is a utd-graph if and only if $T \in \mathcal{T}$.*

Proof. Suppose, there exists a tree that is a utd-graph but not in $\mathcal{T}$. Let T be such a tree of minimal order n. If the diameter of T is less than 5, then $T \in \mathcal{T}_1 \subseteq \mathcal{T}$, by Observation 2.3. Hence, the diameter of T is greater or equal 5. Analogous to the proof of Theorem 2.2 (ii) $\Rightarrow$ (i) we consider one

of these special longest paths P in T and we choice the edge $ab \in E(P)$ by Case I–IV. Thus, we obtain the two components T_1 and T_2 of $T - ab$ such that T_1 and T_2 are utd-graphs of order less than n but at least 3. By the minimality of T and $3 \leq \mathrm{diam(T_1)} \leq 4$, we get that $T_1 \in \mathcal{T}_1$ and $T_2 \in \mathcal{T}_i$ for some $i \geq 1$. Since T fulfills Condition (ii), the following is straightforward to see.
In Case I we obtain that $T = op^1(T_1, T_2)$ or $T = op^2(T_1, T_2)$.
In Case II and Case IV we have $T = op^3(T_1, T_2)$.
In Case III we get that $T = op^1(T_1, T_2)$.
Hence, $T \in \mathcal{T}_{i+1} \subseteq \mathcal{T}$, which is a contradiction.
Now, we prove by induction that for any positive integer i every tree $T \in \mathcal{T}_i$ is a utd-graph. If $i = 1$, then T is a utd-graph, by Observation 2.3. If $i > 1$, then $T = op(T_1, T_2)$ for some trees $T_1 \in \mathcal{T}_1$ and $T_2 \in \mathcal{T}_{i-1}$ and for some operation $op \in \{op^1, op^2, op^3\}$. By the induction hypothesis we get that T_1 and T_2 are utd-graphs. Hence, T_1 and T_2 fulfill Condition (ii) in Theorem 2.2. It is straightforward to see that also $op^i(T_1, T_2)$ fulfills Condition (ii) for any $i = 1, 2, 3$. By Theorem 2.2, the tree $T = op(T_1, T_2)$ is a utd-graph. □

3 The bound on γ_t for utd-graphs

In this section we give an upper bound on the total domination number for utd-graphs, and we characterize utd-graphs having their total domination number equal this upper bound.

Theorem 3.1
a) If G is a unique total domination graph, then $\gamma_t(G) \leq \frac{3}{5}n(G)$.

b) G is a unique total domination graph with $\gamma_t(G) = \frac{3}{5}n(G)$ if and only if $n(G) = 5r$ for some positive integer r and G consists of the disjoint union of r paths $P^1 = v_1^1 v_2^1 \ldots v_5^1$, $P^2 = v_1^2 v_2^2 \ldots v_5^2$, $\ldots$, $P^r = v_1^r v_2^r \ldots v_5^r$ of length 5 and possibly of additional edges between vertices in $\{v_3^1, v_3^2, \ldots, v_3^r\}$.

Proof. Let G be an arbitrary utd-graph, let D be the unique γ_t-set of G and let $\Gamma(G)$ be the set of endvertices of G. Further, let $H \subseteq G$ with $V(H) = D \cup P_t(D, D)$ and $E(H) = \{ab \in E(G) \mid a \in D \text{ and } b \in P_t(a, D)\}$. For every vertex $z \in P_t(D, D) \setminus D$ we have $N_H(z) = \{x\}$ for some vertex $x \in D$. Let $H_1, H_2, \ldots, H_s$ be the components of H. For $i = 1, 2, \ldots, s$ we define $D_i = D \cap V(H_i)$. The induced subgraph $H[D_i]$ is connected for every $i = 1, 2, \ldots, s$. Suppose that some subgraph $H[D_i]$ contains a cycle $v_0 v_1 v_2 v_0$ or a path $v_0 v_1 \ldots v_d$ for some integer $d \geq 3$. Then, $v_1 v_2 \in E(H)$

but $v_1 \notin P_t(v_2, D)$ and $v_2 \notin P_t(v_1, D)$, which is a contradiction. Hence, H is a forest and either $|D_i| = 1$ or $H[D_i] \cong K_{1,|D_i|-1}$ for every $i = 1, 2, \ldots, s$.
a) Without loss of generality let $|D_1| \leq |D_2| \leq \ldots \leq |D_s|$ and let $0 \leq i_1 \leq i_2 \leq s$ such that $|D_i| = 1$ for every $1 \leq i \leq i_1$, $|D_i| = 2$ for every $i_1 < i \leq i_2$ and $|D_i| > 2$ for every $i_2 < i \leq s$. By Lemma 2.1, we have $n(H_i) \geq 2$ for every $1 \leq i \leq i_1$, $n(H_i) \geq 4$ for every $i_1 < i \leq i_2$ and $n(H_i) \geq 2|D_i| - 1$ for every $i_2 < i \leq s$. Thus,

$$\frac{|D_i|}{n(H_i)} \leq \frac{1}{2} < \frac{3}{5} \quad \text{for every } 1 \leq i \leq i_2 \text{ and}$$

$$\frac{|D_i|}{n(H_i)} \leq \frac{|D_i|}{2|D_i| - 1} \leq \frac{3}{5} \quad \text{for every } i_2 < i \leq s.$$

This leads to

$$\gamma_t(G) = |D| = \sum_{i=1}^{s} |D_i| \leq \sum_{i=1}^{s} \frac{3}{5} n(H_i) = \frac{3}{5} n(H) \leq \frac{3}{5} n(G). \tag{1}$$

b) If G is a unique total domination graph with $\gamma_t(G) = 3n(G)/5$, then we have identity in every inequality in (1). This implies that $|D_i| = 3n(H_i)/5$ for every $i = 1, 2, \ldots, s$ and $n(G) = n(H)$. Hence, we have $i_1 = i_2 = 0$ and for every $1 \leq i \leq s$ we get

$$\frac{|D_i|}{n(H_i)} = \frac{|D_i|}{2|D_i| - 1} = \frac{3}{5},$$

which yields $|D_i| = 3$ and $n(H_i) = 5$. Then, for every $1 \leq i \leq s$ we obtain that $H[D_i] \cong K_{1,2} \cong x_1^i x_0^i x_2^i$ and each of the two vertices x_1^i, x_2^i has all its total private neighbors in $V(H_i) \setminus D_i$. By $|V(H_i) \setminus D_i| = 2$ and by Lemma 2.1, the vertex x_j^i has only one total private neighbor y_j^i and $y_j^i \in \Gamma(G)$ for $j = 1, 2$. This implies that $H_i \cong y_1^i x_1^i x_0^i x_2^i y_2^i$ for every $i = 1, 2, \ldots, s$. Thus, $n(G) = 5s$ for the positive integer s and G consists of the disjoint union of the s paths $H_1, H_2, \ldots, H_s$ of length 5 and possibly of additional edges outside $E(H)$. Now, let $ab \in E(G) \setminus E(H)$ arbitrarily. Since $D_i = \{x_0^i, x_1^i, x_2^i\}$ and $y_1^i, y_2^i \in \Gamma(G)$ for every $i = 1, 2, \ldots, s$, we have $\Gamma(G) = \{y_1^i, y_2^i \mid 1 \leq i \leq s\}$ and $a, b \in D = V(G) \setminus \Gamma(G)$. Further, we have $P_t(x_0^i, D) = \{x_1^i, x_2^i\}$ and both vertices $a, b \notin \{x_1^i, x_2^i \mid 1 \leq i \leq s\}$. This yields $a, b \in \{x_0^i \mid 1 \leq i \leq s\}$, which completes this part of the proof.
On the other hand, it is easy to see that every graph G described in b) is a unique total domination graph with $\gamma_t(G) = \frac{3}{5} n(G)$. □

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