Construction and Cataloguing of Nested Partially Balanced Incomplete Block Designs

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Abstract. This article provides some new methods of construction of two and three associate class Nested Partially Balanced Incomplete Block (NPBIB) designs. The methods are based on Latin-square association scheme, rectangular association scheme and triangular association scheme. One method of constructing NPBIB designs has also been given by incorporating a set of new treatments in place of each treatment in a Nested Balanced Incomplete Block (NBIB) design. Exhaustive catalogues of NPBIB designs based on two and three class association schemes with $v \leq 30$ and $r \leq 15$ have also been prepared.

Key Words. Nested partially balanced incomplete block designs, partially balanced incomplete block designs, nested balanced incomplete block designs.

1 Introduction

A design with two systems of blocks such that the second system of blocks is nested within the first system of blocks is called a nested block design. These designs are quite useful for experimental situations in which one nuisance factor is nested within the other nuisance factor, levels of which have been used for blocking. For example, consider a field experiment on some crop conducted using a block design in which the observations are recorded block-wise, *i.e.*, the harvesting of crop is done block-wise. The harvested samples are to be analyzed for their contents on quality indicators *viz.* protein content, *etc.* in the laboratory by different technicians at the same time or by a technician over different periods of time. The variation arising due to technicians or due to different time periods within each block may be controlled by another blocking system. Therefore, technicians or time periods form another system of blocks called sub-blocks that are nested within blocks.

For a nested block design set up, the intra block coefficient matrix of reduced normal equations for estimating the linear function of treatment effects is same as that obtained if blocks are ignored and design is analyzed as sub-blocks. The properties of the coefficient matrix of reduced normal equations are completely determined by treatments *vs* sub-blocks incidence matrix. From this equivalence it follows that the arrangement of treatments in blocks (first system) is of no consequence. Therefore, if we form groups of blocks of a block design in any arbitrary fashion and call such group of blocks as blocks (first system) and blocks of the block design within each group as sub-blocks (second system), then we get a nested block design. The block design ignoring sub-block classification may not possess the desirable characterization properties like connectedness, variance balance, partial balance, *etc.* However, in the example described above, the experimenter may also record yield of the crop in each of the blocks in the field besides recording the characters based on laboratory analysis. The experimenter may wish to compare the treatments based on their yield performance as well as on the characters like protein contents recorded in the laboratory based on the analysis of harvested samples. The yield of the crop obtained from the field is not subjected to laboratory analysis. Therefore, the analysis of the experimental data on yield has to be carried out as per the design adopted for the field experimentation *i.e.* a block design. Therefore, to make a set of possible paired comparisons of treatments based on yield, the block classification ignoring sub-blocks should also possess characterization properties like connectedness, variance balance, partial balance, *etc.* The problem therefore, is to arrange the treatments in blocks and sub-blocks in such a fashion that the block classification ignoring sub-block classification and the sub-block classification ignoring the block classification leaves a block design with desirable properties like connectedness, variance balance, partial balance, *etc.* For such situations, Preece [12] introduced nested balanced incomplete block (NBIB) designs where block classification ignoring sub-blocks is a balanced incomplete block (BIB) design and sub-block classification ignoring blocks is also a BIB design. Jimbo and Kuriki [7], Dey, Das and Banerjee [4], Parsad, Gupta and Srivastava [10] and Morgan, Preece and Rees [9] gave methods of construction of NBIB designs. Morgan, Preece and Rees [9] prepared an exhaustive catalogue of NBIB designs with $v \le 16$ and $r \le 30$. Gupta [5] has introduced nested variance balanced block designs for non-proper settings. A nested variance balanced block design may not exist always or even if it exists may require a large number of replications which the experimenter may not be able to afford. To deal with such situations, Homel and Robinson [6] introduced nested partially balanced incomplete block (NPBIB) designs as defined below:

Definition 1.1: An NPBIB design based on m (≥ 2)-class association scheme defined in v symbols, is an arrangement of v symbols into b_2 sub-blocks of size k_2 nested within b_1 ($= b_2/t$, t is an integer) blocks of size k_1 ($= tk_2 < v$) such that

(i) Every symbol occurs at most once in a block.
(ii) Every symbol appears at most r times in the design.
(iii) If two symbols, say α and β, are i^{th} associates, then they occur together in λ_{1i} blocks and λ_{2i} sub-blocks, the numbers λ_{1i}, λ_{2i} being independent of the particular pair of i^{th} associates α and β, $i = 1, 2 \ldots m$.

The numbers v, b_1, b_2, r, k_1, k_2, λ_{1i}, λ_{2i} ($i = 1, 2 \ldots m$) are called parameters of the design. If $\lambda_{1i} = \lambda_1$ and $\lambda_{2i} = \lambda_2$; $\forall\, i = 1, 2 \ldots m$, then an NPBIB design reduces to

NBIB design. Several methods of construction of NPBIB designs are available in the literature {see, *e.g.* Banerjee and Kageyama [1,2], Kageyama, Philip and Banerjee [8], Philip, Kageyama and Banerjee [11] and Saha, Dey and Midha [13]}. However, no attempts have been made to prepare a catalogue of these designs. Therefore, in this article, we shall obtain some new methods of construction of NPBIB designs in which block classification as well as sub-block classification ignoring the other classification leaves a partially balanced incomplete block (PBIB) design based on same association scheme both on blocks and sub-blocks. Catalogues of two and three associate class NPBIB designs for $v \leq 30$ and $r \leq 15$ have also been presented in the Appendix. Banerjee and Kageyama [2] and Philip, Banerjee and Kageyama [11] have shown that in an NPBIB design the association scheme for the blocks as well as sub-blocks can be different. For example, the design obtained by Method I of Banerjee and Kageyama [2] and Corollary 2.2 of Philip, Banerjee and Kageyama [11] using an irreducible BIB design with 6 treatments is a nested bock design where blocks form a symmetric BIB design while sub-blocks form a PBIB design based on triangular association scheme. Similarly, the nested block design obtained by Method II of Banerjee and Kageyama [2] with $s = 4$ is a symmetric BIB design in blocks and a PBIB design based on L_2 association scheme in sub-blocks. Such designs have also been included in the catalogue and marked with asterisk (*) in the catalogue. The NPBIB designs where the blocks as well as sub-blocks form PBIB designs based on different association schemes have been excluded from the catalogue.

2 Methods of Construction of NPBIB Designs

In this section we shall give some methods of constructions of NPBIB designs based on 2- and 3- class association schemes.

Method 2.1: Let $v=s^2$ symbols be defined on an L_p association scheme. Take all possible pairs of rows of the association scheme. Combine the treatments appearing in paired rows to form blocks and take the treatments from the same row within the blocks as sub-blocks. This process yields $\frac{s(s-1)}{2}$ blocks each of size $2s$, there being 2 sub-blocks each of size s nested within each block. Repeating the same procedure for all possible pairs of columns we get another $\frac{s(s-1)}{2}$ blocks. Now consider $(p-2)$ mutually orthogonal latin squares (MOLS) of side s. From each of these MOLS, form a latin square arrangement by taking the treatments appearing in the positions of a particular symbol as rows of the latin square arrangement. Repeat the process of pairing the rows to form the blocks and sub-blocks on each of the $(p-2)$ Latin square arrangements so formed. Union of all the blocks so obtained gives an NPBIB design based on L_p-

association scheme with parameters $v = s^2$, $b_1 = ps(s-1)/2$, $b_2 = ps(s-1)$, $r = p(s-1)$, $k_1 = 2s$, $k_2 = s$, $\lambda_{11} = s+p-2$, $\lambda_{12} = p$, $\lambda_{21} = s-1$, $\lambda_{22} = 0$.

Example 2.1: Consider an L_3 association scheme defined on $v = 16$ treatments. First associates of a particular treatment will be the treatments appearing in the same rows, same columns and with the same symbols on one of the 3 (4 x 4) mutually orthogonal Latin squares.

$$\text{e.g.} \begin{bmatrix} 1A & 2B & 3C & 4D \\ 5B & 6C & 7D & 8A \\ 9C & 10D & 11A & 12B \\ 13D & 14A & 15B & 16C \end{bmatrix}$$

Following the procedure of Method 2.1, we get an NPBIB design with the following blocks

[(1,2,3,4,),(5,6,7,8)]; [(1,2,3,4),(9,10,11,12)]; [(1,2,3,4),(13,14,15,16)];
[(5,6,7,8),(9,10,11,12)]; [(5,6,7,8),(13,14,15,16)]; [(9,10,11,12),(13,14,15,16)];
[(1,5,9,13),(2,6,10,14)]; [(1,5,9,13),(3,7,11,15)]; [(1,5,9,13),(4,8,12,16)];
[(2,6,10,14),(3,7,11,15)]; [(2,6,10,14),(4,8,12,16)]; [(3,7,11,15),(4,8,12,16)];
[(1,8,11,14),(2,5,12,15)]; [(1,8,11,14), (3,6,9,16)]; [(1,8,11,14), (4,7,10,13)];
[(2,5,12,15),(3,6,9,16)]; [(2,5,12,15), (4,7,10,13)]; [(3,6,9,16), (4,7,10,13)].

The parameters of this design are $v = s^2 = 16$, $b_1 = 18$, $b_2 = 36$, $r = 9$, $k_1 = 8$, $k_2 = 4$, $\lambda_{11} = 5$, $\lambda_{12} = 3$, $\lambda_{21} = 3$, $\lambda_{22} = 0$.

Saha, Dey and Midha [13] gave a procedure to obtain NPBIB design from a PBIB design by replacing each treatment of PBIB design by a new set of treatments and considering this new set of treatments as sub-blocks. As mentioned in Section 1, the intra block coefficient matrix of reduced normal equations for estimating the linear function of treatment effects is same as that obtained if blocks are ignored and design is analyzed with sub-blocks only. Hence, the connectedness property of a nested block design is dependent only on the sub-block structure. Therefore, the designs obtained through their method are disconnected due to disjoint sub-blocks. Hence, we give the following method of construction of 2-associate NPBIB designs using NBIB designs.

Method 2.2: Suppose that an NBIB design D with parameters $v^* = m$, b_1^*, b_2^*, r^*, k_1^*, k_2^*, λ_1^*, λ_2^* exists. In design D replace treatment i by a group of n new treatments, i, $v^* + i$, ..., $(n-1)v^* + i$. Repeating this process for all v^* treatments, we get a 2-associate NPBIB design in which blocks as well as sub blocks form a singular group divisible design. The parameters of this design are

$v = nv^*$, $b_1 = b_1^*$, $b_2 = b_2^*$, $r = r^*$, $k_1 = nk_1^*$, $k_2 = nk_2^*$, $\lambda_{11} = r^*$, $\lambda_{12} = \lambda_1^*$, $\lambda_{21} = r^*$, $\lambda_{22} = \lambda_2^*$, $m = v^*$, $n = n$.

Here n new set of treatments, which replace a particular treatment, will be 1st associate to each other and treatments from the different sets will be 2nd associates.

Example 2.2: Suppose that an NBIB design with parameters $v^* = 5$, $b_1^* = 5$, $b_2^* = 10$, $r^* = 4$, $k_1^* = 4$, $k_2^* = 2$, $\lambda_1^* = 3$, $\lambda_2^* = 1$ exists. The blocks of the design are

[(1,2); (3,5)]; [(2,3); (4,1)]; [(3,4); (5,2)]; [(4,5); (1,3)]; [(5,1); (2,4)].

Replacing each treatment by $n = 3$ new treatments as per procedure of Method 2.2, we get an NPBIB design with two associate classes. The parameters of the design are $v = 15$, $b_1 = 5$, $b_2 = 10$, $r = 4$, $k_1 = 12$, $k_2 = 6$, $\lambda_{11} = 4$, $\lambda_{12} = 3$, $\lambda_{21} = 4$, $\lambda_{22} = 1$, $m = 5$, $n = 3$. The blocks of the design are

[(1,6,11,2,7,12); (3,8,13,5,10,15)]; [(2,7,12,3,8,13); (4,9,14,1,6,11)];
[(3,8,13,4,9,14); (5,10,15,2,7,12)]; [(4,9,14,5,10,15); (1,6,11,3,8,13)];
[(5,10,15,1,6,11); (2,7,12,4,9,14)].

Remark 2.1: The above method can easily be generalized for obtaining $(m + 1)$-associate class NPBIB designs from m-associate class NPBIB designs. As a consequence of this, one can always get NPBIB designs based on 3-class association scheme from the designs catalogued in Table 1 in the Appendix. The parameters of a 3 – associate class NPBIB design obtained from a 2 – associate class NPBIB design with parameters v^*, b_1^*, b_2^*, r^*, k_1^*, k_2^*, λ_{11}^*, λ_{12}^*, λ_{21}^*, λ_{22}^*, are $v = nv^*$, $b_1 = b_1^*$, $b_2 = b_2^*$, $r = r^*$, $k_1 = nk_1^*$, $k_2 = nk_2^*$, $\lambda_{11} = r^*$, $\lambda_{12} = \lambda_{11}^*$, $\lambda_{13} = \lambda_{12}^*$, $\lambda_{21} = r^*$, $\lambda_{22} = \lambda_{21}^*$, $\lambda_{23} = \lambda_{22}^*$. The structure of the 3-class association scheme is described as below:

Denote the v^* treatments of the 2-class association scheme by $1, 2, \ldots, v^*$. Replace each treatment of the 2-class association scheme by a set of n new treatments to get nv^* treatments. Arrange the nv^* treatments into v^* groups of n treatments each. These v^* groups comprise of the following n-tuples: $\{1, v^* + 1, \ldots, (n-1)v^* + 1\}$, $\{2, v^* + 2, \ldots, (n-1)v^* + 2\}$, ..., $\{i, v^* + i, \ldots, (n-1)v^* + i\}$, ..., $\{v^*, 2v^*, \ldots, nv^*\}$. Then we define the 3-class association scheme as follows: two treatments are first associates if they belong to the same n-tuple, the treatments between any two n-tuples are second associates if the original treatments contained in these n-tuples are first associates in the 2-class association scheme and they are third associates if the original treatments contained in these n-tuples are second associates of the 2-class association scheme.

Method 2.3: Consider a rectangular association scheme with $v = mn$ treatments. The mn treatments are arranged in a rectangular array of m rows and n columns. The treatment symbols in the same row are first associates, treatment symbols in the same column are second associates and the remaining are third associates. If $m = n + 1$, then an NPBIB design based on rectangular association scheme may be constructed using the following procedure.

Step 1: Take all the treatments appearing in the i^{th} row of the association scheme into one block, say B_{1i}.

Step 2: Write n-sub-blocks each of size $n = (m - 1)$ by taking treatment symbols in the columns (except the treatment symbols in i^{th} row) as blocks. Number these sub-blocks as $B_{21i}, B_{22i}, \ldots, B_{2ni}$.

Step 3: Form n-blocks in the following manner

$[(B_{1i});(B_{21i})]; [(B_{1i});(B_{22i})];\ldots;[(B_{1i});(B_{2ni})]$

Step 4: Repeat Steps 1 to 3 for all the rows $i = 1, \ldots, m$.

This procedure yields an NPBIB design with rectangular association scheme with parameters as

$v = mn,\ b_1 = mn,\ b_2 = 2mn,\ r = 2n\ ,\ k_1 = 2n,\ k_2 = n,\ \lambda_{11} = m - 1,\ \lambda_{12} = m,\ \lambda_{13} = 2,\ \lambda_{21} = m - 1,\ \lambda_{22} = m - 2,\ \lambda_{23} = 0.$

Example 2.3: Suppose that $v = 12$ treatments are arranged in $m = 4$ rows and $n = 3$ columns as given below

1	5	9
2	6	10
3	7	11
4	8	12

A rectangular association scheme is defined on these 12 treatments. Then applying the procedure of Method 2.3, we get an NPBIB design based on rectangular association scheme with blocks as

[(1,5,9); (2,3,4)];	[(1,5,9); (6,7,8)];	[(1,5,9); (10,11,12)];
[(2,6,10); (1,3,4)];	[(2,6,10); (5,7,8)];	[(2,6,10); (9,11,12)];
[(3,7,11); (1,2,4)];	[(3,7,11); (5,6,8)];	[(3,7,11); (9,10,12)];
[(4,8,12); (1,2,3)];	[(4,8,12); (5,6,7)];	[(4,8,12); (9,10,11)].

The parameters of the above design are $v = 12,\ b_1 = 12,\ b_2 = 24,\ r = 6,\ k_1 = 6,\ k_2 = 3,\ \lambda_{11} = 3,\ \lambda_{12} = 4,\ \lambda_{13} = 2,\ \lambda_{21} = 3,\ \lambda_{22} = 2,\ \lambda_{23} = 0.$

Method 2.4: Let there exists an m-associate class PBIB design with parameters v'_1, b'_1, r'_1, k'_1, λ'_i, $i = 1,2,...,m$, and also there exists an NBIB {or, nested complete block-balanced incomplete sub-block (where blocks ignoring sub-blocks form a complete block design and sub-blocks ignoring blocks form a BIB design)} design with parameters $v^* = k'_1$, b_1^*, b_2^*, r^*, k_1^*, k_2^*, λ_1^*, λ_2^*. Write each of the block contents of PBIB design as NBIB {or, nested complete block-balanced incomplete sub-block} design. This procedure yields an NPBIB design with parameters $v = v'_1$, $b_1 = b'_1 b_1^*$, $b_2 = b'_1 b_2^*$, $r = r'_1 r^*$, $k_1 = k_1^*$, $k_2 = k_2^*$, $\lambda_{1i} = \lambda'_i \lambda_1^*$, $\lambda_{2i} = \lambda'_i \lambda_2^*$; $i = 1,2...m$; with common association scheme for both blocks and sub-blocks.

Example 2.4: Consider a singular group divisible design S6 of Clatworthy [3] with parameters $v'_1 = 8$, $b'_1 = 6$, $r'_1 = 3$, $k'_1 = 4$, $\lambda'_1 = 3$, $\lambda'_2 = 1$ with $m = 4$, $n = 2$. The block contents of the design are

(1, 2, 5, 6); (1, 3, 5, 7); (1, 3, 5, 8); (2, 3, 6, 7); (2, 4, 6, 8); (3, 4, 7, 8).

There also exists a nested complete block-balanced incomplete sub-block design with parameters $v^* = 4$, $b_1^* = 3$, $b_2^* = 6$, $r^* = 3$, $k_1^* = 4$, $k_2^* = 2$, $\lambda_1^* = 3$, $\lambda_2^* = 1$. The blocks and sub-block contents are [(A, B); (C, D)]; [(A, C); (B, D)]; [(A, D); (B, C)].

Writing each block of the PBIB design by the nested complete block-balanced incomplete sub-block design, we get an NPBIB design with parameters $v = 8$, $b_1 = 18$, $b_2 = 36$, $r = 9$, $k_1 = 4$, $k_2 = 2$, $\lambda_{11} = 9$, $\lambda_{12} = 3$, $\lambda_{21} = 3$, $\lambda_{22} = 1$.

The blocks and sub-block contents of this design are

[(1, 2); (5, 6)];	[(1, 5); (2, 6)];	[(1, 6); (2, 5)];	[(1, 3); (5, 7)];
[(1, 5); (3, 7)];	[(1, 7); (3, 5)];	[(1, 3); (5, 8)];	[(1, 5); (3, 8)];
[(1, 8); (3, 5)];	[(2, 3); (6, 7)];	[(2, 6); (3, 7)];	[(2, 7); (3, 6)];
[(2, 4); (6, 8)];	[(2, 6); (4, 8)];	[(2, 8); (4, 6)];	[(3, 4); (7, 8)];
[(3, 7); (4, 8)];	[(3, 8); (4, 7)].		

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