Sums of powers of binomial coefficients via Legendre polynomials

H. W. Gould
Department of Mathematics
West Virginia University, PO Box 6310
Morgantown, WV 26506-6310
email: gould@math.wvu.edu

ABSTRACT

Define

$$S_n(p,x) = \sum_{k=0}^{n} \binom{n}{k}^p x^k, \text{ where } n \geq 0,$$

Then it is well-known that $S_n(1,x)$, $S_n(2,1)$, $S_n(2,-1)$, and $S_n(3,-1)$ can be exhibited in closed form. The formula

$$S_{2n}(3,-1) = (-1)^n \binom{2n}{n} \binom{3n}{n}$$

was discovered by A. C. Dixon in 1891. L. Carlitz [Mathematics Magazine, Vol. 32(1958), 47-48] posed the formulas

$$S_n(3,1) = ((x^n)) \, (1 - x^2)^n \, P_n\left(\frac{1+x}{1-x}\right)$$

and

$$S_n(4,1) = ((x^n)) \, (1 - x)^{2n} \left\{P_n\left(\frac{1+x}{1-x}\right)\right\}^2 ,$$

where $((x^n))\, f(x)$ means the coefficient of x^n in the series expansion of f(x). We use Legendre polynomials to get the analogous formulas

$$S_n(3,-1) = ((x^n)) \, (1 - x)^{2n} \, P_n\left(\frac{1+x}{1-x}\right),$$

and

$$S_n(5,1) = ((x^n)) \, (1 - x)^n \, P_n\left(\frac{1+x}{1-x}\right) S_n(3,x).$$

We obtain some partial results for $S_n(p,x)$ when p is arbitrary, and also give a new proof of Dixon's formula.

1. Introduction. It is well-known that the sums of powers of binomial coefficients of the form

$$S_n(p,x) = \sum_{k=0}^{n} \binom{n}{k}^p x^k, \text{ where } n \geq 0, \tag{1.1}$$

have "closed form" formulas only for certain values of n, p and x. In particular

$$S_n(1,x) = (x+1)^n, \quad \text{for all complex } x, \tag{1.2}$$

$$S_n(2,1) = \binom{2n}{n}, \tag{1.3}$$

$$S_{2n}(2,-1) = (-1)^n \binom{2n}{n}, \tag{1.4}$$

and Dixon's [3] formula

$$S_{2n}(3,-1) = (-1)^n \binom{2n}{n} \binom{3n}{n}. \tag{1.5}$$

These are discussed and proved throughout the mathematical literature. They are among the 550 formulas tabulated in my book [5] as formulas (1.1), (3.66), (3.80)/(3.81), and (6.6) to be specific.

A metatheory of the existence of "closed forms" has been attempted in various ways. De Bruijn [1, p. 72] made an argument based on asymptotic expansions that suggests that we should not expect to find $S_n(p,-1)$ in closed form for values of $p > 3$ other than as in the known examples listed above. De Bruijn says that a simple way to determine whether or not such formulas exist is to determine the asymptotic behavior of the series as $n \to \infty$ for fixed p and see whether this corresponds with the behavior of multiplicative combinations of factorials. The asymptotic expansion of $S_{2n}(p,-1)$ involves $(\cos \frac{\pi}{2p})^{2np}$. The number $(\cos \frac{\pi}{2p})^{2p}$ is rational for $p = 2$ or 3. When $p > 3$ this is not true, and it follows that $(\cos \frac{\pi}{2p})^{2np}$ does not occur in the Stirling approximation formulas for $n!$, $(2n)!$, $(3n)!$, . . . , and therefore we cannot expect simple extensions of Dixon's formula when $p > 3$.

It may be also be argued from the point of view, perhaps, of the existence of certain closed form expressions for the generalized hypergeometric function which contains $S_n(p,x)$ in its special cases. Kenneth Stolarsky wrote to me many years ago about the problem of finding a general metatheorem telling when sums of ratios of products of integers may be expressed in closed form using ratios of integers. But I do not know of any useful metatheorem that has been developed to assure us that (1.2) through (1.5) are in fact the only possible cases.

In the present paper we shall show how in a curious way we may find (1.3)-(1.5) by elementary series manipulations using just the Legendre polynomials, and also get partial results when $p > 3$. The remarks below flow from discussions over the years with my late mentor Leonard Carlitz. In passing we also offer a new proof of Dixon's formula (1.5).

2. Carlitz's formulas and generalizations. Carlitz [2] posed the following formulas:

$$\sum_{k=0}^{n} \binom{n}{k}^3 = ((x^n))\ (1 - x^2)^n\ P_n\left(\frac{1+x}{1-x}\right) \tag{2.1}$$

and

$$\sum_{k=0}^{n} \binom{n}{k}^4 = ((x^n))\ (1 - x)^{2n} \left\{P_n\left(\frac{1+x}{1-x}\right)\right\}^2, \tag{2.2}$$

where the notation $((x^n))\ f(x)$ means the coefficient of x^n in the series expansion of the function $f(x)$. This is a convenient notation that was widely used in the literature many years ago that we feel is still quite useful.

From the point of view of using Legendre polynomials (2.1) and (2.2) offer similar closed forms for both $p = 3$ and $p = 4$. These two do not yield closed formulas when we try to eliminate the Legendre polynomials.

We shall prove (2.1) and (2.2) by methods different from those used by Wang [7]. In analogy to Carlitz's formulas we show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 = ((x^n))\ (1 - x)^{2n}\ P_n\left(\frac{1+x}{1-x}\right), \tag{2.3}$$

which simplifies to yield (1.5), and

$$\sum_{k=0}^{n} \binom{n}{k}^5 = ((x^n))\ (1 - x)^n\ P_n\left(\frac{1+x}{1-x}\right) \sum_{k=0}^{n} \binom{n}{k}^3 x^k. \tag{2.4}$$

For the proofs we use several standard forms for the Legendre polynomials as exhibited in my first publication [3] and elementary formulas in my book.

3. Standard forms for the Legendre polynomials. The following are taken directly from pages 38-39 of my book [5].

$$P_n(x) = \frac{1}{2^n n!} D_x^n (x^2 - 1)^n, \tag{3.1}$$

$$\sum_{n=0}^{\infty} t^n P_n(x) = (1 - 2xt + t^2)^{-1/2}, \tag{3.2}$$

$$P_n(x) = F(n+1, -n, 1; \frac{1-x}{2}), \tag{3.3}$$

in terms of the hypergeometric function,

$$P_n(x) = 2^{-n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}, \tag{3.4}$$

$$P_n(x) = \left(\frac{x-1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x+1}{x-1}\right)^k, \tag{3.5}$$

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \left(\frac{x-1}{2}\right)^k = \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} \left(\frac{x-1}{2}\right)^{n-k}, \tag{3.6}$$

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} 2^{-k} (x^2-1)^{k/2} (x - (x^2-1)^{1/2})^{n-k}, \tag{3.7}$$

$$P_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{2k} \binom{2k}{k} 2^{-2k} x^{n-2k} (x^2 - 1)^k, \tag{3.8}$$

$$2^{2n} x^n P_n(\tfrac{1}{2}(x + x^{-1})) = \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} x^{2k}, \tag{3.9}$$

In our work here we will not use all of these. We list all of them, however, because they exhibit the versatility of the Legendre polynomial as a way to study sums of products of binomial coefficients. Relations (3.4), (3.5), and (3.6) played a prominent role in my proof [3] of Grosswald's formula.

4. Proofs of the formulas in Section 2. We first prove (2.1). From (3.5) we have

$$P_n\left(\frac{1+x}{1-x}\right) = (1-x)^{-n} \sum_{k=0}^{n} \binom{n}{k}^2 x^k \tag{4.1}$$

$$= \frac{(1+x)^n}{(1-x^2)^n} \sum_{k=0}^{n} \binom{n}{k}^2 x^k$$

$$= \frac{1}{(1-x^2)^n} \sum_{j=0}^{n} \binom{n}{j} x^j \sum_{k=0}^{n} \binom{n}{k}^2 x^k$$

$$= \frac{1}{(1-x^2)^n} \sum_{k=0}^{2n} x^k \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-j}^2 ,$$

and therefore

$$(1-x^2)^n P_n\left(\frac{1+x}{1-x}\right) = \sum_{k=0}^{2n} x^k \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-j}^2 , \tag{4.2}$$

from which we have at once that the coefficient of x^n is the sum of the cubes of $\binom{n}{j}$ which proves the first formula (2.1) of Carlitz.

We next prove (2.2). We have again by (3.5) that

$$P_n\left(\frac{1+x}{1-x}\right) = (1-x)^{-n} \sum_{k=0}^{n} \binom{n}{k}^2 x^k ,$$

so that

$$(1-x)^{2n}\left\{P_n\left(\frac{1+x}{1-x}\right)\right\}^2 = \left(\sum_{k=0}^{n}\binom{n}{k}^2 x^k\right)^2$$

$$= \sum_{k=0}^{2n} x^k \sum_{j=0}^{k}\binom{n}{j}^2\binom{n}{k-j}^2, \qquad (4.3)$$

from which it is clear that the coefficient of x^n is the sum of the fourth powers of $\binom{n}{j}$, which proves (2.2).

Next we prove (2.3). Again by means of (3.5) we have

$$(1-x)^{2n} P_n\left(\frac{1+x}{1-x}\right) = (1-x)^n \sum_{k=0}^{n}\binom{n}{k}^2 x^k$$

$$= \sum_{j=0}^{n}(-1)^j\binom{n}{j} x^j \sum_{k=0}^{n}\binom{n}{k}^2 x^k$$

$$= \sum_{k=0}^{2n} x^k \sum_{j=0}^{k}(-1)^j\binom{n}{j}^3,$$

from which it is evident that the coefficient of x^n is the alternating sum of the cubes of $\binom{n}{j}$ and so we have the proof of (2.3).

Our proof of (2.4) proceeds as follows. We observe first that

$$\sum_{k=0}^{n}\binom{n}{k}^5 = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n}{k}^3 = \sum_{j=0}^{n}\binom{n}{j}^2\binom{n}{n-j}^3.$$

Then

$$\sum_{k=0}^{n} \binom{n}{k}^2 x^k \sum_{j=0}^{n} \binom{n}{j}^3 x^j = \sum_{k=0}^{2n} x^k \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n}{n-j}^3 ,$$

which tells us by means of (4.1) that

$$(1 - x)^n P_n\left(\frac{1+x}{1-x}\right) \sum_{j=0}^{n} \binom{n}{j}^3 x^j$$

$$= \sum_{k=0}^{2n} x^k \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n}{n-j}^3 , \qquad (4.4)$$

so that the coefficient of x^n in the left hand member is just the sum of the fifth powers of $\binom{n}{j}$ which proves (2.4).

Similar but more complicated expansions are possible when $p > 5$.

5. A new proof of Dixon's formula and some variations. In my paper [4] I expressed Dixon's series (1.5) as a sum of all positive terms, in fact a sum of the product of four binomial coefficients and the series was a convolution. Then by means of an application of the Vandermonde convolution I was able to deduce the sum in closed form. The work there also used several of the standard forms for the Legendre polynomial.

Now we give still another proof of Dixon's formula. By the standard formula (3.4), which is the one usually shown in texts where the Legendre polynomial is first introduced, we find

$$(1 - x)^{2n} P_n\left(\frac{1+x}{1-x}\right)$$

$$= 2^{-n}(1-x)^n \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \left(\frac{1+x}{1-x}\right)^{n-2k} .$$

Expanding this and recalling (2.3) we find that

$$((x^{2n}))\ (1-x)^{4n}\ P_{2n}\left(\frac{1+x}{1-x}\right)$$

$$= 2^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{2n}{k} \binom{4n-2k}{2n} \binom{2n-2k}{n}, \qquad (5.1)$$

so that we have

$$\sum_{k=0}^{2n} (-1)^{n-k} \binom{2n}{k}^3$$

$$= 2^{-n} \sum_{k=0}^{n} (-1)^{k} \binom{2n}{k} \binom{4n-2k}{2n} \binom{2n-2k}{n}$$

$$= 2^{-n} \binom{3n}{n} \sum_{k=0}^{n} (-1)^{n-k} \binom{2n}{k} \binom{4n-2k}{3n}. \qquad (5.2)$$

To use this to prove (1.5) we need to show that

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{2n}{k} \binom{4n-2k}{3n} = 2^{n} \binom{2n}{n}. \qquad (5.3)$$

Now, identity (3.102) in my book [5] states that

$$\sum_{k=0}^{[n/2]} (-1)^{k} \binom{n}{k} \binom{2n-2k}{n+r} = 2^{n-r} \binom{n}{r}.$$

Replacing n by $2n$ and setting $r = n$, we obtain (5.3), and our proof of (1.5) is done.

Now we give a variation of Dixon's formula. By means of the standard form (3.6) above we have

$$(1-x)^{2n} P_n\left(\frac{1+x}{1-x}\right)$$

$$= (1-x)^{2n} \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} \left(\frac{x}{1-x}\right)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} x^{n-k} (1-x)^{n+k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} x^{n-k} \sum_{j=0}^{n+k} (-1)^{n+k-j} \binom{n+k}{j} x^{n+k-j}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} \sum_{j=0}^{n+k} (-1)^{n+k-j} \binom{n+k}{j} x^{2n-j},$$

from which we have

$$((x^n))\ (1-x)^{2n} P_n\left(\frac{1+x}{1-x}\right)$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-k}{n} \binom{n+k}{n}. \qquad (5.4)$$

Thus by (2.3) we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-k}{n} \binom{n+k}{n}. \qquad (5.5)$$

This is formula (6.20) in my book [5].

6. Some polynomial extensions. Relation (4.1) tells us that

$$\sum_{k=0}^{n} \binom{n}{k}^2 x^k = (1-x)^n P_n\left(\frac{1+x}{1-x}\right) \qquad (6.1)$$

so that

$$\sum_{k=0}^{n} \binom{n}{k}^2 (tx)^k = (1-tx)^n P_n\left(\frac{1+tx}{1-tx}\right). \qquad (6.2)$$

From this we get

$$\sum_{k=0}^{n} \binom{n}{k}^4 t^n$$

$$= ((x^n))\ (1 - x)^n\ P_n\left(\frac{1+x}{1-x}\right)(1-tx)^n\ P_n\left(\frac{1+tx}{1-tx}\right) \qquad (6.3)$$

as an extension of Carlitz's (2.2).

In general. if we let A be defined by

$$A = \sum_{j=0}^{n} \binom{n}{j}^p (xu)^j \sum_{k=0}^{n} \binom{n}{k}^q (xv)^k$$

$$= \sum_{k=0}^{2n} x^k \sum_{j=0}^{k} \binom{n}{j}^p \binom{n}{k-j}^q u^j v^{k-j}, \qquad (6.4)$$

then we have

$$\sum_{k=0}^{n} \binom{n}{j}^{p+q} u^k v^{n-k} = ((x^n))A. \qquad (6.5)$$

This is the formula that motivates everything.

Using $p = 1$ and $q = 2$ we get two variant extensions of (2.1):

$$\sum_{k=0}^{n} \binom{n}{k}^3 v^{n-k} = ((x^n))\ (1+x)^n (1-xv)^n\ P_n\left(\frac{1+xv}{1-xv}\right) \qquad (6.6)$$

and

$$\sum_{k=0}^{n} \binom{n}{k}^3 u^k = ((x^n))\ (1+xu)(1-x)^n P_n\left(\frac{1+x}{1-x}\right). \qquad (6.7)$$

Formula (6.5) may be contrasted with the notable formula of Nanjundiah [6], [5, formula (X.15)], which says that

$$\sum_{k=0}^{n} \binom{n}{j}^{p} u^{k} v^{n-k}$$

$$= \sum_{k=0}^{[n/2]} C(p,n,k) \binom{n-k}{k} (u+v)^{n-2k} (uv)^{k}, \qquad (6.8)$$

where the C coefficients are defined recursively by

$$C(p+1,n,k) = \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} C(p,n,j), \qquad (6.9)$$

with $C(0,n,k) = (-1)^{k}$.

Other values: $C(1,n,k) = \binom{0}{k}$; $C(2,n,k) = \binom{n}{k}$; $C(3,n,k) = \binom{n}{k}\binom{n+k}{k}$.

References

1. N. G. De Bruijn, Asymptotic Methods in Analysis, North Holland Publ. Co., Amsterdam, 1958. Later editions 1961 and 1970.

2. L. Carlitz, Problem 352, Mathematics Magazine, Vol. 32(1958), 47-48.

3. A. C. Dixon, On the sum of cubes of the coefficients in a certain expansion by the binomial theorem, Messenger of Math., Vol. 20(1891), 79-80.

3. H. W. Gould, A note on a paper of Grosswald, Amer. Math. Monthly, Vol. 61(1954), 251-253.

4. H. W. Gould, Dixon's series expressed as a convolution, Nordisk Matematisk Tidskrift, Vol. 7(1959), 73-76.

5. H. W. Gould, Combinatorial Identities, A Standardized Set of Tables Listing 500 Binomial Coefficient Summations, Revised Edition, Published by the author Morgantown, W. Va. 1972.

6. T. S. Nanjundiah, On a formula of A. C. Dixon, Proc. Amer. Math. Soc. 9(1958), 308-311.

7. Chih-yi Wang, Solution of Problem 352, Mathematics Magazine, Vol. 32(1959), 224-226.