On the existence of trades of large volumes

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Abstract

Gray and Ramsay [5], showed that for any $s \geq (2t-1)2^t$, a t-(v,k) trade of volume s exists. In this note we improve their bound and show that for $t \geq 3$, a given k, and $s \geq (t-2)2^t + 2^{t-1} + 2$, there exists a simple t-(v,k) trade of volume s.

1. Introduction

Let $0 < t < k < v$ be natural numbers and let X be a v-set. For any $i, 0 \leq i \leq v, P_i(X)$ denotes the set of all i-subsets of X. The k-subsets of X are called *blocks*.

Let T_1 and T_2 be a pair of disjoint collections of blocks such that every element of $P_t(X)$ appears in precisely the same number of blocks of T_1 and T_2. Then $T = \{T_1, T_2\}$ is called a t-(v,k) *trade* or simply a t-trade. (Some authors use the notation (v,k,t) trade, instead.)

Let $T = \{T_1, T_2\}$ be a t-(v,k) trade. Clearly $|T_1| = |T_2|$. The *volume* of T, *vol*(T), is $|T_1|$. The *foundation* of T, *found*(T), is the subset of X whose elements appear in T_1 and T_2. Trades with repeated blocks are allowed, and trades with no repeated block is said to be *simple*. In what follows we only deal with simple trades.

In [6], it is shown that for any t-(v,k) trade T, $\text{vol}(T) \geq 2^t$ and $|\text{found}(T)| \geq k+t+1$. Any trade with $\text{vol}(T) = 2^t$ and $|\text{found}(T)| = k+t+1$ is called a *minimal trade*. Any minimal trade, up to isomorphism, has a unique structure. To every minimal t-(v,k) trade, a polynomial of the following form is associated:

$$T = (x_1 - x_2)(x_3 - x_4) \cdots (x_{2t+1} - x_{2t+2}) x_{2t+3} x_{2t+4} \cdots x_{k+t+1},$$

where $x_i \in$ found (T). Multiplying out, the monomials of positive sign are considered as blocks of T_1 and those of negative sign as blocks of T_2. It is well known that every t-trade is a linear combination of minimal trades

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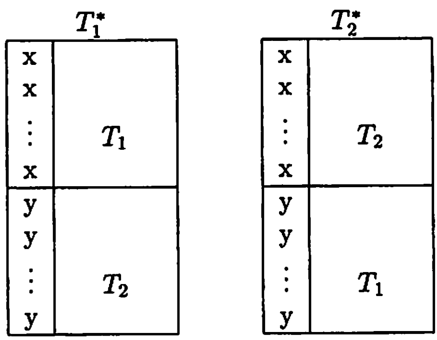
over $\mathbb{Z}$ [2,3,6]. For a rather comprehensive review on trades, the reader is referred to [7].

The following lemmas appear in the literature, and we include them here for the sake of completeness.

Lemma 1 has appeared in [8] and demonstrates a simple construction.

Lemma 1. If a t-(v, k) trade T of volume s exists, then a $(t+1)$-$(v', k+1)$ trade of volume $2s$ exists.

Proof. Suppose that $T = \{T_1, T_2\}$ is a t-(v, k) trade. By augmenting the blocks of T_1 and T_2 with new elements x and y according to the following scheme, a $(t+1)$-trade $T^* = \{T_1^*, T_2^*\}$ with block size $k+1$ and volume $2s$ can be constructed. If T is a simple trade, then T^* is also simple. □



Lemma 2 is a special case of Lemma 3.6 of [4].

Lemma 2. If a t-$(v,\ k)$ trade T of vol$(T) = s$ exists, then a t-(v', k) trade T^* with vol$(T^*) = s + 2^t - 1$ exists.

Proof. Let $T = \{T_1, T_2\}$ and let $x_1x_2\cdots x_k$ be a block of T_2. Now, if for $i = 1, \cdots, t+1, a_i \notin$ found(T), then

$$T^* = T + (x_1 - a_1)\cdots(x_{t+1} - a_{t+1})x_{t+2}\cdots x_k$$

is a t-trade with blocksize k and vol$(T^*) = s + 2^t - 1$, since the block $x_1x_2\cdots x_k$ cancels out. We note that if T is a simple trade, then T^* would be simple too. □

Lemma 3 is well known (see [1]).

Lemma 3. There exist 3-$(v, 4)$ trades of volume s for $s > 13$.

Proof. In [7], 2-trades with $k = 3$ and volumes $s = 6, 7, 8, 9, 10$ have been listed. By Lemma 1, 3-$(v, 4)$ trades of volumes $s = 12, 14, 16, 18, 20$ exist. On the other hand, by Lemma 2, 3-$(v, 4)$ trades of volumes $8 + 2^3 - 1 = 15, 12 + 2^3 - 1 = 19$, and $14 + 2^3 - 1 = 21$ exist too. Below, a 3-trade with $k = 4$ and volume 17 is given:

	1	1	1	1	1	1	1	1	1	2	2	2	2	3	3	4	6
T_1	2	2	2	2	2	2	3	4	5	3	4	4	5	4	4	6	7
	3	3	4	6	7	8	6	7	7	6	5	6	6	5	5	7	8
	4	5	5	7	8	9	8	9	8	8	7	9	8	6	8	8	9
	1	1	1	1	1	1	1	1	1	2	2	2	2	3	3	4	4
T_2	2	2	2	2	2	2	3	6	7	3	4	6	6	4	5	5	6
	3	3	4	4	5	5	4	7	8	4	5	7	8	6	6	7	7
	6	8	7	9	7	8	5	8	9	5	6	8	9	8	8	8	9

Thus, 3-trades with $k = 4$ and $14 \leq s \leq 21$ exist. Now, by adding sufficient copies of minimal 3-trades with $k = 4$ and with disjoint foundations to the 3-trades of volumes $14 \leq s \leq 21$, any 3-$(v, 4)$ trade of volume $s, s \geq 14$ is obtained. □

2. Main Result

Lemma 4. If, for any $s, s \geq s_0$, a t-$(v, t+1)$ trade of volume s exists, then for any $s \geq 2s_0 + 2^{t+1} - 2$, a $(t+1)$-$(v', t+2)$ trade of volume s exists.

Proof. If a t-$(v, t+1)$ trade of volume s exists, then by Lemma 1, a $(t+1)$-$(v', t+2)$ trade of volume $2s$ can be constructed. Therefore, for any $s \geq s_0$, if a t-trade with $k = t+1$ and volume s exists, then for any s even such that $s \geq 2s_0$, a $(t+1)$-trade with $k = t+2$ of volume s exists. Consequently, by Lemma 2, we conclude that for any s odd such that $s \geq 2s_0 + 2^{t+1} - 1$, a $(t+1)$-trade with $k = t+2$ of volume s does exist. Therefore, for any $s, s \geq 2s_0 + 2^{t+1} - 1 - 1$, a $(t+1)$-$(v', t+2)$ trade of volume s exists. □

Theorem 5. For $t \geq 3$ and for any s, $s \geq (t-2)2^t + 2^{t-1} + 2$, a t-(v, k) trade of volume s exists.

Proof. If a t-trade with $k = t+1$ exists, then by adding $k - (t+1)$ new elements to all the blocks of the trade, a t-trade with $k > t+1$ is obtained. (This method has already been used in [4].) Therefore, it suffices to prove the theorem for $k = t+1$. We prove this by induction on t.

For $t = 3$, Lemma 3 establishes the base case. Now assume that a t-$(v, t+1)$ trade of volume s such that $s \geq (t-2)2^t + 2^{t-1} + 2 = s_0$ exists, then by Lemma 4, a $(t+1)$-$(v', t+2)$ trade of volume s such that

$$\begin{aligned} s \geq 2s_0 + 2^{t+1} - 2 &= (t-2)2^{t+1} + 2^t + 4 + 2^{t+1} - 2 \\ &= ((t+1)-2)2^{t+1} + 2^{(t+1)-1} + 2, \end{aligned}$$

exists. Now the proof is complete. □

Example. For $t = 4$, by Theorem 5, a 4-trade with a given $k \geq 5$ and volume s, $s \geq 42$, exists. To construct a 4-trade with $k = 5$ with $s = 43$, we observe that $s = 43 = 28 + 15 = 2 \times 2 \times 7 + 2^4 - 1$. We start with a 2-$(v, 3)$ trade of volume 7 and by employing Lemma 1 twice and Lemma 2 once, the desired trade is obtained:

$$T = (8 - A)(9 - B)(123 + 124 + 157 + 167 + 267 + 347 + 456 - 126 - 127 - 137 - 145 - 234 - 467 - 567) + (1 - C)(2 - D)(6 - E)(8 - F)(9 - G).$$

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