New graceful families on bipartite graphs

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Abstract

A graph with a graceful labeling (an α-labeling) is called a graceful (λ-graceful) graph. In this paper six methods for constructing bigger graceful graphs from a given graceful graph or a set of given λ-graceful graphs are provided. Two of which generalize Koh and others Theorems in [2, 3].

1. Introduction

Let $G = (V, E)$ be a connected graph with n (≥ 1) edges. A *graceful labeling* of $G = (V, E)$ is an injection $\theta : V \to \{0, 1, \cdots, n\}$ such that the corresponding induced function given by $\theta^*(e) = |\theta(u) - \theta(v)|$, for each edge $e = (u, v)$, is 1-1. A graceful labeling is an *α-labeling* if there is an integer λ such that, for each edge (u, v), either $\theta(u) \leq \lambda < \theta(v)$ or $\theta(v) \leq \lambda < \theta(u)$ (i.e., the larger label on the edge is bigger than λ and the smaller label is at most λ). For convenience, an α-labeling is said to be a λ-graceful labeling. The integer λ will be called the *boundary value* of the λ-graceful labeling. Clearly, a graph admitting a λ-graceful labeling is necessarily bipartite. For results on the construction of graceful graphs, the reader can refer to [2, 3, 4, 5, 6, 7, 10, 11] or a dynamic survey [1].

Let a_i and b_i ($1 \leq i \leq n$) be distinct elements of Z_{n+1}, the set of all residue classes modulo $n + 1$. A sequence $\{(a_1, b_1), \cdots, (a_n, b_n)\}$ is called a *graceful sequence* $S(n)$ if $|a_i - b_i| = i$ for $1 \leq i \leq n$. Each (a_i, b_i) ($1 \leq i \leq n$) is a *term* of a graceful sequence. Clearly, $(a_n, b_n) = (n, 0)$. We shall denote by $E(S(n))$ the set of distinct elements of terms in $S(n)$. The following result is readily given and its proof is omitted (refer to Figure 1).

Lemma 1.1. [11] *There is a one-to-one correspondence between a graph G with n edges having a graceful labeling θ and a graceful sequence $\{(a_1, b_1), \cdots, (a_n, b_n)\}$ with n terms. The correspondence is given by*

$$a_i = max\{\theta(u), \theta(v)\},$$
$$b_i = min\{\theta(u), \theta(v)\}, \qquad 1 \leq i \leq n,$$

where u, v are the ends of the edge labeled i.

A graph corresponding to a graceful sequence $S(n)$ is called an *induced graceful graph* of $S(n)$, denoted $G(S(n))$; a graceful sequence corresponding to a graph G with n edges and a graceful labeling is said to be the *induced graceful sequence* of G. For example, the induced graceful graph $G(S(8))$ corresponding a graceful sequence $S(8) = \{(1, 0), (8, 6), (6, 3), (5, 1), (6, 1), (8, 2), (7, 0), (8, 0)\}$ is shown in Figure.1. Two graceful sequences $S(n)$ and $S'(n)$ are said to be *equivalent* (written $S(n) \sim S'(n)$) if there is a bijection $\theta : E(S(n)) \to E(S'(n))$ such that $(a_i, b_i) \in S(n)$ if and only if $(\theta(a_i), \theta(b_i)) \in S'(n)$. By inspection, we have the following:

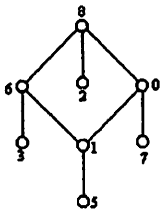


Figure 1.

Lemma 1.2. [11] *$S(n) \sim S'(n)$ if and only if $G(S(n)) \cong G(S'(n))$.*

Let G be a graph with n edges and a graceful labeling θ. The *complementary labeling* θ_c (see [8, 9]) of G is defined as $\theta_c(v) = n - \theta(v)$ for each vertex $v \in V$. In order to define another labeling associated with a λ-graceful labeling, let the graph G be bipartite with bipartition (A, B), and let θ be a λ-graceful labeling of G, where $\theta(v) \le \lambda$, if $v \in A$ and $\theta(v) > \lambda$, if $v \in B$. The *reverse labeling* θ_r (see [9]) of G is given by

$$\theta_r(v) = \begin{cases} \lambda - \theta(v), & if\ v \in A, \\ n + \lambda + 1 - \theta(v), & if\ v \in B. \end{cases}$$

Observe that if θ is a λ-graceful labeling of G, then θ_c is also a λ-graceful labeling of G with $\lambda = n - \lambda - 1$, and θ_r itself is still a λ-graceful labeling of G. It is easy to see that if $\theta(v) = 0$ for some vertex $v \in V$, then $\theta_c(v) = n$, and if $\theta(u) = \lambda$ and $\theta(v) = \lambda + 1$ for some vertex $u \in A$ and $v \in B$, then $(\theta_r\theta_c)(u) = \theta_r(v) = n$. Notice that the vertex in a graceful graph with label n plays an important role for the construction of new graceful graphs in Section 2.

2. The main results

From now on, we will assume that G is a bipartite graph with n edges. For brevity, the graph G with n edges having a graceful labeling or a λ-graceful labeling will be denoted by $G(n)$ or $G(n, \lambda)$. A graph G is called *graceful* (*λ-graceful*) if G has a graceful (λ-graceful) labeling. By $d(n, a)$ denote the *length* of the shortest path joining the vertex with label n and the vertex with label a in $G(n)$; in particular, $d(n, n) = 0$. Observing the vertices of $G(n)$ and $G(n, \lambda)$, we have that if θ is a graceful labeling of $G(n)$, there are at least two distinct vertices u, v in G such that $\theta(u) = \theta_c(v) = n$, and if θ is a λ-graceful labeling of $G(n)$, there are vertices v_i $(1 \le i \le 4)$ in G such that $\theta(v_4) = \theta_c(v_1) = \theta_r(v_3) = (\theta_c\theta_r)(v_2) = (\theta_r\theta_c)(v_2) = n$, where $\theta(v_1) = 0$, $\theta(v_2) = \lambda$, $\theta(v_3) = \lambda + 1$, and $\theta(v_4) = n$. In what follows, we will assume that the vertices w_0 and w_i in $G(n)$ and $G_i(n, \lambda_i)$ $(1 \le i \le p)$ are labeled n, respectively. By $\cup_{i=1}^{p} G_i$ we mean the union of vertex-disjoint graphs $G_1, G_2, \ldots, G_p$. We begin with introducing the gracefulness of three distinct classes of graphs.

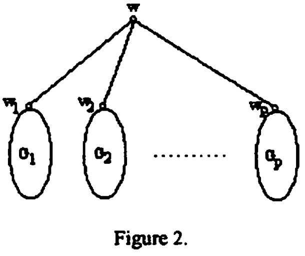


Figure 2.

Construction I: Suppose that G_i $(1 \le i \le p)$ are connected graphs. Let w_i be an arbitrary vertex of G_i. Adjoin to the graph $\cup_{i=1}^{p} G_i$ a new vertex w accompanied with p edges $ww_1, ww_2, \cdots, ww_p$. Then we obtain a new graph, denoted $\oplus_w(G_1, G_2, \cdots, G_p)$, as shown in Figure 2.

Theorem 2.1. *Let $G_1, G_2, \cdots, G_p$ $(p \ge 1)$ be disjoint isomorphic copies of $G(n)$, and let w_i be the isomorphic image of w_0 in $G(n)$ for $1 \le i \le p$. Then the graph $\oplus_w(G_1, G_2, \cdots, G_p)$ is graceful.*

Proof. Suppose that $S(n) = \{(a_1, b_1), \cdots, (a_n, b_n)\}$ is the induced graceful sequence of $G(n)$ with the property that $d(n, a_i)$ is even (≥ 0) and $d(n, b_i)$ is odd for $1 \leq i \leq n$. Note that $a_n = n$ and $b_n = 0$. Set $S_i(n) = \{(i(n+1) - 1 - a_1, (p+1-i)(n+1) - 1 - b_1), \cdots, (i(n+1) - 1 - a_n, (p+1-i)(n+1) - 1 - b_n)\}$, $1 \leq i \leq p$, and set $S_0(p) = \{(p(n+1), 0), (p(n+1), n+1), \cdots, (p(n+1), (p-1)(n+1))\}$. Clearly, $|\oplus_w(G_1, G_2, \cdots, G_p)| = p(n+1) = e$, where $|\oplus_w(G_1, G_2, \cdots, G_p)|$ denote the number of edges in $\oplus_w(G_1, G_2, \cdots, G_p)$. Let $S^*(e)$ be a sequence defined by

$$S^*(e) = S_1(n) \cup S_2(n) \cup \cdots \cup S_p(n) \cup S_0(p)$$
$$= \{(c_1, d_1), \cdots, (c_i, d_i), \cdots, (c_e, d_e)\}.$$

It is sufficient to prove that $S^*(e)$ is a graceful sequence and the induced graceful graph $G(S^*(n))$ of $S^*(e)$ is isomorphic to $\oplus_w(G_1, G_2, \cdots, G_p)$. We divide the proof into two cases, depending on whether p is even or odd.

Case 1: p is even, say $p = 2k$.

(1) If $i = t(n+1)$ for $1 \leq t \leq p$, then $(c_i, d_i) = (p(n+1), (p-t)(n+1))$.

(2) If $i = (2t-1)(n+1) \pm j$ for $1 \leq t \leq k$ and $1 \leq j \leq n$, then

$$\begin{cases} (c_{(2t-1)(n+1)+j}, d_{(2t-1)(n+1)+j}) = ((k+1-t)(n+1) - 1 - a_j, (p+t-k)(n+1) - 1 - b_j) \\ (c_{(2t-1)(n+1)-j}, d_{(2t-1)(n+1)-j}) = ((p+t-k)(n+1) - 1 - a_j, (k+1-t)(n+1) - 1 - b_j) \end{cases}, \text{ if } a_j > b_j,$$

or

$$\begin{cases} (c_{(2t-1)(n+1)+j}, d_{(2t-1)(n+1)+j}) = ((p+t-k)(n+1) - 1 - a_j, (k+1-t)(n+1) - 1 - b_j) \\ (c_{(2t-1)(n+1)-j}, d_{(2t-1)(n+1)-j}) = ((k+1-t)(n+1) - 1 - a_j, (p+t-k)(n+1) - 1 - b_j) \end{cases}, \text{ if } a_j < b_j.$$

A routine verification shows that $|d_i - c_i| = i$ for $1 \leq i \leq e$, and hence, $S^*(e)$ is a graceful sequence.

To prove that $G(S^*(e)) \cong \oplus_w(G_1, G_2, \cdots, G_p)$, define, first, bijections θ_i ($1 \leq i \leq p$): $E(S(n)) \to E(S_i(n))$ with $\theta_i(j) = i(n+1) - 1 - j$ if $d(n, j)$ is even (≥ 0), and $\theta_i(j) = (p+1-i)(n+1) - 1 - j$ if $d(n, j)$ is odd. For each term (a_m, b_m) in $S(n)$, term $(a_m, b_m) \in S(n)$ if and only if $(\theta_i(a_m), \theta_i(b_m)) = (i(n+1) - 1 - a_m, (p+1-i)(n+1) - 1 - b_m) \in S_i(n)$. It follows from Lemma 1.2 that $S(n) \sim S_i(n)$ and so $G(S(n)) \cong G(S_i(n))$; i.e., $G(n) \cong G_i$ ($1 \leq i \leq p$).

Now to prove that the graphs $G(S_i(n))$ ($1 \leq i \leq p$) are mutually disjoint, it suffices to show that $E(S_i(n)) \cap E(S_j(n)) = \varnothing$ for $1 \leq i < j \leq p$. Partition the set $E(S(n))$ into two independent subsets $E_1(S(n))$ and $E_2(S(n))$, where $E_1(S(n)) = \{r \mid d(n, r) \text{ is even } (\geq 0)\}$ and $E_2(S(n)) = \{s \mid d(n, s) \text{ is odd}\}$.

If $S_i(r_1) = S_j(r_2)$ or $S_i(s_1) = S_j(s_2)$ for $r_1, r_2 \in E_1(S(n))$ and $s_1, s_2 \in E_2(S(n))$, then $(j-i)(n+1) = r_2 - r_1$ or $s_1 - s_2$, which implies that $r_1 = r_2$ or $s_1 = s_2$ and hence $j = i$, a contradiction.

If $S_i(r_1) = S_j(s_2)$ or $S_i(s_1) = S_j(r_2)$ for $r_1, r_2 \in E_1(S(n))$ and $s_1, s_2 \in E_2(S(n))$, then $(p+1-j-i)(n+1) = s_2 - r_1$ or $s_1 - r_2$, contradicting the facts that $s_i - r_i \neq 0$ and $-n \leq s_i - r_i < n$, $i = 1, 2$.

Therefore the graphs $G(S_i(n))$ ($1 \leq i \leq p$) are mutually disjoint. Moreover, since $\theta_i(n) = (i-1)(n+1)$ and $(p(n+1), (i-1)(n+1)) \in S_0(p)$ for $1 \leq i \leq p$, it is not difficult to see that the induced graceful graph of $S^*(e)$ is the graph $\oplus_w(G_1, G_2, \cdots, G_p)$ (the graph $\oplus_w(G_1, G_2, G_3, G_4)$ in Figure 3 is an example, where $G(8)$ is the graph of Figure 1).

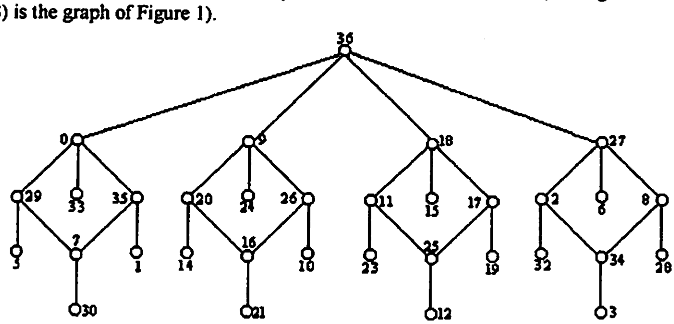


Figure 3. $\oplus_w(G_1, G_2, G_3, G_4)$.

Case 2: p is odd.

The proof is similar to that of case 1 and is omitted. □

Theorem 2.1 generalizes a theorem in [2], which states that if $G(n)$ is a graceful tree, say $G_i = T_i$, then the tree $\oplus_w(T_1, T_2, \cdots, T_p)$ is graceful for $p \geq 1$. In Theorem 2.1, the graphs G_i $(1 \leq i \leq p)$ are all isomorphic to $G(n)$. The next result permits G_i $(1 \leq i \leq p)$ to be non-isomorphic if each graph G_i is λ_i-graceful and each pair of graphs $G_i(n, \lambda_i)$ and $G_{p-i+1}(n, \lambda_{p-i+1})$ $(1 \leq i \leq \lfloor p/2 \rfloor)$ has the same boundary value (i.e., $\lambda_i = \lambda_{p-i+1}$).

Theorem 2.2. *If each pair of graphs* $G_i(n, \lambda_i)$ and $G_{p-i+1}(n, \lambda_{p-i+1})$ $(1 \leq i \leq \lfloor p/2 \rfloor)$ *has the same boundary value, then the graph* $\oplus_w(G_1(n, \lambda_1), G_2(n, \lambda_2), \cdots, G_p(n, \lambda_p))$ *is graceful*, $p \geq 1$.

Proof. Suppose that $S_i(n) = \{(a_{i,1}, b_{i,1}), \cdots, (a_{i,n}, b_{i,n})\}$ is the induced graceful sequence of $G_i(n, \lambda_i)$ satisfying that $d(n, a_{i,j})$ (≥ 0) is even and $d(n, b_{i,j})$ is odd for $1 \leq i \leq p$ and $1 \leq j \leq n$. Note that $|a_{i,j} - b_{i,j}| = a_{i,j} - b_{i,j} = j$. Set $S_i'(n) = \{(i(n+1) - 1 - a_{i,1}, (p+1-i)(n+1) - 1 - b_{i,1}), \cdots, (i(n+1) - 1 - a_{i,n}, (p+1-i)(n+1) - 1 - b_{i,n})\}$, $1 \leq i \leq p$, and set $S_0(p) = \{(p(n+1), 0), (p(n+1), n+1), \cdots, (p(n+1), (p-1)(n+1))\}$. Obviously, $|\oplus_w(G_1(n,\lambda_1), G_2(n,\lambda_2), \cdots, G_p(n,\lambda_p))| = p(n+1) = e$. Let $S^*(e)$ be a sequence given by

$$S^*(e) = S_1'(n) \cup S_2'(n) \cup \cdots \cup S_p'(n) \cup S_0(p)$$
$$= \{(c_1, d_1), \cdots, (c_i, d_i), \cdots, (c_e, d_e)\}.$$

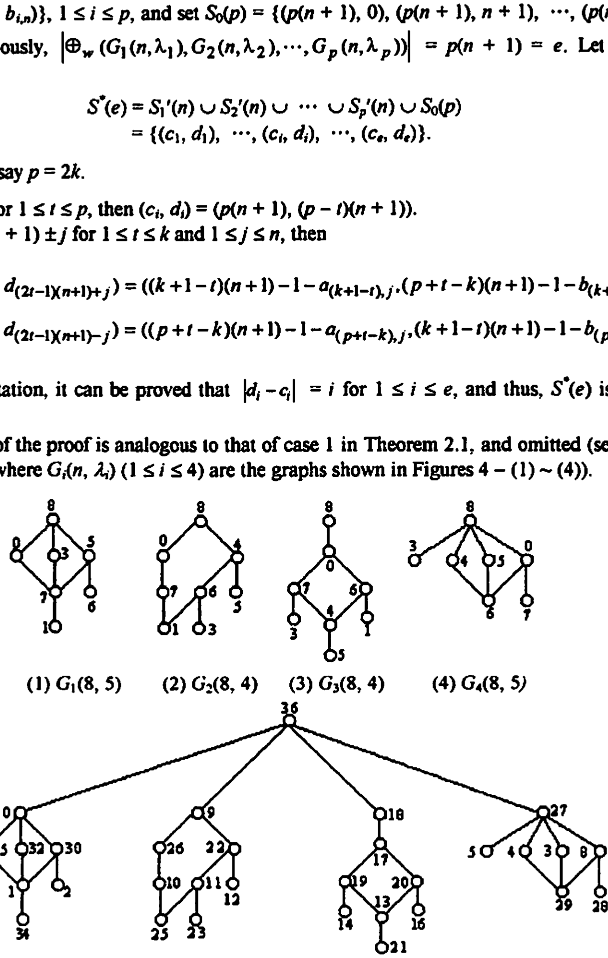
Case 1: p is even, say $p = 2k$.

(1) If $i = t(n+1)$ for $1 \leq t \leq p$, then $(c_i, d_i) = (p(n+1), (p-t)(n+1))$.
(2) If $i = (2t-1)(n+1) \pm j$ for $1 \leq t \leq k$ and $1 \leq j \leq n$, then

$$\begin{cases} (c_{(2t-1)(n+1)+j}, d_{(2t-1)(n+1)+j}) = ((k+1-t)(n+1) - 1 - a_{(k+1-t),j}, (p+t-k)(n+1) - 1 - b_{(k+1-t),j}), \\ (c_{(2t-1)(n+1)-j}, d_{(2t-1)(n+1)-j}) = ((p+t-k)(n+1) - 1 - a_{(p+t-k),j}, (k+1-t)(n+1) - 1 - b_{(p+t-k),j}). \end{cases}$$

By routine computation, it can be proved that $|d_i - c_i| = i$ for $1 \leq i \leq e$, and thus, $S^*(e)$ is a graceful sequence.

The remainder of the proof is analogous to that of case 1 in Theorem 2.1, and omitted (see the graph of Figure 4 – (5), where $G_i(n, \lambda_i)$ $(1 \leq i \leq 4)$ are the graphs shown in Figures 4 – (1) ~ (4)).



(1) $G_1(8, 5)$ (2) $G_2(8, 4)$ (3) $G_3(8, 4)$ (4) $G_4(8, 5)$

(5) $\oplus_w(G_1(8, 5), G_2(8, 4), G_3(8, 4), G_4(8, 5))$

Figure 4.

Case 2: p is odd.

Similar to case 1 and omitted. □

Construction II: Suppose that the graphs G_i ($1 \le i \le p$ and $p \ge 2$) are connected. Let w_i be an arbitrary vertex of G_i. Adjoin to the graph $\cup_{i=1}^{p} G_i$ the $p - 1$ edges $w_1w_2, w_2w_3, \cdots, w_{p-1}w_p$. The graph constructed is denoted by $\Theta\,(G_1, G_2, \cdots, G_p)$ and depicted in Figure 5.

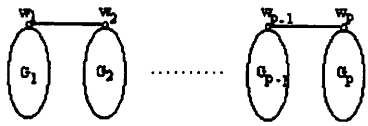


Figure 5. $\Theta\,(G_1, G_2, \cdots, G_p)$.

According to Theorems 2.1 and 2.2, the graphs $\oplus_w(G_1, G_2, \cdots, G_p)$ and $\oplus_w(G_1(n, \lambda_1), G_2(n, \lambda_2), \cdots, G_p(n, \lambda_p))$ are graceful. First remove from the graph $\oplus_w(G_1, G_2, \cdots, G_p)$ (or $\oplus_w(G_1(n, \lambda_1), G_2(n, \lambda_2), \cdots, G_p(n, \lambda_p))$) the vertex with label $p(n + 1)$ and all edges incident with it. Then adjoin to the resulting graph either the $p - 1$ edges

$\{(0, (p-1)(n+1)), ((p-1)(n+1), n+1), (n+1, (p-2)(n+1)), ((p-2)(n+1), 2(n+1)), \cdots, ((r-2)(n+1), (r+1)(n+1)), ((r+1)(n+1), (r-1)(n+1)), ((r-1)(n+1), r(n+1))\}$ if p is even, say $p = 2r$,

or the $p - 1$ edges

$\{(0, (p-1)(n+1)), ((p-1)(n+1), n+1), (n+1, (p-2)(n+1)), ((p-2)(n+1), 2(n+1)), \cdots, ((r+2)(n+1), (r-1)(n+1)), ((r-1)(n+1), (r+1)(n+1)), ((r+1)(n+1), r(n+1))\}$ if p is odd, say $p = 2r + 1$.

Then we have the following two results.

Theorem 2.3. *Let $G_1, G_2, \cdots, G_p$ be disjoint isomorphic copies of $G(n)$, and let w_i be the isomorphic image of w_0 in $G(n)$ for $1 \le i \le p$ and $p \ge 2$. Then the graph $\Theta\,(G_1, G_2, \cdots, G_p)$ is graceful.*

Theorem 2.4. *If each pair of graphs $G_i(n, \lambda_i)$ and $G_{p-i+1}(n, \lambda_{p-i+1})$ ($1 \le i \le \lfloor p/2 \rfloor$) has the same boundary value, then the graph $\Theta\,(G_1(n, \lambda_1), G_p(n, \lambda_p), G_2(n, \lambda_2), G_{p-1}(n, \lambda_{p-1}), \cdots, G_{\lfloor (p+2)/2 \rfloor}(n, \lambda_{\lfloor (p+2)/2 \rfloor}))$ is graceful, $p \ge 2$.*

The graph of Figure 6 is an example, which is obtained from Figure 4 by the method described above.

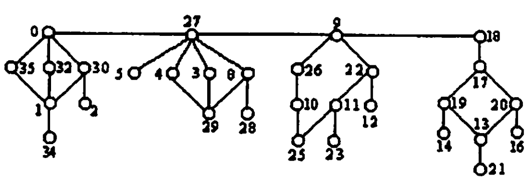


Figure 6. $\Theta\,(G_1(8, 5), G_4(8, 5), G_2(8, 4), G_3(8, 4))$

Construction III: Suppose that graphs G_i ($1 \le i \le p$) are connected. Let w_i be an arbitrary vertex of G_i. Identify the vertices $w_1 = w_2 = \cdots = w_p$ on the graph $\cup_{i=1}^{p} G_i$. See Figure 7. Denote the resulting graph by $\odot\,(G_1, G_2, \cdots, G_p)$.

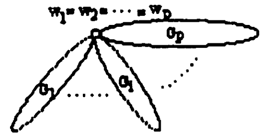


Figure 7.

For a graceful graph G with n edges, let $adj(n) = \{j \mid d(n, j) = 1\}$. If $adj(n) \subseteq \{0\} \cup \{n - j \mid j \in adj(n)\}$, then we call $adj(n)$ an *adjacency set* of n on G.

The following has been proved by K. M. Koh et al [5], but a proof is included here to keep the paper self-contained.

Theorem 2.5. *Let $G_1, G_2, \cdots, G_p$ be disjoint isomorphic copies of $G(n)$, and let w_i be the isomorphic image of w_0 in $G(n)$ for $1 \le i \le p$ and $p \ge 2$. If $adj(n)$ is an adjacency set of n on $G(n)$, then the graph $\odot(G_1, G_2, \cdots, G_p)$ is graceful.*

Proof. Let $adj(n) = \{u_1, u_2, \cdots, u_k\}$ $(k \ge 1)$, and let $S(n)$ be the induced graceful sequence of $G(n)$. Suppose that $S(n) = \{(a_1, b_1), \cdots, (a_n, b_n)\} = \{(a_{r_1}, b_{r_1}), \cdots, (a_{r_{n-k}}, b_{r_{n-k}}), (n, u_1), \cdots, (n, u_k)\}$ with the property that $d(n, a_i)$ is even and $d(n, b_i)$ is odd for $i = r_1, r_2, \cdots, r_{n-k}$. Note that $|a_{r_s} - b_{r_s}| = r_s$ for $1 \le s \le n - k$. Set $S_i(n) = \{((p - i)n + a_{r_1}, (i - 1)n + b_{r_1}), \cdots, ((p - i)n + a_{r_{n-k}}, (i - 1)n + b_{r_{n-k}}), (pn, (i - 1)n + u_1), \cdots, (pn, (i - 1)n + u_k)\}$, $1 \le i \le p$. Clearly, $|\tilde{\oplus}_w(G_1, G_2, \cdots, G_p)| = pn$. Define a sequence $S^*(pn)$ as follows:

$$S^*(pn) = S_1(n) \cup S_2(n) \cup \cdots \cup S_p(n) = \{(c_1, d_1), \cdots, (c_i, d_i), \cdots, (c_{pn}, d_{pn})\}.$$

Case 1: p is even, say $p = 2e$.

(1) If $i = (p - m + 1)n - u_t$ for $1 \le m \le p$ and $1 \le t \le k$, then $(c_i, d_i) = (pn, (m - 1)n + u_t)$.

(2) If $i = (p - 2m + 1)n \pm r_s$ for $1 \le m \le e$ and $1 \le s \le n - k$, then

$$\begin{cases} (c_{(p-2m+1)n+r_s}, d_{(p-2m+1)n+r_s}) = ((p-m)n + a_{r_s}, (m-1)n + b_{r_s}) \\ (c_{(p-2m+1)n-r_s}, d_{(p-2m+1)n-r_s}) = ((m-1)n + a_{r_s}, (p-m)n + b_{r_s}) \end{cases}, \text{ if } a_{r_s} > b_{r_s},$$

or

$$\begin{cases} (c_{(p-2m+1)n+r_s}, d_{(p-2m+1)n+r_s}) = ((m-1)n + a_{r_s}, (p-m)n + b_{r_s}) \\ (c_{(p-2m+1)n-r_s}, d_{(p-2m+1)n-r_s}) = ((p-m)n + a_{r_s}, (m-1)n + b_{r_s}) \end{cases}, \text{ if } a_{r_s} < b_{r_s}.$$

Observe that it is impossible that $(p - m + 1)n - u_t = (p - 2m + 1)n \pm r_s$. Since if it holds, then the only possibility is that, when $m = 1$ and $r_s > 0$, $n - u_t = r_s$, which contradicts the hypothesis. Note that $\{r_s \mid 1 \le s \le n - k\} \cup \{n - u_t \mid 1 \le t \le k\} = \{1, 2, \cdots, n\}$, and $\{n - r_s \mid 1 \le s \le n - k\} \cup \{n - u_t \mid 1 \le t \le k\} = \{1, 2, \cdots, n\}$. A routine verification shows that $|c_i - d_i| = i$ for $1 \le i \le pn$, and hence, $S^*(pn)$ is a graceful sequence.

Next, we shall show that $S(n) \sim S_i(n)$ and then $G(S(n)) \cong G(S_i(n))$ for $1 \le i \le p$. This can be done as follows. Define bijections θ_i $(1 \le i \le p)$: $E(S(n)) \to E(S_i(n))$ with $\theta_i(n) = pn$, $\theta_i(q) = (p - i)n + q$ if $d(n, q)$ is even, and $\theta_i(q) = (i - 1)n + q$ if $d(n, q)$ is odd. For each term (a_{r_s}, b_{r_s}) $(1 \le s \le n - k)$, $(a_{r_s}, b_{r_s}) \in S(n)$ if and only if $(\theta_i(a_{r_s}), \theta_i(b_{r_s})) = ((p - i)n + a_{r_s}, (i - 1)n + b_{r_s}) \in S_i(n)$; for each term (n, u_t) $(1 \le t \le k)$, $(n, u_t) \in S(n)$ if and only if $(\theta_i(n), \theta_i(u_t)) = (pn, (i - 1)n + u_t) \in S_i(n)$.

To prove that graphs $G(S_i(n))$ and $G(S_j(n))$ $(1 \le i < j \le p)$ have only one vertex with label pn in common, it remains to show that $E(S_i(n)) \cap E(S_j(n)) = pn$. It is clear that $pn \in E(S_i(n))$ for $1 \le i \le p$. Partition the set $E(S(n))$ into two independent subsets $E_1(S(n))$ and $E_2(S(n))$, where $E_1(S(n)) = \{r \mid d(n, r)$ is even$\}$ and $E_2(S(n)) = \{s \mid d(n, s)$ is odd$\}$.

If $S_i(r_1) = S_j(r_2)$ or $S_i(s_1) = S_i(s_2)$ for $r_1, r_2 \in E_1(S(n))$ and $s_1, s_2 \in E_2(S(n))$, then $(j - i)(n + 1) = r_2 - r_1$ or $s_1 - s_2$, which implies that $r_1 = r_2$ or $s_1 = s_2$ and hence $j = i$, a contradiction.

If $S_i(r_1) = S_j(s_2)$ or $S_i(s_1) = S_j(r_2)$ for $r_1, r_2 \in E_1(S(n))$ and $s_1, s_2 \in E_2(S(n))$, then $(p - j - i + 1)(n + 1) = s_2 - r_1$ or $s_1 - r_2$, contradicting the facts that $s_i - r_i \neq 0$ and $-n < s_i - r_i < n$, $i = 1, 2$.

Thus the graceful graph induced by $S^*(pn)$ is indeed the graph $\circledcirc\,(G_1, G_2, \cdots, G_p)$ (see the graph of Figure 8, where $G(8)$ is the graph of Figure 1).

Case 2: p is odd.

Similar to case 1 and omitted. □

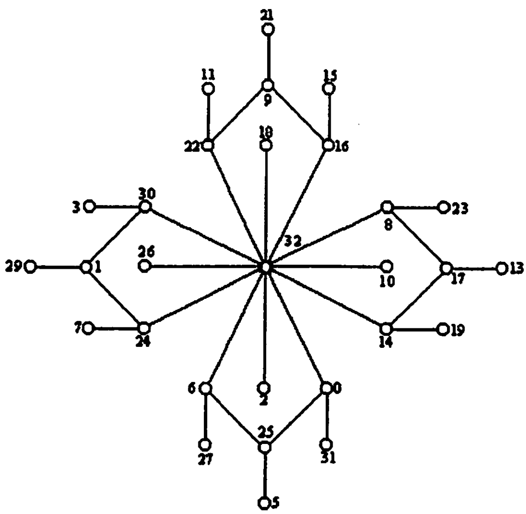


Figure 8. $\circledcirc\,(G_1, G_2, G_3, G_4)$.

Remark. Theorem 2 of [3] is a special case of Theorem 2.5.

Theorem 2.6. *If $adj_1(n)$ and $adj_i(n)$ are the adjacency sets of n on $G_1(n, \lambda_1)$ and $G_i(n, \lambda_i)$ $(2 \le i \le p)$, respectively, and if each pair of graphs $G_i(n, \lambda_i)$ and $G_{p-i+1}(n, \lambda_{p-i+1})$ $(1 \le i \le \lfloor p/2 \rfloor)$ has the same boundary value, then the graph $\circledcirc\,(G_1(n, \lambda_1), G_2(n, \lambda_2), \cdots, G_p(n, \lambda_p))$ is graceful.*

Proof. Let $adj_1(n) = \{u_1, u_2, \cdots, u_r\}$, and let $adj_i(n) = \{v_1, v_2, \cdots, v_s\}$. Suppose that $S_1(n) = \{(a_{1,1}, b_{1,1}), \cdots, (a_{1,n}, b_{1,n})\} = \{(a_{1,y_1}, b_{1,y_1}), \cdots, (a_{1,y_{n-r}}, b_{1,y_{n-r}}), (n, u_1), \cdots, (n, u_r)\}$ is the induced graceful sequence of $G_1(n, \lambda_1)$ satisfying that $d(n, a_{1,y_i})$ is even and $d(n, b_{1,y_i})$ is odd for $1 \le i \le n - r$. Let $S_i(n) = \{(a_{i,1}, b_{i,1}), \cdots, (a_{i,n}, b_{i,n})\} = \{(a_{i,x_1}, b_{i,x_1}), \cdots, (a_{i,x_{n-s}}, b_{i,x_{n-s}}), (n, v_1), \cdots, (n, v_s)\}$ be the induced graceful sequence of $G_i(n, \lambda_i)$ with the property that $d(n, a_{i,x_j})$ is even and $d(n, b_{i,x_j})$ is odd for $2 \le i \le p$ and $1 \le j \le n - s$. Note that $\left|a_{1,y_k} - b_{1,y_k}\right| = y_k$ and $\left|a_{i,x_j} - b_{i,x_j}\right| = x_j$ for $1 \le k \le n - r$ and $1 \le j \le n - s$. Set $S_1'(n) = \{((p-1)n + a_{1,y_1}, b_{1,y_1}), \cdots, ((p-1)n + a_{1,y_{n-r}}, b_{1,y_{n-r}}), (pn, u_1), \cdots, (pn, u_r)\}$, and for $2 \le i \le p$ set $S_i'(n) = \{(p-i)n + a_{i,x_1}, (i-1)n + b_{i,x_1}\}, \cdots, ((p-i)n + a_{i,x_{n-s}}, (i-1)n + b_{i,x_{n-s}}), (pn, (i-1)n + v_1), \cdots, (pn, (i-1)n + v_s)\}$. Evidently, $\left|\widetilde{\oplus}_w(G_1(n, \lambda_1), G_2(n, \lambda_2), \cdots, G_p(n, \lambda_p))\right| = pn$. Let us introduce a sequence $S^*(pn)$ defined by

$$\begin{aligned} S^*(pn) &= S_1'(n) \cup S_2'(n) \cup \cdots \cup S_p'(n) \\ &= \{(c_1, d_1), \cdots, (c_i, d_i), \cdots, (c_{pn}, d_{pn})\}. \end{aligned}$$

Case 1: p is even, say $p = 2e$.

(1) If $i = tn - v_k$ for $1 \le t \le p - 1$ and $1 \le k \le s$, then $(c_i, d_i) = (pn, (p-t)n + v_k)$.

(2) If $i = (2t - p - 1)n - x_j$ for $e + 1 \le t \le p$, then $(c_i, d_i) = ((p-t)n + a_{t,x_j}, (t-1)n + b_{t,x_j})$.

(3) If $i = (p - 2t + 1)n + x_j$ for $2 \le t \le e$, then $(c_i, d_i) = ((p - t)n + a_{t,x_j}, (t - 1)n + b_{t,x_j})$.

(4) If $i = pn - u_\ell$ for $1 \le \ell \le r$, then $(c_i, d_i) = (pn, u_\ell)$.

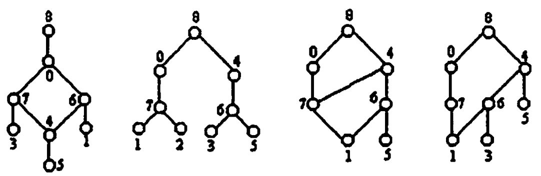
(5) If $i = (p - 1)n + y_j$, then $(c_i, d_i) = ((p - 1)n + a_{1,y_t}, b_{1,y_t})$.

Similarly, it can be checked that $|c_i - d_i| = i$, $1 \le i \le pn$, and so $S^*(pn)$ is a graceful sequence.

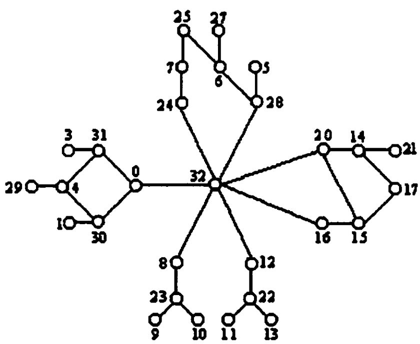
The rest of the proof is also analogous to that of case 1 in Theorem 2.5, and omitted (the graph, as shown in Figure 9, is an example).

Case 2: p is odd.

Similar to case 1 and omitted. □



(1) $G_1(8, 4)$ (2) $G_2(8, 5)$ (3) $G_3(8, 5)$ (4) $G_4(8, 4)$



(5) ⊚ $(G_1(8, 4), G_2(8, 5), G_3(8, 5), G_4(8, 4))$

Figure 9.

Remark. Some results similar to Theorem 2.6 had been obtained by Koh and Rogers [7].

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