Obstruction Sets for Outer-Bananas-Surface Graphs

Luis Boza
Departamento de Matemática Aplicada I. Univ. de Sevilla.
Avda Reina Mercedes 2, 41012-SEVILLA. E-mail: boza@us.es
Eugenio M. Fedriani
Departamento de Economía y Empresa. Univ. Pablo de Olavide.
Ctra. de Utrera, Km.1. 41013-SEVILLA. E-mail: efedmar@dee.upo.es
Juan Núñez
Departamento de Geometría y Topología. Univ. de Sevilla.
Apdo. 1160. 41080-SEVILLA. E-mail: jnvaldes@us.es

Abstract

Let B_2 be the bananas surface arising from the torus by contracting two different meridians of the torus to a simple point each. It was proved in [8] that there is not a finite Kuratowski theorem for B_2.

A graph is outer-bananas-surface if it can be embedded in B_2 so that all its vertices lie on the same face. In this paper, we prove that the class of the outer-B_2 graphs is closed under minors. In fact, we give the complete set of 38 minor-minimal non-outer-B_2 graphs and we also characterize these graphs by a finite list of forbidden topological minors.

We also extend outer embeddings to other pseudosurfaces. The S pseudosurfaces treated are spheres joined by points in such a way that each sphere has two singular points. We give an excluded minor characterization of outer-S graphs and we also give an explicit and finite list of forbidden topological minors for these pseudosurfaces.

1 Introduction

Robertson and Seymour proved in [8] that if a class of graphs is closed under minors, it is possible to give a finite list of forbidden minors. Then the class of the graphs which admit an embedding in a surface has got a finite list of forbidden minors. As can be read in [4] you can obtain a finite list of forbidden topological minors when you have a finite list of forbidden minors. But there are pseudosurfaces in which it is not possible to deal with a finite list of forbidden minors. In fact, Širáň and Gvozdjak [9] showed this property using *bananas surface*. Later, we will turn to this pseudosurface.

A planar graph is outerplanar if it can be embedded in the plane so that all its vertices lie on the same face. Chartrand and Harary showed in [5] that a graph is outerplanar if and only if it does not contain a subdivision of either K_4 or $K_{2,3}$.

Outerplanar embeddings also extend to any surface and any pseudosurface: a graph is outer-S if it embeds in S so that there is one face whose boundary contain all the vertices. Several generalizations of Chartrand and Harary's Theorem appear from this definition. We are going to study some embeddings as the ones described in [3], where pinch points can be out of the set of vertices of the graph.

If the graphs with embeddings in S are characterized in terms of forbidden topological minors, it is feasible to deduce the characterization for outer-S graphs. Cáceres does this in [4]. So the classification of outer-projective-planar graphs is due to Cáceres [4], Archdeacon, Hartsfield, Little, and Mohar [2], and Revuelta [7] using 32 forbidden minors and 45 forbidden topological minors.

Another extension of outerplanarity is the concept of k-outerplanar graph: a planar graph that admits an embedding with all vertices in k faces at most.

2-outerplanar graphs were characterized by Cáceres [4], Archdeacon, Bonnington, Dean, Hartsfield, and Scott [1], and Revuelta [7]. They gave 38 forbidden minors and 56 forbidden topological minors. These forbidden minors are K_5, $K_{3,3}$ and the following ones in Figure 1.

And the 56 forbidden topological minors are the 38 preceding minors and the 18 graphs given in Figure 2.

In the following sections we connect 2-outerplanar graphs with outer-S graphs, S being some pseudosurfaces. Then new characterization in several pseudosurfaces will be allowed.

2 Outer-Bananas-Surface graphs

Bananas surface, B_2, is the 2-amalgamation of two spheres sharing the two singular points P_1 and P_2 (as in Figure 3). An excluded minor characterization of graphs with an embedding in B_2 cannot be given and there is not a finite list of forbidden topological minors. In order to give a characterization of outer-B_2-graphs we use the following result.

Proposition 2.1 *Let G be a connected graph. G is outer-B_2 if and only if G is 2-outerplanar.*

Proof. Sufficiency: If G is a 2-outerplanar graph, there is an embedding in the sphere so that all its vertices lie in one or two faces. If you choose

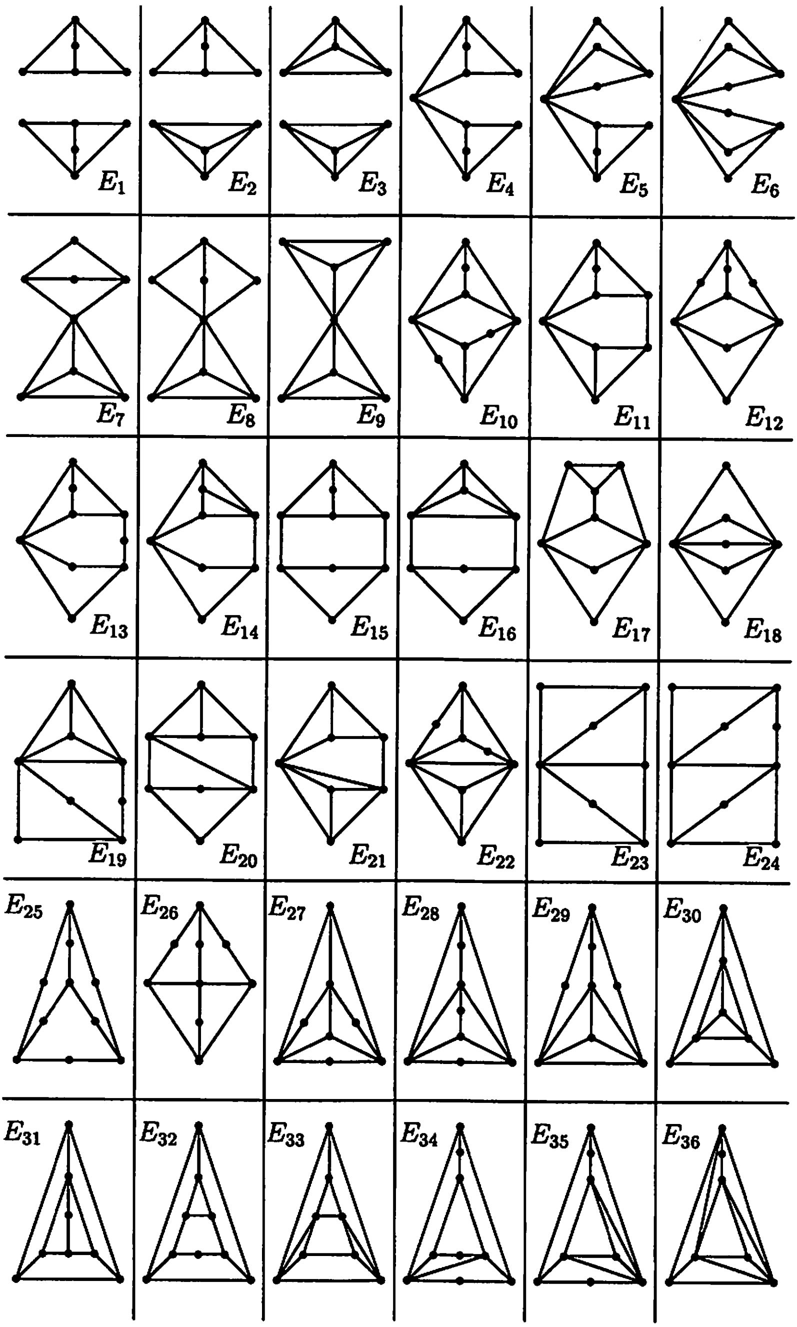

Figure 1: Planar forbidden minors for 2-outerplanarity.

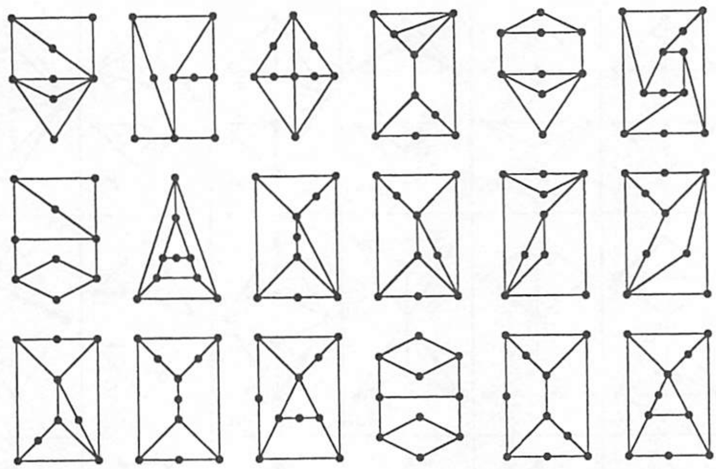

Figure 2: Other forbidden topological minors for 2-outerplanarity.

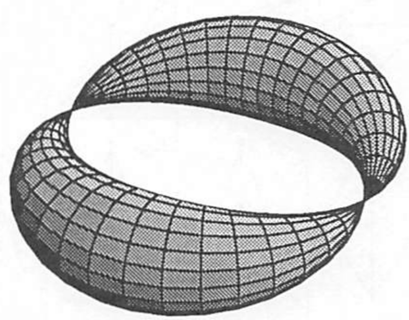

Figure 3: B_2.

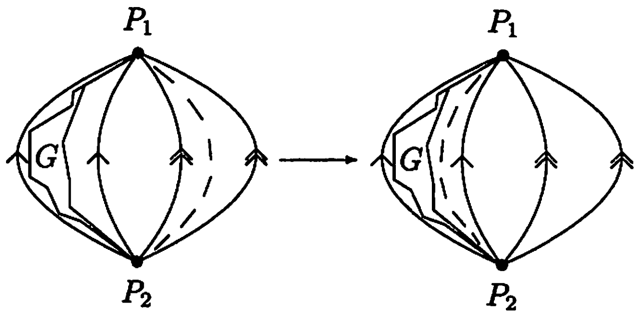

Figure 4: Case 3b for proving G is 2-outerplanar.

one point in each of these faces and you glue a sphere by both points, G will be embedded on B_2 with all its vertices in the same face.

Necessity: We consider an outer-B_2 embedding Γ of G, and we call P_1 and P_2 the singular points of B_2 and C the face of Γ where all the vertices of G are. We can distinguish three cases:

1) Neither P_1 nor P_2 are in G. In this case, G is embedded in one sphere with all its vertices in at most two faces.

2) P_1 is in G and P_2 is not (or vice-versa). So G is formed by two graphs sharing one vertex: one of them is outerplanar and the other is outerplanar except on this vertex (at most). Therefore G is outerplanar.

3) P_1 and P_2 are in G. We do not lose any piece of information although we suppose P_1 and P_2 are vertices. Obviously one sphere has not got any vertex out of P_1 and P_2 because C is entirely contained in the other sphere. Here we have two possibilities:

3a) P_1 and P_2 are not adjacent. Then G is embedded in one sphere with all its vertices in one face.

3b) The edge P_1P_2 is present in G. In one sphere there is at most one edge (P_1P_2).

P_1 and P_2 are in C. Then there exists a path from P_1 to P_2 into face C, so you can use this path to draw P_1P_2 if necessary (see Figure 4). Then G is 2-outerplanar because only two new faces form C. □

We are describing the general situation:

Theorem 2.2 *Let G be a graph. G is outer-B_2 if and only if G is 2-outerplanar.*

Proof. We can remove every outerplanar connected component of G without modifying either the 2-outerplanarity of G or its condition of outer-B_2.

If G has more than one non-outerplanar connected component, the graph will not be 2-outerplanar or outer-B_2. Hence this demonstration holds only studying the connected case proved in Proposition 2.1. □

Corollary 2.3 *Let G be a graph. The following statements are equivalent:*

1. *G is outer-B_2.*
2. *G has not got any minor among K_5, $K_{3,3}$ or one of the 36 minors given in Figure 1.*
3. *G has no subgraph homeomorphic to one of the 38 minors or one of the 18 subgraphs given in Figure 2.*

□

We have given a finite and excluded subgraph characterization of outer-B_2 graphs although there is not a finite number of forbidden topological minors which characterize the graphs that admit an embedding in B_2 (see [9]).

According to the preceding results it is easy to characterize outer-B_n graphs.

3 Outer-embeddings in other pseudosurfaces

Let B_n be the pseudosurface arising from the torus by contracting each of n different meridians of the torus to a simple point. These n points are called *singular points of B_n* and denoted by $P_1, P_2, \ldots, P_n$.

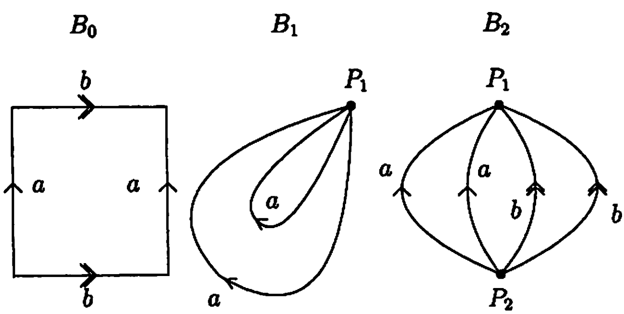

Figure 5: The simplest pseudosurfaces.

For instance, B_3 is given in Figure 6.

We are going to characterize outer-B_n graphs.

Theorem 3.1 *Let G be a graph. G is outer-B_n ($n \geq 2$) if and only if G is 2-outerplanar.*

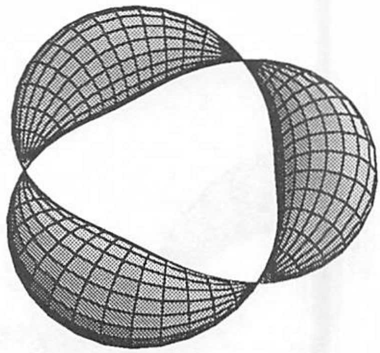

Figure 6: B_3.

Proof. If G is connected, it is easy to prove the sufficiency by induction in n. When proving the necessity we should distinguish how many singular points there are in G, as was done in the case of B_2.

If G is non-connected, you can use the proof given in Theorem 2.2. □

Let L_n denote ($n \geq 2$) the connected pseudosurface with only two singular points and with n spheres which each have both singular points. Now we are going to characterize the graphs which have an embedding in L_n.

Obviously, every outer-B_2 graph is outer-L_n. But there are outer-L_n non outer-B_2 graphs. For instance, we consider L_3 and let G be the graph $G = K_4 \cup K_4$ (disjointed union as in Figure 8). G is not 2-outerplanar since G has two non-outerplanar connected components. However, it is intuitively obvious that G can be embedded in B_2 with all its vertices in two faces which each have one singular point. So G is outer-L_3 but it is not 2-outerplanar.

We can say something else about 2-connected graphs:

Lemma 3.2 *Let G be a 2-connected graph. G is outer-L_n ($n \geq 2$) if and only if G is 2-outerplanar.*

Proof. If G is a 2-connected graph and it admits an embedding in L_n, then

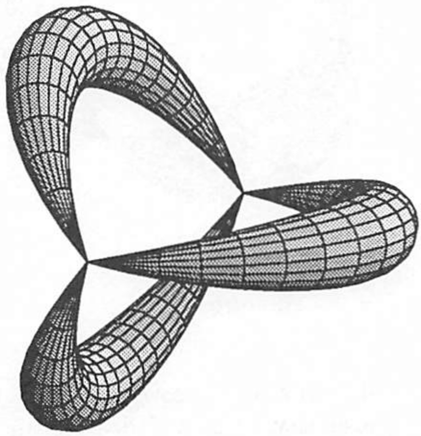

Figure 7: L_3.

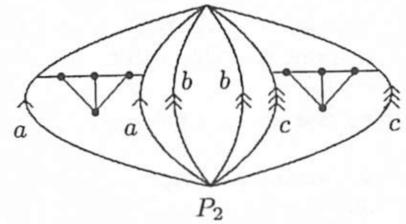

Figure 8: $K_4 \cup K_4$ in L_3.

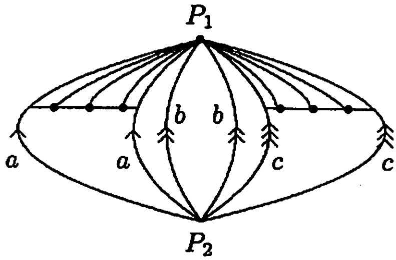

Figure 9: Two K_4 sharing one vertex.

this embedding uses one sphere and another sphere only for one edge. So G is outer-L_n ($n \geq 2$) if and only if G is outer-L_2. □

As we will see, similarly to Theorem 2.2, it is possible to characterize outer-L_n graphs ($n > 2$) using the following lemmas:

Lemma 3.3 *Let G be a graph with $n-1$ 2-outerplanar connected components ($n \geq 2$). If the other connected components in G are outerplanar, G is outer-L_n.* □

Lemma 3.4 *Let G be a graph with n non-outerplanar connected components ($n \geq 2$). Then G is not outer-L_n.* □

Lemma 3.5 *E_5, E_6, and E_8 from Figure 1 are not outer-L_n ($n \geq 2$).* □

We now present, for the sake of an example, another outer-L_3 graph that is not a 2-outerplanar graph:

Let G be the graph obtained with two K_4s sharing one vertex. This is not 2-outerplanar (see the list given before). But there exists an L_3-embedding with all its vertices in only one face.

We find it convenient to consider the 3 minors in Figure 10. As it will be seen, we denote this family of graphs $\mathcal{F}_2 = \{F_{2,0}, F_{2,1}, F_{2,2}\}$, where $F_{i,j}$ has i blocks and $2j$ vertices that are not adjacent to a central vertex.

To classify outer-L_3 graphs, we need the following result:

Lemma 3.6 *Let G be a connected graph. If G is outer-L_3 but it is not 2-outerplanar, it has got at least one of the 3 subgraphs in $\mathcal{F}_2 = \{F_{2,0}, F_{2,1}, F_{2,2}\}$.*

Proof. If G is not 2-outerplanar, then it has one subgraph from the list given in Figure 1 and Figure 2. By Lemmas 3.2 and 3.3, only E_1, E_2, E_3, $E_4 = F_{2,2}$, $E_7 = F_{2,1}$, and $E_9 = F_{2,0}$ in this list are outer-L_3. But there is

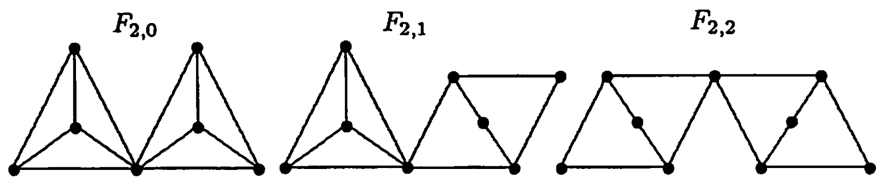

Figure 10: Outer-L_3 non 2-outerplanar graphs.

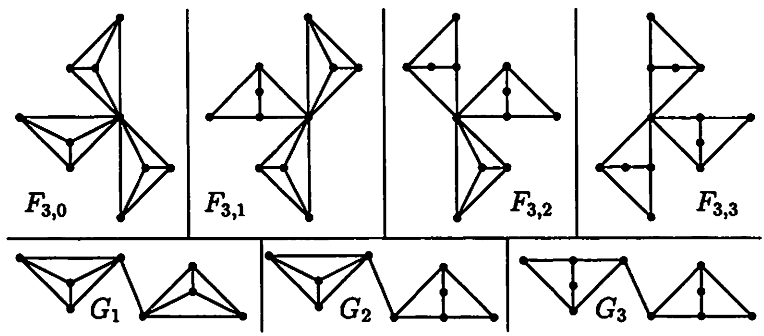

Figure 11: Some non outer-L_3 graphs.

not any connected and outer-L_3 graph with one subgraph in $\{E_1, E_2, E_3\}$. □

As a consequence of the preceding lemmas, we have the main result for outer-L_3 graphs:

Theorem 3.7 *Let G be a graph. G is outer-L_3 if and only if it has not got any minor among the following ones: K_5, $K_{3,3}$, $K_4 \cup F_{2,i}$, or $K_{2,3} \cup F_{2,i}$ (with $i \in \{0,1,2\}$), E_5, E_6, E_8, E_j from Figure 1 (with $10 \le j \le 36$), the disjointed union of three graphs chosen between $\{K_4, K_{2,3}\}$ and, finally, none of the 7 minors given in Figure 11: $F_{3,0}$, $F_{3,1}$, $F_{3,2}$, $F_{3,3}$, G_1, G_2, and G_3.*

Proof. Let G be a non outer-L_3 graph. It must happen at least one of the following possibilities:

- G has got more than two non-outerplanar connected components. Then there is a minor of G among $K_4 \cup K_4 \cup K_4$, $K_4 \cup K_4 \cup K_{2,3}$, $K_4 \cup K_{2,3} \cup K_{2,3}$, and $K_{2,3} \cup K_{2,3} \cup K_{2,3}$.
- G has exactly two non-outerplanar components and one of them is not 2-outerplanar. This component has got one minor among the 36

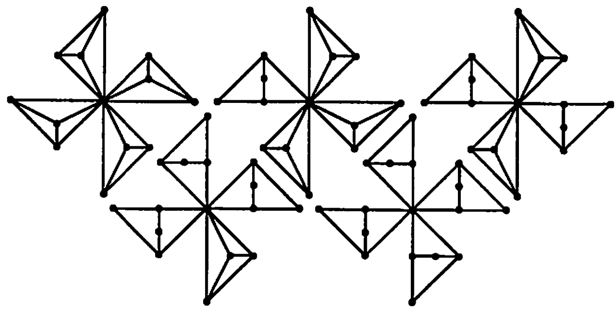

Figure 12: Some non outer-L_4 graphs.

forbidden minors for 2-outerplanarity and the other component has got a minor between K_4 and $K_{2,3}$, thus there is a minor of G among one of 72 graphs although some of them can be redundant.

- G has only one non-outerplanar connected component. This component is not 2-outerplanar, therefore it has got a forbidden minor for 2-outerplanarity. If this minor is 2-connected, it is one E_j (with $10 \leq j \leq 36$) from Figure 1. In the opposite case, G has got one minor in $E_1, \ldots, E_{10}$, but some of these minors are outer-L_3. Lemma 3.6 helps us to add only ten minors: $F_{3,0}$, $F_{3,1}$, $F_{3,2}$, $F_{3,3}$, G_1, G_2, G_3 (from Figure 11), E_5, E_6, and E_8 (from Figure 1). E_5, E_6, and E_8 are neither 2-outerplanar nor outer-L_3 because they are in Figure 1 and they do not belong to $\mathcal{F}_2$. The graphs from Figure 11 are the effect when adding something to E_1, E_2, E_3, E_4, E_7, or E_9 making minimal non-outer-L_3-graphs.

Deleting spare minors, the Theorem holds. □

Remark 3.8 Three of these minors have bridges which can not be removed or contracted.

As in Lemma 3.6, we can consider the set of the connected outer-L_4 graphs which are not 2-outerplanar. If G is a graph in this set, it will have one minor in $\mathcal{F}_2$ or even in $\mathcal{F}_3$ (the four graphs $F_{3,i}$ with $0 \leq i \leq 3$ in Figure 11).

For L_5, we need to define $\mathcal{F}_4$ as the set of the 5 graphs in Figure 12.

As a general rule, a graph in $\mathcal{F}_n$ is formed by n blocks chosen in $\{K_4, K_{2,3}\}$ and sharing one vertex (which have minimum degree in $K_{2,3}$ blocks).

Now an excluded minor characterization of outer-L_n graphs is also known when $n \geq 4$.

Theorem 3.9 *Let G be a graph. G is outer-L_n if and only if it has not got any minor among the following ones:*

- K_5;
- $K_{3,3}$;
- *the* $n+1$ *minors in* $\mathcal{F}_n$;
- *the* $n+1$ *non-connected minors formed by* n *components in* $\{K_4, K_{2,3}\}$;
- *the 6 non-connected minors* $K_4 \cup F_{2,i}$, $K_{2,3} \cup F_{2,i}$, *with* $i \in \{0,1,2\}$ *($F_{2,i}$ were given in Lemma 3.6);*
- *the 33 connected minors* G_1, G_2, G_3 *(from Figure 11),* E_5, E_6, E_8, E_j *with* $10 \leq j \leq 36$ *(from Figure 1).*

Proof. Let G be a non-outer-L_n graph. There are three possibilities:

- G has more than $n-1$ non-outerplanar connected components. Then it has got one minor that is the disjointed union of n graphs in $\{K_4, K_{2,3}\}$ (there are $n+1$ forbidden minors here).
- G has got a non-2-outerplanar component and other component that is not outerplanar. In this case, there exists a minor of G with two connected components: the first is one forbidden minor for 2-outerplanarity and the second is K_4 or $K_{2,3}$ (there are 76 minors here but lots of them can be spare minors in our list).
- G has got a non-2-outerplanar component out of $\mathcal{F}_j$ $(1 \leq j \leq n-1)$. If it is not 2-connected, then there exists a minor of G that either is in $\mathcal{F}_n$ or is a graph of $\mathcal{F}_2$ with an edge in the midst of its blocks.

Finally, deleting the spare graphs, we have the forbidden minors for outer-L_n graphs. □

That is $43+2n$ forbidden minors when $n \geq 3$ (for $n = 2$, there are only 38 forbidden minors).

As we have seen, the class of outer-L_n graphs is closed under minors. We have given the complete set of $43+2n$ minor-minimal non-outer-L_n graphs.

Using the method given in [7], we can find a list of forbidden topological minors. For L_n $(n \geq 2)$, you only need to add the 18 subgraphs from Figure 2 to the list of forbidden minors given in each case.

References

[1] D. Archdeacon, C.P. Bonnington, N. Dean, N. Hartsfield, and K. Scott, "Obstruction sets for outer-cylindrical graphs". Pending publication, 1997.

[2] D. Archdeacon, N. Hartsfield, C.H.C. Little, and B. Mohar, "Obstruction sets for outer-proyective-planar graphs. *Ars Combinatoria* **49** (1998), 113–127.

[3] L. Boza, E.M. Fedriani, and J. Núñez, "The problem of outer-embeddings in pseudosurfaces". *Ars Combinatoria*, in press.

[4] J. Cáceres, "Diversos tipos de planaridad en grafos". Ph.D. dissertation, Dpto. de Geometría y Topología, Universidad de Almería (Spain), 1996.

[5] G. Chartrand and F. Harary, "Planar permutation graphs". *Ann. Inst. H. Poincaré Sect. B.* **3** (1967), 433–438.

[6] F. Harary, "Graph Theory". *Addison Wesley Reading Mass*, 1969.

[7] M.P. Revuelta, "Inmersiones de grafos en superficies tubulares de género finito". Ph.D. dissertation, Dpto. de Matemática Aplicada I. Univ. de Sevilla (Spain), 1999. 1999.

[8] N. Robertson and P.D. Seymour, "Graph minors VIII. A Kuratowski Theorem for general surfaces". *J. Combinatorial Theory Ser. B*, **48** (1990), 255–288.

[9] J. Širáň and P. Gvozdjak, "Kuratowski-type theorems do not extend to pseudosurfaces". *Journal of Combinatorial Theory Ser. B*, **54** (1992), 209–212.