Ranks of Regular Graphs Under Certain Unary Operations

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Abstract

We consider the rank of the adjacency matrix of some classes of regular graphs that are transformed under certain unary operations. In particular, we study the ranks of the subdivision graph, the connected cycle graph, the connected subdivision graph and the total graph of the following families of graphs: cycles, complete graphs, complete bipartite and multipartite graphs, circulant graphs of degrees three and four and some Cartesian graph products.

1 Introduction and Preliminary Results.

Ranks of regular graphs under the following unary operations are considered in the present work. The *subdivision graph* $S(G)$ of a graph G is obtained by subdividing every edge of G; that is, for every edge $e = uv \in G$, we add a new vertex w and replace the edge uv by the edges uw and wv. The *connected cycle graph* $R(G)$ of a graph G is obtained by adding a new vertex corresponding to every edge and adding two new edges from each new vertex to the endpoints of the corresponding edge. Note that this operation creates a triangle on every edge of G. The *connected subdivision graph* $Q(G)$ of a graph G is obtained by subdividing every edge and connecting the pairs of new vertices that lie on adjacent edges of G. Finally, if $G = (\mathcal{V}, \mathcal{E})$ then the *total graph* $T(G)$ has as its set of vertices $\mathcal{V} \cup \mathcal{E}$ with vertices connected by an edge if and only if the corresponding elements of G are either adjacent or incident.

Let $P_G(\lambda)$ be the characteristic polynomial of the adjacency matrix of a regular graph G of degree r on n vertices with $m = \frac{1}{2}nr$ edges and let its spectrum consist of the numbers $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$. The spectrum is also denoted by $\text{spec}(G) = \left\{\lambda_1^{(\mu_1)}, \lambda_2^{(\mu_2)}, \lambda_3^{(\mu_3)}, ..., \lambda_k^{(\mu_k)}\right\}$ where λ_j is a distinct eigenvalue of G with multiplicity μ_j, for $j = 1, 2, ..., k$. Note that since G is regular, r is the largest eigenvalue of G and its multiplicity is 1. Thus, it is customary in the literature to denote $\lambda_1 = r$ and to omit the superscript, which is understood to indicate a multiplicity of 1.

The lemmas below, found in [2], relate the eigenvalues of a regular graph to the eigenvalues of the graph under each operation.

Lemma 1 *If G is a regular graph of degree r with n vertices, m edges and characteristic polynomial $P_G(\lambda)$, then $S(G)$ has $(m+n)$ vertices, $2m$ edges and the characteristic polynomial*

$$P_{S(G)}(\lambda) = \lambda^{m-n} P_G(\lambda^2 - r).$$

Equivalently, the spectrum of $S(G)$ consists of $(m-n)$ numbers equal to zero and $2n$ numbers given by

$$\pm\sqrt{\lambda_i + r} \tag{1}$$

where λ_i is an eigenvalue of G for $i = 1, 2, ..., n$.

Lemma 2 *If G is a regular graph of degree r with n vertices, m edges and characteristic polynomial $P_G(\lambda)$, then $R(G)$ has $(m+n)$ vertices, $3m$ edges and the characteristic polynomial*

$$P_{R(G)}(\lambda) = \lambda^{m-n}(\lambda+1)^n P_G\left(\frac{\lambda^2 - r}{\lambda + 1}\right).$$

Equivalently, the spectrum of $R(G)$ consists of $(m-n)$ numbers equal to zero and $2n$ numbers given by the expression

$$\frac{1}{2}\left(\lambda_i \pm \sqrt{\lambda_i^2 + 4(r + \lambda_i)}\right) \tag{2}$$

where λ_i is an eigenvalue of G for $i = 1, 2, ..., n$.

Lemma 3 *If G is a regular graph of degree r with n vertices, m edges and characteristic polynomial $P_G(\lambda)$, then $Q(G)$ has $(m+n)$ vertices, $m(r+1)$ edges and the characteristic polynomial*

$$P_{Q(G)}(\lambda) = (\lambda+1)^m P_G\left(\frac{\lambda^2-(r-2)\lambda-r}{\lambda+1}\right).$$

Equivalently, the spectrum of $Q(G)$ consists of $(m-n)$ numbers equal to -1 and $2n$ numbers given by

$$\frac{1}{2}\left(\lambda_i+r-2\pm\sqrt{\lambda_i^2+2r\lambda_i+r^2+4}\right) \tag{3}$$

where λ_i is an eigenvalue of G for $i = 1, 2, ..., n$.

Lemma 4 *If G is a regular graph of degree r with n vertices and m edges, then $T(G)$ has $(m+n)$ vertices, $r(m+n)$ edges and $(m-n)$ eigenvalues equal to -2 and $2n$ eigenvalues given by the expression*

$$\frac{1}{2}(2\lambda_i+r-2\pm\sqrt{4\lambda_i+r^2+4}) \tag{4}$$

where λ_i is an eigenvalue of G for $i = 1, 2, ..., n$.

As an example of the application of the preceding lemmas, we examine the Petersen graph P. It is well known that P is 3-regular, has 10 vertices and 15 edges and that its spectrum is $\text{spec}(P) = \{3, 1^{(5)}, -2^{(4)}\}$. By Lemma 1, $\text{spec}(S(P)) = \left\{-\sqrt{6}, -2^{(5)}, -1^{(4)}, 0^{(5)}, 1^{(4)}, 2^{(5)}, \sqrt{6}\right\}$ so $\text{rank}(S(P)) = 20$. Similar computation of eigenvalues by the results of Lemmas 2 and 3 reveal that $\text{rank}(R(P)) = 20$ and $\text{rank}(Q(G)) = 25$.

Finally, we state a useful result for conditions under which an eigenvalue of $T(G)$ will be zero.

Corollary 1 *If G is a regular graph of degree r with n vertices and m edges, then $T(G)$ will have zero eigenvalues if and only if the numbers given by*

$$\frac{1}{2}\left(-r+3\pm\sqrt{r^2-2r+9}\right) \tag{5}$$

are included in the spectrum of G.

Proof. Setting (4) equal to zero and solving for λ_i gives the result. ■

Again, we illustrate this by examining the Petersen graph. Since $r = 3$, (5) indicates that the numbers $\pm\sqrt{3}$ should be in spec(P) if $T(P)$ is to have any zero eigenval ies. Clearly, this is not the case. Hence, rank($T(P)$) = 25.

The speccrum of a graph is known to indicate some fundamental structural properties of a graph. This is the purpose of the next section.

2 Bipartite Graphs

The following theorem from [3], concerning bipartite graphs, is useful in determining the structure of some graphs transformed by the unary operations under consideration.

Theorem 1 *Let G be a connected graph. Then the following statements are equivalent.*

1) *G is bipartite.*
2) *If r is the largest eigenvalue of G, then $-r$ is an eigenvalue of G.*
3) *The eigenvalues of G are symmetric about zero on the real line. In other words, if λ is an eigenvalue of G, then so is $-\lambda$ with the same multiplicity.*

Theorem 1 combined with Lemma 1 gives the following corollary.

Corollary 2 *$S(G)$ is bipartite for any connected regular graph G.*

Proof. By Lemma 1, the nonzero eigenvalues of $S(G)$ are given by $\lambda = \pm\sqrt{\lambda_i + r}$, where $\lambda_i \in$ spec(G) and r is the degree of G. Hence, the eigenvalues of $S(G)$ are symmetric about zero. Therefore, $S(G)$ is bipartite. ■

Because we seek the rank, we determine the number of zero eigenvalues of the graph. This is more easily done by setting the eigenvalue expressions in the four lemmas in Section 1 equal to zero and solving for λ_i. Solutions to those equations are values of the original graph that will be transformed to zero eigenvalues of the graph under the unary operations. Noting that if a graph G is r-regular and bipartite then its largest and smallest eigenvalues are r and $-r$ motivates the following theorem.

Theorem 2 *Let G be a connected regular graph of degree r with n vertices, m edges and eigenvalues $\lambda_1 = r$, λ_2, λ_3, ..., λ_n. Then the following statements are equivalent.*

1) *G is bipartite.*
2) *rank($S(G)$) $= 2n - 2$.*
3) *rank($R(G)$) $= 2n - 1$.*

4) $rank(Q(G)) = m + n - 1$.

Proof. Let G be a connected bipartite r-regular graph with n vertices, m edges and eigenvalues $\lambda_1 = r$, λ_2, λ_3, ..., λ_n. Setting each of the expressions (1), (2) and (3) for the eigenvalues of $S(G)$, $R(G)$ and $Q(G)$ equal to zero we obtain the following equations.

$$\pm\sqrt{\lambda_i + r} = 0 \tag{1'}$$

$$\lambda_i \pm \sqrt{\lambda_i^2 + 4(r + \lambda_i)} = 0 \tag{2'}$$

$$\lambda_i + r - 2 \pm \sqrt{\lambda_i^2 + 2r\lambda_i + r^2 + 4} = 0 \tag{3'}$$

All three equations have only one solution: $\lambda_i = -r$. The multiplicity determined by those solutions is, however, different.

For $S(G)$, both positive and negative roots satisfy (1′). Hence, there are two eigenvalues equal to zero in addition to the $(m - n)$ given by $P_{S(G)}(\lambda)$. Thus $\text{rank}(S(G)) = (m + n) - (m - n) - 2 = 2n - 2$.

Only the negative root satisfies (2′), so there is only one additional zero eigenvalue of $R(G)$. There are already $(m - n)$ eigenvalues of zero given by $P_{R(G)}(\lambda)$. Hence, $\text{rank}(R(G)) = (m + n) - (m - n) - 1 = 2n - 1$.

For $Q(G)$ also, only the negative root satisfies (3′). Thus, there is only one zero eigenvalue and so $\text{rank}(Q(G)) = m + n - 1$.

Now, assume that G is a connected r-regular graph with n vertices, m edges and eigenvalues $\lambda_1 = r$, λ_2, λ_3, ..., λ_n. Also, let $\text{rank}(S(G)) = 2n - 2$. We know there are $(m - n)$ zero eigenvalues given by the characteristic polynomial of $S(G)$. Thus there are two additional zero eigenvalues in the spectrum of $S(G)$ that have been transformed from a nonzero eigenvalue of G. Since the other numbers in the spectrum are given by (1), it must be that $\lambda_i = -r$. Hence, by Theorem 1, G is bipartite. Since G is bipartite, it must follow that $\text{rank}(R(G)) = 2n - 1$ and $\text{rank}(Q(G)) = m + n - 1$.

Assuming the rank of $R(G)$ or $Q(G)$ instead of that of $S(G)$ also establishes that G is bipartite, and is proved similarly. ■

Notice that Theorem 2 does not imply rank conditions for a non-bipartite graph. For that, we need the next theorem.

Theorem 3 *Let G be a connected regular graph of degree r with n vertices, m edges and eigenvalues $\lambda_1 = r$, λ_2, λ_3, ..., λ_n. Then the following statements are equivalent.*

1) *G is not bipartite.*
2) *rank*$(S(G)) = 2n$.
3) *rank*$(R(G)) = 2n$.
4) *rank*$(Q(G)) = m + n$.

Proof. Let G be a connected non-bipartite r-regular graph with n vertices, m edges and eigenvalues $\lambda_1 = r$, λ_2, λ_3, ..., λ_n. Then, since G is not bipartite, $-r \notin$ spec(G) by Theorem 1. Thus there are no solutions to $(1')$ which implies that there are only $m - n$ zero eigenvalues in $S(G)$. Hence, rank$(S(G)) = 2n$. Similarly, there are no solutions to $(2')$ or $(3')$. Thus, rank$(R(G)) = 2n$ and rank$(Q(G)) = m + n$.

Now, assume that $S(G)$ has full rank. Then there are no eigenvalues of G that satisfy equation $(1')$, which implies that $-r$ is not an eigenvalue of G. Hence, G is not bipartite by Theorem 1. Assuming that $R(G)$ and $Q(G)$ have full rank instead of $S(G)$ is established similarly. ■

In the sections that follow, we investigate the rank of a regular graph under each of the four operations. We consider cycles, complete graphs, the n-dimensional hypercube, the cocktail-party graph, complete bipartite and multipartite graphs, circulant graphs of degrees three and four and some Cartesian graph products.

3 Cycles, Completes, Cubes and Cocktails

A cycle C_n is 2-regular, has n vertices and n edges, for $n \geq 3$.

Theorem 4 *For $n \geq 3$,*

$$rank(S(C_n)) = \begin{cases} 2n - 2 & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd} \end{cases}$$

$$rank(R(C_n)) = rank(Q(C_n)) = \begin{cases} 2n - 1 & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd} \end{cases}$$

$$rank(T(C_n)) = \begin{cases} 2n - 3 & \text{if } n \equiv 0 \bmod 3 \\ 2n - 1 & \text{otherwise} \end{cases}$$

Proof. If n is even, C_n is bipartite, and for n odd, C_n is not bipartite. Thus the ranks of $S(C_n)$, $R(C_n)$ and $Q(C_n)$ follow from Theorem 2 and 3.

Since $r = 2$, (5) gives that the numbers 2 and -1 should be included in spec(C_n) for there to be any zero eigenvalues. Since the spectrum of C_n includes the numbers $\lambda_j = 2\cos(2\pi j/n)$ for $j = 1, 2, ..., n$, (from [2]) we set λ_j equal to 2 and then to -1 and solve.

$$\begin{aligned} 2\cos(2\pi j/n) &= 2 \\ 2\pi j/n &= \pi(2k),\ k \in \mathbb{Z} \\ j &= \frac{n}{2}(2k),\ k \in \mathbb{Z} \\ j &\equiv 0 \bmod n \end{aligned}$$

Thus, $2 \in \text{spec}(C_n)$ for every n and will give one zero eigenvalue of $T(C_n)$. This should be obvious, since $2 = r = \lambda_1$.

$$\begin{aligned} 2\cos(2\pi j/n) &= -1 \\ 2\pi j/n &= \frac{2\pi}{3}(3k \pm 1),\ k \in \mathbb{Z} \\ j &= \frac{n}{3}(3k \pm 1),\ k \in \mathbb{Z} \\ j &\equiv \pm\frac{n}{3} \bmod n \end{aligned}$$

Hence, $-1^{(2)} \in \text{spec}(C_n)$ only for n divisible by 3, and so will give two zero eigenvalues of $T(C_n)$. The result follows. ■

We now turn to the complete graph K_n. The complete graph on $n \geq 2$ vertices is $(n-1)$-regular with $\frac{1}{2}n(n-1)$ edges.

Theorem 5 *For $n \geq 2$,*

$$rank(S(K_n)) = \begin{cases} 2 & \text{if } n = 2 \\ 2n & \text{otherwise} \end{cases}$$

$$rank(R(K_n)) = \begin{cases} 3 & \text{if } n = 2 \\ 2n & \text{otherwise} \end{cases}$$

$$rank(Q(K_n)) = \begin{cases} 2 & \text{if } n = 2 \\ \frac{1}{2}n(n+1) & \text{otherwise} \end{cases}$$

$$rank(T(K_n)) = \begin{cases} 3 & \text{if } n = 3 \\ \frac{1}{2}n(n+1) & \text{otherwise} \end{cases}$$

Proof. The graph K_n is bipartite only for $n = 2$, otherwise K_n is not bipartite; hence, the ranks of $S(K_n)$, $R(K_n)$ and $Q(K_n)$ follow from Theorems 2 and 3.

For the rank of $T(K_n)$ we examine (5) upon substitution of $r = n - 1$:

$$\frac{1}{2}\left(-n + 4 \pm \sqrt{n^2 - 4n + 12}\right) \tag{6}$$

$\mathrm{Spec}(K_n) = \{n-1,\ -1^{(n-1)}\}$ (from [2]). Thus we set (6) equal to each of $n-1$ and -1 to examine which values of n give eigenvalues that will be transformed to zero in the total graph. Solving expression (6) equal to $n-1$ gives $n = 3$, and solving equal to -1 gives $n = 3$. Hence, $n = 3$ gives the only instance of rank deficiency, in which case $\mathrm{spec}(K_3) = \{2,\ -1^{(2)}\}$ so that all three eigenvalues become zeros of $T(K_n)$. The result follows. ■

Next we examine the n-dimensional hypercube Q_n, for $n \geq 2$. This graph is defined recursively by $Q_1 = K_2$ and $Q_{n+1} = Q_n \times K_2$, where ($\times$) indicates the Cartesian product. Q_n is n-regular, has 2^n vertices and $2^{n-1}n$ edges.

Theorem 6 *For $n \geq 2$,*

$$rank(S(Q_n)) = 2(2^n - 1),$$

$$rank(R(Q_n)) = 2^{n+1} - 1,$$

$$rank(Q(Q_n)) = 2^{n-1}(n+2) - 1,$$

$$rank(T(Q_n)) = \begin{cases} 7 & \text{if } n = 2 \\ 2^{n-1}(n+2) & \text{otherwise} \end{cases}.$$

Proof. The ranks of $S(Q_n)$, $R(Q_n)$ and $Q(Q_n)$ follow from Theorem 2 since Q_n is bipartite for all $n \geq 2$.

The spectrum of Q_n consists of the numbers $n - 2k$, each with multiplicity $\binom{n}{k}$, for $k = 0, 1, \ldots, n$ (from [1]). Substituting $r = n$ in (5) gives $\frac{1}{2}\left(-n + 3 \pm \sqrt{n^2 - 2n + 9}\right)$. We set this expression equal to $n - 2k$ to determine conditions for zero eigenvalues in $T(Q_n)$.

$$\begin{aligned}
-n + 3 \pm \sqrt{n^2 - 2n + 9} &= 2n - 4k \\
\pm\sqrt{n^2 - 2n + 9} &= 3n - 4k - 3 \\
n^2 - 2n + 9 &= 9n^2 + 16k^2 - 24nk - 18n + 24k + 9 \\
0 &= 8n^2 + 16k^2 - 24nk - 16n + 24k \\
0 &= n^2 - n(3k+2) + 2k^2 + 3k \\
n &= \frac{1}{2}\left(3k + 2 \pm \sqrt{k^2 + 4}\right)
\end{aligned}$$

The only rational value of k that gives a rational value of n is $k = 0$, which implies that $n = 2$. Thus, if $n = 2$, the eigenvalue corresponding to $k = 0$ is n, with multiplicity $\binom{n}{0} = 1$, and is transformed to a zero eigenvalue of $T(Q_n)$. The result follows. ■

The final graph examined in this section is the cocktail-party graph, $CP(k)$, $k \geq 2$, which is a complete graph on $n = 2k$ vertices with a perfect matching removed. Here, $2k^2 - 2k$ is the number of edges. This graph is the only $(n-2)$-regular graph on n vertices.

Theorem 7 *For $k \geq 2$,*

$$rank(S(CP(k))) = \begin{cases} 6 & \text{if } k = 2 \\ 4k & \text{otherwise} \end{cases}$$

$$rank(R(CP(k))) = \begin{cases} 7 & \text{if } k = 2 \\ 4k & \text{otherwise} \end{cases}$$

$$rank(Q(CP(k))) = rank(T(CP(k))) = \begin{cases} 7 & \text{if } k = 2 \\ 2k^2 & \text{otherwise} \end{cases}$$

Proof. From [2] we have that $\text{spec}(CP(k)) = \{2k-2, -2^{(k-1)}, 0^{(k)}\}$. Clearly, the eigenvalues are symmetric about zero on the real line only for $k = 2$. Thus, only for $k = 2$ is $CP(k)$ bipartite by Theorem 1, and for any other value of k, $CP(k)$ is then not bipartite. By Theorems 2 and 3, the ranks of $S(CP(k))$, $R(CP(k))$ and $Q(CP(k))$ follow.

To find rank($T(CP(k))$), we substitute $r = 2k - 2$ in (5). This gives an expression for the eigenvalues of $CP(k)$ that will be transformed to zero eigenvalues in $T(CP(k))$: $\frac{1}{2}\left(-2k + 5 \pm \sqrt{4k^2 - 12k + 17}\right)$. Next, we set this expression equal to each of the three eigenvalues, beginning with $2k - 2$. Recall that $k \geq 2$.

$$\begin{aligned} -2k + 5 \pm \sqrt{4k^2 - 12k + 17} &= 4k - 4 \\ 4k^2 - 12k + 17 &= 36k^2 - 108k + 81 \\ 0 &= 32k^2 - 96k + 64 \\ 0 &= k^2 - 3k + 2 \\ k &= 2 \end{aligned}$$

The remaining two equations do not give viable solutions ($k = 8/3$ and $k = 1$); hence, there is one zero eigenvalue of $T(CP(k))$ only for $k = 2$. Therefore, the result is established. ■

4 Complete Multipartite Graphs

We begin this section with the complete bipartite graph $K_{n,n}$. This n-regular graph has $2n$ vertices and n^2 edges for $n \geq 1$.

Theorem 8 *For $n \geq 1$,*

$$rank(S(K_{n,n})) = 4n - 2,$$

$$rank(R(K_{n,n})) = 4n - 1,$$

$$rank(Q(K_{n,n})) = n^2 + 2n - 1,$$

$$rank(T(K_{n,n})) = \begin{cases} 7 & \text{if } n = 2 \\ n^2 + 2n & \text{otherwise} \end{cases}.$$

Proof. Obviously, $K_{n,n}$ is bipartite for all values of $n \geq 1$. Thus the ranks of $S(K_{n,n})$, $R(K_{n,n})$ and $Q(K_{n,n})$ follow directly from Theorem 2.

Since $\text{spec}(K_{n,n}) = \{n, 0^{(2n-2)}, -n\}$ (found in [2]), we need only substitute $r = n$ in (5) and solve equal to the three distinct eigenvalues. Thus, the equation

$$-n + 3 \pm \sqrt{n^2 - 2n + 9} = 2n$$

gives only $n = 2$ as a solution for the positive root. The other equations using 0 and $-n$ give a solution of $n = 0$, which is nonsense. Hence, $\text{rank}(T(K_{n,n}))$ is deficient by one only for $n = 2$, and the result follows. ■

Next we examine the complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$ where the n_i's are all equal for $i = 1, 2, \ldots, k$, $k \geq 3$. Letting $n = n_i$, we have that this graph is $n(k-1)$-regular and has nk vertices and $\frac{1}{2}n^2k(k-1)$ edges.

Theorem 9 *For $n \geq 2$ and $k \geq 3$,*

$$rank(S(K_{n_1,n_2,\ldots,n_k})) = rank(R(K_{n_1,n_2,\ldots,n_k})) = 2nk,$$

$$rank(Q(K_{n_1,n_2,\ldots,n_k})) = rank(T(K_{n_1,n_2,\ldots,n_k})) = \frac{1}{2}nk(nk - n + 2).$$

Proof. Since $K_{n_1,n_2,\ldots,n_k}$ is clearly not bipartite, the ranks of $S(K_{n_1,n_2,\ldots,n_k})$, $R(K_{n_1,n_2,\ldots,n_k})$ and $Q(K_{n_1,n_2,\ldots,n_k})$ follow from Theorem 3.

To establish the rank of $T(K_{n_1,n_2,\ldots,n_k})$, we examine the spectrum. Again, from [2] we have $\text{spec}(K_{n_1,n_2,\ldots,n_k}) = \{n(k-1),\ 0^{(nk-n)},\ -n^{(k-1)}\}$. Hence, we substitute $r = n(k-1)$ and set (5) equal to $n(k-1)$, 0 and $-n$ in turn. We then have the three equations below.

$$n^2(k-1)^2 - 2n(k-1) + 9 = (3n(k-1) - 3)^2$$

$$n^2(k-1)^2 - 2n(k-1) + 9 = (n(k-1) - 3)^2$$

$$n^2(k-1)^2 - 2n(k-1) + 9 = (n(k-3) - 3)^2$$

The solutions to these equations are $k = 1$, $k = 1 + 2/n$ and $k = \frac{2(n-2)}{n+1}$ which are not acceptable solutions since $k \geq 3$ and $n \geq 2$. Hence, there are no zero eigenvalues in $T(K_{n_1,n_2,\ldots,n_k})$, and the rank is $\frac{1}{2}nk(nk - n + 2)$. ■

5 Circulants of Degrees Three and Four

Developed in this section are the ranks of 3- and 4-circulants under the unary operations. We begin with 3-circulant graphs. Recall from [4] that a three-circulant is defined by a jump set $S = \{a,\ \frac{n}{2},\ n-a\}$; in other words, vertex i is adjacent to vertex j if and only if $(|i-j| \bmod n) \in S$. We denote a 3-circulant on n vertices with jump set $S = \{a,\ \frac{n}{2},\ n-a\}$ by $3C_n(a)$. A three-circulant is 3-regular and has n vertices and $3n/2$ edges. Note that n must be even. The following theorem from [8] gives a formula for the eigenvalues of a circulant matrix.

Theorem 10 *If A is an $n \times n$ circulant matrix with first row $[c_1,\ c_2,\ \ldots,\ c_n]$, then the eigenvalues of A are given by $\lambda_p = \sum_{i=1}^{n} c_i\omega^{(i-1)p}$, $p = 0, 1, \ldots, n-1$, where $\omega = e^{2\pi i/n}$.*

Now we determine the ranks of circulants under the unary operations. In the following, we assume a connected 3-circulant. This implies $\gcd(a, n) = 1$ (see [4]).

Theorem 11 *For even $n \geq 4$ and $gcd(a, n) = 1$,*

$$rank(S(3C_n(a))) = \begin{cases} 2n-2 & \text{if } n/2 \text{ is odd} \\ 2n & \text{if } n/2 \text{ is even} \end{cases}$$

$$rank(R(3C_n(a))) = \begin{cases} 2n-1 & \text{if } n/2 \text{ is odd} \\ 2n & \text{if } n/2 \text{ is even} \end{cases}$$

$$rank(Q(3C_n(a))) = \begin{cases} 5n/2-1 & \text{if } n/2 \text{ is odd} \\ 5n/2 & \text{if } n/2 \text{ is even} \end{cases}$$

$$rank(T(3C_n(a))) = 5n/2.$$

Proof. In light of Theorems 1, 2, and 3, there will be rank deficiency if and only if $3C_n(a)$ is bipartite, and $3C_n(a)$ is bipartite if and only if $-r = -3 \in \text{spec}(3C_n(a))$. Thus, we examine conditions under which and eigenvalue of $3C_n(a)$ is -3.

By Theorem 10, $\lambda_p = \omega^{ap} + \omega^{(n/2)p} + \omega^{(n-a)p} = -3$ if and only if

$$2\cos(2\pi ap/n) + \cos(\pi p) = -3 \tag{7}$$

From this, two cases arise: p even and p odd. If p is even, (7) reduces to $2\cos(2\pi ap/n) = -4$, which has no solution. Thus, p must be odd; so (7) reduces to $2\cos(2\pi ap/n) = -2$, which implies $ap \equiv n/2 \bmod n$.

Since connectivity is assumed, $\gcd(a, n) = 1$, which implies a is odd, since n is even. Thus both a and p are odd, forcing $n/2$ to be odd for there to be a solution to the congruence; that solution being $p = n/2$. Hence, $3C_n(a)$ is bipartite for $n/2$ odd, and the results for the ranks of $S(3C_n(a))$, $R(3C_n(a))$ and $Q(3C_n(a))$ hold.

For $T(3C_n(a))$, we substitute $r = 3$ in (5). This gives that $\pm\sqrt{3}$ should be in $\text{spec}(3C_n(a))$ for there to be a zero eigenvalue in the total graph of $3C_n(a)$. Thus we examine

$$2\cos(2\pi ap/n) + \cos(\pi p) = \pm\sqrt{3}$$

For p even, this reduces to $\cos(2\pi ap/n) = (-1+\sqrt{3})/2$ and for p odd, it reduces to $\cos(2\pi ap/n) = (1-\sqrt{3})/2$. Neither of these equations have solutions for which ap/n is rational. Thus, there are no eigenvalues of $3C_n(a)$ equal to $\pm\sqrt{3}$, which implies no zero eigenvalues in the total graph. ■

Now we consider the four-circulant graph. Recall from [5] that a four-circulant is defined by a jump set $S = \{a, b, n-b, n-a\}$; in other words, vertex i is adjacent to vertex j if and only if $(|i-j| \bmod n) \in S$. We denote a 4-circulant on n vertices with jump set $S = \{a, b, n-b, n-a\}$ by $4C_n(a, b)$. A four-circulant is 4-regular and has n vertices and $2n$ edges. Again, we assume the 4-circulant is connected; then, $\gcd(a, b, n) = 1$ (see [5]).

Theorem 12 *Let $d_1 = gcd(a, n)$ and $d_2 = gcd(b, n)$. Then for $n \geq 5$ and $gcd(a, b, n) = 1$,*

$$rank(S(4C_n(a, b))) = \begin{cases} 2n-2 & \text{if } n \text{ is even and } 2 \text{ divides } (d_2 - d_1) \\ 2n & \text{otherwise} \end{cases}$$

$$rank(R(4C_n(a, b))) = \begin{cases} 2n-1 & \text{if } n \text{ is even and } 2 \text{ divides } (d_2 - d_1) \\ 2n & \text{otherwise} \end{cases}$$

$$rank(Q(4C_n(a, b))) = \begin{cases} 3n-1 & \text{if } n \text{ is even and } 2 \text{ divides } (d_2 - d_1) \\ 3n & \text{otherwise} \end{cases}$$

$$rank(T(4C_n(a, b))) = 3n.$$

Proof. Proceeding in a similar fashion to Theorem 11, we find conditions under which an eigenvalue of $4C_n(a, b)$ will be -4. By Theorem 10, $\lambda_p = \omega^{ap} + \omega^{bp} + \omega^{(n-b)p} + \omega^{(n-a)p} = -4$ if and only if

$$2\cos(2\pi ap/n) + 2\cos(2\pi bp/n) = -4. \tag{8}$$

Clearly, this can only happen when each cosine term is equal to -1. Hence, it must be that $ap \equiv n/2 \bmod n$ and $bp \equiv n/2 \bmod n$. Note that this implies that n must be even. Let $d_1 = \gcd(a, n)$ and $d_2 = \gcd(b, n)$. By the Generalized Chinese Remainder Theorem (see [10]), a simultaneous solution for the two congruencies exists if and only if $\gcd(n/d_1, n/d_2)$ divides $n(d_2 - d_1)/2d_1d_2$. But since the graph is connected, $\gcd(d_1, d_2) = 1$; and so $\gcd(n/d_1, n/d_2) = n/\text{lcm}(d_1, d_2) = n/d_1d_2$ divides $n(d_2 - d_1)/2d_1d_2$. This implies that $(d_2 - d_1)/2$ must be an integer. Therefore, $4C_n(a, b)$ is bipartite if and only if n is even and 2 divides $d_2 - d_1$. Hence, the results for the ranks of $S(4C_n(a, b))$, $R(4C_n(a, b))$ and $Q(4C_n(a, b))$ hold.

Now for the total graph of $4C_n(a, b)$, we substitute $r = 4$ in (5). This gives that $\frac{1}{2}(-1 \pm \sqrt{17})$ should be in $\text{spec}(4C_n(a, b))$ for there to be a zero eigenvalue in the total graph of $4C_n(a, b)$. Thus we examine

$$2\cos(2\pi ap/n) + 2\cos(2\pi bp/n) = \frac{1}{2}(-1 \pm \sqrt{17})$$

This equation has no solutions for which ap/n and bp/n are rational. Thus, there are no eigenvalues of $4C_n(a, b)$ equal to $\frac{1}{2}(-1 \pm \sqrt{17})$, which implies no zero eigenvalues in the total graph. ■

6 Some Cartesian Graph Products

We begin by recalling some basic facts concerning the Cartesian product of two graphs. The Cartesian product of graphs G and H is a graph that has vertex set $\mathcal{V}(G \times H) = \mathcal{V}(G) \times \mathcal{V}(H)$ and edge set $\mathcal{E}(G \times H) = \{\{\{u, v_1\}, \{u, v_2\}\} \mid u \in \mathcal{V}(G) \text{ and } \{v_1, v_2\} \in \mathcal{E}(H)\} \cup \{\{\{u_1, v\}, \{u_2, v\}\} \mid \{u_1, u_2\} \in \mathcal{E}(G) \text{ and } v \in \mathcal{V}(H)\}$. The next result is well-known and can be found in various works, including [2].

Theorem 13 *If graph G has eigenvalues $\lambda_1, ..., \lambda_n$ and graph H has eigenvalues $\mu_1, ..., \mu_m$, then the Cartesian graph product $G \times H$ has eigenvalues $\lambda_i + \mu_j$ for $i = 1, ..., n$ and $j = 1, ..., m$.*

The first product we shall consider is that of a cycle on n vertices and a path on 2 vertices. This graph has $2n$ vertices and $3n$ edges, and is 3-regular for $n \geq 3$.

Theorem 14 *For $n \geq 3$,*

$$rank(S(C_n \times P_2)) = \begin{cases} 4n - 2 & \text{if } n \text{ is even} \\ 4n & \text{if } n \text{ is odd} \end{cases}$$

$$rank(R(C_n \times P_2)) = \begin{cases} 4n - 1 & \text{if } n \text{ is even} \\ 4n & \text{if } n \text{ is odd} \end{cases}$$

$$rank(Q(C_n \times P_2)) = \begin{cases} 5n - 1 & \text{if } n \text{ is even} \\ 5n & \text{if } n \text{ is odd} \end{cases}$$

$$rank(T(C_n \times P_2)) = 5n.$$

Proof. By Theorems 1, 2 and 3, there will be rank deficiency if and only if $C_n \times P_2$ is bipartite, and $C_n \times P_2$ is bipartite if and only if $-r = -3 \in \text{spec}(C_n \times P_2)$. Thus, we examine conditions under which $\lambda_i = -3$.

Using the result of Theorem 13, the eigenvalues are given by $2\cos(2\pi j/n) - 1$ and $2\cos(2\pi j/n) + 1$ for $j = 1, 2, \ldots, n$. Thus $\lambda_i = -3$ if and only if either $2\cos(2\pi j/n) = -2$ or $2\cos(2\pi j/n) = -4$. The latter equation has no real solution; the former implies a solution exists if and only if $j \equiv n/2 \bmod n$. Thus, n must be even for a solution to exist. Hence, $C_n \times P_2$ is bipartite if and only if n is even. The results for $S(C_n \times P_2)$, $R(C_n \times P_2)$ and $Q(C_n \times P_2)$ follow.

Next, we substitute $r = 3$ in (5) to obtain that $\pm\sqrt{3}$ must be in the spectrum of $C_n \times P_2$ for there to be any zero eigenvalues in the total graph. Since neither $2\cos(2\pi j/n) + 1 = \pm\sqrt{3}$ nor $2\cos(2\pi j/n) - 1 = \pm\sqrt{3}$ have solutions for which j/n is rational, there are no zero eigenvalues in $T(C_n \times P_2)$. ■

Next, we examine $K_n \times P_2$, which has $2n$ vertices, n^2 edges, and is n-regular.

Theorem 15 *For $n \geq 2$,*

$$rank(S(K_n \times P_2)) = \begin{cases} 6 & \text{if } n = 2 \\ 4n & \text{otherwise} \end{cases}$$

$$rank(R(K_n \times P_2)) = \begin{cases} 7 & \text{if } n = 2 \\ 4n & \text{otherwise} \end{cases}$$

$$rank(Q(K_n \times P_2)) = rank(T(K_n \times P_2)) = \begin{cases} 7 & \text{if } n = 2 \\ n^2 + 2n & \text{otherwise} \end{cases}$$

Proof. We determine conditions under which all eigenvalues of $K_n \times P_2$ will be symmetric about zero, since this implies $K_n \times P_2$ is bipartite. There is no need to solve any eigenvalue equations, since from [2] we have that the spectrum of $K_n \times P_2$ is $\{n,\ n-2,\ 0^{(n-1)},\ -2^{(n-1)}\}$. Clearly, only $n = 2$ will give a symmetric set of eigenvalues. Hence, the first three results follow from Theorems 2 and 3.

For $T(K_n \times P_2)$, we substitute $r = n$ in (5) to get that an eigenvalue of $K_n \times P_2$ must be equal to $\frac{1}{2}\left(-n + 3 \pm \sqrt{n^2 - 2n + 9}\right)$ for there to be zero eigenvalues in the total graph. We set this expression equal to each of the four distinct eigenvalues in the spectrum in turn. Solving the four resulting equations reveals that only $n = 2$ is a feasible solution. Therefore, the rank of $T(K_n \times P_2)$ follows. ■

The product of a cycle on m vertices and a cycle on n vertices is 4-regular, has mn vertices and $2mn$ edges.

Theorem 16 *For m, $n \geq 3$,*

$$rank(S(C_m \times C_n)) = \begin{cases} 2mn - 2 & m \text{ and } n \text{ both even} \\ 2mn & \text{otherwise} \end{cases}$$

$$rank(R(C_m \times C_n)) = \begin{cases} 2mn - 1 & m \text{ and } n \text{ both even} \\ 2mn & \text{otherwise} \end{cases}$$

$$rank(Q(C_m \times C_n)) = \begin{cases} 3mn - 1 & m \text{ and } n \text{ both even} \\ 3mn & \text{otherwise} \end{cases}$$

$$rank(T(C_m \times C_n)) = 3mn.$$

Proof. Since a graph is bipartite if and only if the graph contains no odd cycles, $C_m \times C_n$ is bipartite if and only if m and n are both even. Then by Theorems 2 and 3, the results for the ranks of $S(C_m \times C_n)$, $R(C_m \times C_n)$ and $Q(C_m \times C_n)$ hold.

For the total graph of $C_m \times C_n$, we substitute $r = 4$ in (5). This gives that $\frac{1}{2}(-1 \pm \sqrt{17})$ should be in spec$(C_m \times C_n)$ for there to be a zero eigenvalue in the total graph of $C_m \times C_n$. However, just as in Theorem 11, there are no eigenvalues of $C_m \times C_n$ equal to $\frac{1}{2}(-1 \pm \sqrt{17})$, which implies no zero eigenvalues in the total graph. ■

Next, we examine the Cartesian product of a complete graph on m vertices and a complete graph on n vertices for both $m, n \geq 3$. This graph is $(m + n - 2)$-regular, and has mn vertices and $\frac{1}{2}mn(m + n - 2)$ edges.

Theorem 17 *For m, $n \geq 3$,*

$$rank(S(K_m \times K_n)) = rank(R(K_m \times K_n)) = 2mn,$$

$$rank(Q(K_m \times K_n)) = rank(T(K_m \times K_n)) = \frac{1}{2}mn(m + n).$$

Proof. Since K_n is not bipartite for $n \geq 3$, the Cartesian product of two complete graphs cannot be bipartite; the results then follow from Theorem 3.

$\text{Spec}(K_m \times K_n) = \{m+n-2,\ m-2^{(n-1)},\ n-2^{(m-1)},\ -2^{(m-1)(n-1)}\}$ (from [2]). For the total graph, we substitute $r = m+n-2$ in (5) and set equal to each of the four eigenvalues. Tedious computations reveal that no solutions satisfy the requirement that both m and n are greater than 3. Hence, the total graph has full rank. ■

Finally, we look at the Cartesian product of a cycle on m vertices and a complete graph on n vertices for both $m,\ n \geq 3$. This graph is $(n+1)$-regular, and has mn vertices and $\frac{1}{2}mn(n+1)$ edges.

Theorem 18 *For $m,\ n \geq 3$,*

$$rank(S(C_m \times K_n)) = rank(R(C_m \times K_n)) = 2mn,$$

$$rank(Q(C_m \times K_n)) = rank(T(C_m \times K_n)) = \frac{1}{2}mn(n+3).$$

Proof. Since K_n is not bipartite for $n \geq 3$, the Cartesian product of a complete graph and a cycle cannot be bipartite. The results then follow from Theorem 3.

For the total graph, we substitute $r = n+1$ in (5) and set it equal to each of the eigenvalues of $C_m \times K_n$. By Theorem 13, the spectrum of $C_m \times K_n$ consists of the numbers $2\cos(2\pi j/m) + n - 1$ for $j = 1, 2, ..., m$ along with $n-1$ each of the numbers $2\cos(2\pi j/m) - 1$ for $j = 1, 2, ..., m$. Therefore (5) implies that

$$\cos(2\pi j/m) = \frac{1}{4}(4 - 3n \pm \sqrt{n^2+8})$$

or

$$\cos(2\pi j/m) = \frac{1}{4}(4 - n \pm \sqrt{n^2+8})$$

However, the only solutions for which j/m is rational occur only when $n = 1$. Thus, no solutions satisfy the requirement that both m and n are greater than 3. Hence, the total graph of $C_m \times K_n$ has full rank. ■

7 A Special Non-Regular Graph

We begin this section with a result from [2] for the connected subdivision graph of a non-regular graph. This relates the eigenvalues of $Q(G)$ to those of the line graph of G, denoted $L(G)$.

Theorem 19 *Let G be a graph with n vertices and m edges. Then*

$$P_{Q(G)}(\lambda) = \lambda^{n-m}(\lambda+1)^m P_{L(G)}\left(\frac{\lambda^2-2}{\lambda+1}\right).$$

The path P_n is a well-known non-regular graph. Due to the unique structure of this graph, it is a straightforward procedure to establish basic results for P_n under the unary operations.

Theorem 20 *For $n \geq 2$, $Rank(S(P_n)) = rank(Q(P_n)) = 2n - 2$.*

Proof. Since $S(P_n)$ is another path on $m + n = (n-1) + n = 2n - 1$ vertices, the $\text{rank}(S(P_n)) = \text{rank}(P_{2n-1})$. It is well-known that $\text{rank}(P_n) = n - 1$ if n is odd. Hence, because $2n - 1$ is odd for all n, $\text{rank}(P_{2n-1}) = \text{rank}(S(P_n)) = (2n-1) - 1 = 2n - 2$.

For $Q(P_n)$, note that $L(P_n) \simeq P_{n-1}$. Now, from Theorem 18, simple algebra reveals that the eigenvalues of P_{n-1} are transformed by the operation of connected subdivision into numbers given by the expression $\frac{1}{2}\left(\lambda_i \pm \sqrt{\lambda_i^2 + 4\lambda_i + 8}\right)$, where $\lambda_i \in \text{spec}(P_{n-1})$ for $i = 1, 2, ..., n-1$. Setting this expression equal to zero determines that only an eigenvalue of -2 will be transformed into an eigenvalue of zero in $Q(P_{n-1})$. Now, eigenvalues of P_{n-1} are given by $2\cos(\pi i/[n-1+1]) = 2\cos(\pi i/n)$ for $i = 1, 2, ..., n-1$. Thus, we determine which eigenvalues are equal to -2. But $\cos(\pi i/n) = -1$ has no solution since $i \neq n$. So $-2 \notin \text{spec}(P_{n-1})$ which implies additional zero eigenvalues, other than the $n - m = n - (n-1) = 1$ given by the characteristic polynomial, do not exist. The result follows. ■

, To treat $R(P_n)$, we require some results from matrix theory and the block matrix representation of the adjacency matrix of $R(G)$ for some graph G. Recall that for every graph G on n vertices, where each vertex has degree d_i for $i = 1, 2, ..., n$, we have the valency (or degree) matrix $D = \text{diag}[d_1, d_2, ..., d_n]$. The following lemma can be found in [2].

Lemma 5 *If A is the $n \times n$ adjacency matrix of graph G and D is the valency matrix of G, then the adjacency matrix of $R(G)$ is of the block form*

$$\begin{bmatrix} 0_m & R^T \\ R & A \end{bmatrix}$$

where $RR^T = A + D$ and 0_m denotes the $m \times m$ zero matrix.

Lemmas 6 and 7 can be found in [9].

Lemma 6 *If A is a nonsingular matrix and D is square, then*

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||D - CA^{-1}B|.$$

Lemma 7 *If a square matrix A of order n is tridiagonal—that is, of the form*

$$A = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & & 0 \\ c_1 & a_2 & b_2 & & & \vdots \\ 0 & c_2 & a_3 & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & & 0 \\ & & & \ddots & a_{n-1} & b_{n-1} \\ 0 & \cdots & & 0 & c_{n-1} & a_n \end{bmatrix}$$

—then $|A_n| = a_n|A_{n-1}| - b_{n-1}c_{n-1}|A_{n-2}|$, *where* A_{k-1} *is* A_k *with the kth row and kth column deleted.*

Theorem 21 *For* $n \geq 2$, $rank(R(P_n)) = 2n - 1$.

Proof. By Lemma 6, the adjacency matrix of $R(P_n)$ is $\begin{bmatrix} 0_{n-1} & R^T \\ R & A(P_n) \end{bmatrix}$. Since this operation adds $n - 1$ vertices to the path on n vertices, the zero matrix is of dimension $n - 1$. Hence, the characteristic polynomial of the adjacency matrix of $R(P_n)$, denoted $A(R(P_n))$, is

$$\begin{aligned}
&|\lambda I_{2n-1} - A(R(P_n))| \\
&= \begin{vmatrix} \lambda I_{n-1} & -R^T \\ -R & \lambda I_n - A(P_n) \end{vmatrix} \\
&= |\lambda I_{n-1}|\left|\lambda I_n - A(P_n) - R\left(\frac{I_{n-1}}{\lambda}\right)R^T\right| \\
&= \lambda^{n-1}\left|\lambda I_n - A(P_n) - \frac{1}{\lambda}RR^T\right| \\
&= \lambda^{n-1-n}|\lambda^2 I_n - \lambda A(P_n) - RR^T| \\
&= \lambda^{-1}|\lambda^2 I_n - \lambda A(P_n) - A(P_n) - D(P_n)| \\
&= \lambda^{-1}|\lambda^2 I_n - (\lambda + 1)A(P_n) - D(P_n)| \\
&= \frac{1}{\lambda}\begin{vmatrix} \lambda^2 - 1 & -\lambda - 1 & 0 & & \cdots & 0 \\ -\lambda - 1 & \lambda^2 - 2 & -\lambda - 1 & & & \\ 0 & -\lambda - 1 & \lambda^2 - 2 & \ddots & & \vdots \\ & & \ddots & \ddots & & 0 \\ \vdots & & & & \lambda^2 - 2 & -\lambda - 1 \\ 0 & \cdots & & 0 & -\lambda - 1 & \lambda^2 - 1 \end{vmatrix}
\end{aligned}$$

where all entries are zero except on the sub- and super-diagonal (where entries are all $-\lambda - 1$) and the main diagonal. We denote this tridiagonal matrix B_n.

By Lemma 7, $|B_n| = (\lambda^2 - 1)|B_{n-1}| - (\lambda + 1)^2|B_{n-2}|$. Note that B_{n-1} and B_{n-2} are tridiagonal; moreover, all entries are identical to B_n except for the last diagonal entry which is $\lambda^2 - 2$. To simplify the following argument, we denote B_{n-1} by A_N. Thus we have $|B_n| = (\lambda^2 - 1)|A_N| - (\lambda + 1)^2|A_{N-1}|$.

The determinant of B_n results in a characteristic polynomial of degree $2n$; denote it by $p(\lambda)$. Upon multiplication of $p(\lambda)$ by $1/\lambda$, the result must be a polynomial of degree $2n - 1$. Therefore, there will be a constant term in the polynomial $p(\lambda)/\lambda$ if and only if the λ term of $p(\lambda)$ is nonzero. Thus we examine the λ term of $|B_n|$. This term is obtained by the multiplication of the λ terms of both $|A_N|$ and $|A_{N-1}|$ by -1 and the constant term of $|A_{N-1}|$ by -2λ. We show that none of these terms is nonzero; in fact, we show by induction that the constant term of $|A_N|$ is $(-1)^N$, the λ term of $|A_N|$ is $(-1)^{N-1}N(N-1)$, and that the λ term of $|B_n|$ is given by $2(-1)^{n-1}(n-1)$.

We begin with $|A_N|$. Note that

$$\begin{aligned}|A_3| &= (\lambda^2 - 2)|A_2| - (\lambda + 1)^2|A_1| \\ &= (\lambda^2 - 2)\big[(\lambda^2 - 2)(\lambda^2 - 1) - (\lambda + 1)^2\big] - (\lambda + 1)^2(\lambda^2 - 1) \\ &= \lambda^6 - 7\lambda^4 - 4\lambda^3 + 9\lambda^2 + 6\lambda - 1.\end{aligned}$$

Assume that the constant term of $|A_k|$ is $(-1)^k$ and the λ term of $|A_k|$ is $(-1)^{k-1}k(k-1)$ for all $k \leq N$. Let $N = k + 1$. Then the constant term of $|A_{k+1}|$ is given by $-2(-1)^k - (-1)^{k-1} = (-1)^{k+1}$ and the λ term is given by $-2(-1)^{k-1}k(k-1) - [(-1)^{k-2}(k-1)(k-2) + 2(-1)^{k-1}] = (-1)^k k(k+1)$.

Now we examine $|B_n|$. Note that

$$\begin{aligned}|B_3| &= (\lambda^2 - 1)|A_2| - (\lambda + 1)^2|A_1| \\ &= (\lambda^2 - 1)\big[(\lambda^2 - 2)(\lambda^2 - 1) - (\lambda + 1)^2\big] - (\lambda + 1)^2(\lambda^2 - 1) \\ &= \lambda^6 - 6\lambda^4 - 4\lambda^3 + 5\lambda^2 + 4\lambda.\end{aligned}$$

Assume the λ term of $|B_k|$ is $2(-1)^{k-1}(k-1)$ for all $k \leq n$. Let $n = k + 1$. Then the λ term of $|B_{k+1}|$ is given by $-(-1)^{k-1}k(k-1) - [(-1)^{k-2}(k-1)(k-2) + 2(-1)^{k-1}] = 2(-1)^k k$.

Hence, $p(\lambda)$ has a nonzero λ term, which implies $p(\lambda)/\lambda$ has a nonzero constant term. Since the constant term is the product of the roots (the eigenvalues) of the characteristic polynomial of $R(P_n)$, and the constant term is nonzero, there cannot be any zero eigenvalues of $R(P_n)$. Thus, its rank is $2n - 1$. ■

8 Concluding Remarks

The direction of research is to determine which structural properties of a graph are related to the rank of its adjacency matrix. Notice that the bipartite property has an important role in the rank of regular graph under the unary operations specified. The authors have previously investigated the more familiar unary operations: complements and line graphs of regular graphs ([6] and [7]). The next possible step includes determining ranks of binary operations on two graphs, such as the complete product, the strong product, more Cartesian products, the bipartite product of a graph with itself, and the composition of two graphs. More results involving specific graphs, such as the generalized Petersen graphs, cages and Cayley graphs, are also possible studies. Eventually, we hope that all results will be summarized, and a relationship between a graph and its rank will be found.

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