Dissolved Graphs and Strong Perfect Graph Conjecture

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Abstract

We define a new graph operation called "dissolve $N(v)$ into v" where $N(v)$ is the set of vertices adjacent to a vertex v and characterize odd cycles of length greater than 5 in terms of p-critical graphs using this operation. This enable us to re-phrase the Strong Perfect Graph Conjecture.

Introduction

We consider only simple graphs and refer to [[1], [2], [3]] for terminology. In a graph $G(V, E)$, let $N(v)$ denote the set of all vertices adjacent to v in G and $N[v] := N(v) \bigcup \{v\}$, $N_2(v) := \{x \in V(G)$: the shortest path joining x and v is of length 2 in $G\}$ and $N_c(v) := V(G) - N[v] - N_2(v)$. Let G^c denote the complement of G. As usual let $\alpha(G)$, $\omega(G)$ and $\chi(G)$ respectively denote the independence number, clique number and the vertex chromatic number of G.

A graph G is perfect if $\chi(H) = \omega(H)$ for all induced subgraphs H of G. A graph G is p-critical if G is not perfect but every proper induced subgraph of G is perfect. The celebrated Strong Perfect Graph Conjecture (SPGC) of C. Berge states that the only p-critical graphs are C_{2n+1} and C^c_{2n+1}, $n \geq 2$.

We require a few properties of p-critical graphs [[3], [4], [5], [6]].

Every p-critical graph G on n vertices with $\alpha = \alpha(G)$ and $\omega = \omega(G)$, has the following properties.

(1) $n = \alpha\omega + 1$; $\alpha, \omega \geq 2$.

(2) It has exactly n maximum cliques and n maximum independent sets.

(3) Every vertex of G is contained in exactly ω maximum cliques and in exactly α maximum independent sets.

(4) Each maximum clique intersects all but one maximum independent set and each maximum independent set intersects all but one maximum clique.

(5) For any vertex v in G, $|N(v)| \geq 2\omega - 2$, $\omega(G - v) = \omega(G)$ and $\alpha(G - v) = \alpha(G)$.

(6) $\omega(G) = 2$ if and and only if G is an odd cycle of length greater than 3.

In this paper, we define a new graph operation " dissolve $N(v)$ into v" and characterize odd cycles of length greater than 5 in terms of p-critical graphs.

Definition: Let $G(V, E)$ be a graph and $v \in V(G)$. Let G_v be the graph obtained from G after *dissolving* $N(v)$ into v, where $V(G_v) = V(G) - N(v)$ and $E(G_v) = (E(G) - \{xy \in E(G) : x \in N[v]\}) \bigcup \{vy : y \in N_2(v)\}$. The graph G_v is called the dissolved graph of G with respect to v (see figures 1 and 2).

We note that the dissolved graph of a bipartite (chordal) graph is bipartite(chordal).

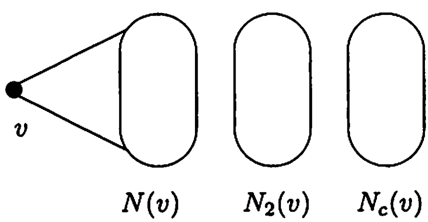

Figure 1 : G

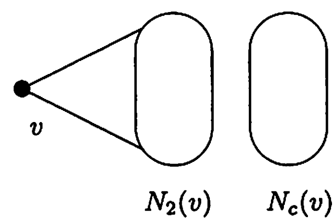

Figure 2 : G_v

Theorem 1 *Let G be a graph and $v \in V(G)$. Then G and G_v are p-critical if and only if $G \cong C_{2n+1}$, $n \geq 3$.*

Proof:

If $G \cong C_{2n+1}$, $n \geq 3$, then $G_v \cong C_{2n-1}$, $n \geq 3$ and so G and G_v are p-critical.

Next, suppose that G and G_v are p-critical. By the property(3) of p-critical graphs, $\alpha(G - N[v]) = \alpha(G) - 1$. We split $V(G) - \{v\}$ into 3 parts: $N(v)$, $N_2(v)$ and $N_c(v)$ (see figures 1 and 2). $N_c(v) \neq \phi$; else G_v is perfect, since it is then obtained by joining v with every vertex of $G[N_2(v)]$, a contradiction. Next we claim that $\alpha(G_v) = \alpha(G) - 1$. Clearly, $\alpha(G_v) \geq \alpha(G - N[v]) = \alpha(G) - 1$. If G_v has an independent set of size greater than $\alpha(G) - 1$, then $\alpha(G - N[v]) = \alpha(G_v - v) = \alpha(G_v) > \alpha(G) - 1$, a contradiction. For future use we note that $\alpha(G) = \alpha(G_v) + 1 \geq 3$.

We next to show that $\omega(G_v) = \omega(G)$. Since $N_c(v) \neq \phi$, we have $\omega(G - N[v]) = \omega(G)$ by property (3) of p-critical graphs. So $\omega(G_v) = \omega(G_v - v) = \omega(G - N[v]) = \omega(G)$.

Using property (1) and equations $\alpha(G_v) = \alpha(G) - 1$ and $\omega(G_v) = \omega(G)$, we deduce that

(7) $\quad |V(G_v)| = \alpha(G_v)\omega(G_v) + 1 = (\alpha(G) - 1)\omega(G) + 1$

In G_v, there are $\alpha(G) - 1$ independent sets of size $\alpha(G) - 1$ containing v. So there are $\alpha(G) - 1$ independent sets of size $\alpha(G) - 2$ in the subgraph induced by $N_c(v)$; let I_1 , I_2 be two of them.

Let C be a clique of size $\omega(G)$ containing v in G. Then $N(v) - C$ is a clique, else there exist x, y in $N(v) - C$ such that $xy \notin E(G)$ and we arrive at a contradiction. The sets $I^j = I_j \bigcup \{x, y\}$, $j = 1, 2$ are distinct independent sets of size $\alpha(G)$. But $C \bigcap I^j = \phi$, $j = 1, 2$, a contradiction to property (4).

Since $N(v) - C$ is a clique and $|N(v)| \geq 2\omega(G) - 2$ (by property (5)), $|N[v] - C| = \omega(G) - 1$. So $|N(v)| = |C| - 1 + |N[v] - C| = 2\omega(G) - 2$. Since $|V(G)| = |V(G_v)| + |N(v)|$, by (7) we have $\alpha(G)\omega(G) + 1 = (\alpha(G) - 1)\omega(G) + 1 + 2\omega(G) - 2$, and hence $\omega(G) = 2$. By property (6), G is an odd cycle. Since $\alpha(G) \geq 3$, $G \cong C_{2n+1}$, $n \geq 3$.

□

Theorem 2 *The following statements are equivalent.*

(a) If G is p-critical, then $G \cong C_{2n+1}$ or C^c_{2n+1}, $n \geq 2$.

(b) If G is p-critical and $\alpha(G) \geq 3$, then G_v is p-critical, $v \in V(G)$.

Proof:

By Lovász's perfect graph theorem ([4]), G is p-critical if and only if G^c is p-critical. Using this fact and property (6), it is easy to show that (a) and (b) are equivalent.

□

References

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