Maximum Number of Contractible Edges on Longest Cycles of a 3-Connected Graph

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Abstract

We show that if G is a 3-connected graph of order at least 5, then there exists a longest cycle C of G such that the number of contractible edges of G which are on C is greater than or equal to $\frac{|V(C)|+9}{8}$.

1 Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges.

A graph G is called *3-connected* if $|V(G)| \geq 4$ and $G - S$ is connected for any subset S of $V(G)$ having cardinality 2. An edge e of a 3-connected graph G is called *contractible* if the graph which we obtain from G by contracting e (and replacing each of the resulting pairs of parallel edges by a simple edge) is 3-connected; otherwise e is called *noncontractible*. We let $E_c(G)$ denote the set of contractible edges of G and $E_{nc}(G)$ denote the set of noncontractible edges of G; thus $E(G) = E_c(G) \cup E_{nc}(G)$ (disjoint union). In [8], Tutte proved that if G is a 3-connected graph other than K_4, then $E_c(G) \neq \phi$. In [2], Dean, Hemminger and Ota proved that if G is a 3-connected graph other than K_4 or $K_2 \times K_3$, then every longest cycle C of G satisfies $|E(C) \cap E_c(G)| \geq 3$. In [3], Ellingham, Hemminger nd Johnson proved that if G is a nonhamiltonian 3-connected graph, hen every longest cycle C of G satisfies $|E(C) \cap E_c(G)| \geq 6$. Further the lassification of those pairs (G,C) of a 3-connected graph G and a longest ycle C of G such that $|E(C) \cap E_c(G)| \leq 5$ has been completed by [1], [7], 4] and [6]. On the other hand, in [7], Ota made a conjecture that there xists a constant $\alpha > 0$ such that if G is a 3-connected graph of order at least 5, then G has a longest cycle C such that $|E(C) \cap E_c(G)| \geq \alpha |E(C)|$. In [5], we showed that such a constant exists if we restrict ourselves to 3-connected hamiltonian graphs:

Theorem ([5]). *Let G be a 3-connected hamiltonian graph of order at least 5. Then there exists a hamiltonian cycle C of G such that* $|E(C) \cap E_c(G)| \geq \lceil \frac{1}{8}|E(C)| + \frac{9}{8} \rceil$.

In this paper, we show that the same conclusion holds for 3-connected graphs in general, i.e., we prove the following theorem:

Main Theorem. *Let G be a 3-connected graph of order at least 5. Then there exists a longest cycle C of G such that* $|E(C) \cap E_c(G)| \geq \lceil \frac{1}{8}|E(C)| + \frac{9}{8} \rceil$.

Let α_0 denote the supremum of those real numbers α which make true the aforementioned conjecture of Ota. The above theorem shows $\alpha_0 \geq \frac{1}{8}$. On the other hand, $\alpha_0 \leq \frac{1}{3}$. To see this, let G be the line graph of a graph obtained from a 3-regular 3-connected graph by subdividing all edges once. Then G is 3-connected, and $|E(C) \cap E_c(G)| = \frac{|E(C)|}{3}$ for every longest cycle C of G.

The organization of this paper is as follows. Section 2 contains fundamental results concerning noncontractible edges lying on a longest cycle of a 3-connected graph. In Section 3, we state two propositions, Propositions 1 and 2, and show that the Main Theorem follows from Proposition 2. In Section 4, we define an *admissible partition*, which is an indispensable concept for the proof of Proposition 2, and we prove Proposition 2 in Section 5.

Our notation and terminology are standard except possibly for the following. Let G be a graph. For $U \subseteq V(G)$, we let $\langle U \rangle = \langle U \rangle_G$ denote the graph induced by U in G. For U, $V \subseteq V(G)$, we let $E(U,V)$ denote the set of edges of G which join a vertex in U and a vertex in V; if $V = \{v\}$ ($v \in V(G)$), we write $E(U,v)$ for $E(U,\{v\})$. A subset S of $V(G)$ is called a *cutset* if $G - S$ is disconnected; thus G is 3-connected if and only if $|V(G)| \geq 4$ and G has no cutset of cardinality 2. If G is 3-connected, then for $e = uv \in E(G)$, we let $K(e) = K(u,v)$ denote the set of vertices x of G

such that $\{u, v, x\}$ is a cutset; thus e is contractible if and only if $K(e) = \phi$. If $e = uv$ is noncontractible, then for each $x \in K(e)$, $\{u, v, x\}$ is called a *cutset associated with* e. For an x-$z$path X of G, we let $\overline{X}$ denote the z-x path such that $V(\overline{X}) = V(X)$ and $E(\overline{X}) = E(X)$. For a cycle C of G and for u, $v \in V(C)$ with $u \neq v$, we let $\mathscr{P}_C(u,v) \stackrel{\text{def}}{=} \{ P \subsetneq G \,|\, P$ is an u-v path and $(V(P) - \{u,v\}) \cap V(C) = \phi\}$, $\widetilde{\mathscr{P}_C}(u,v) \stackrel{\text{def}}{=} \mathscr{P}_C(u,v) - E(G)$ and $\widehat{\mathscr{P}_C}(u,v) \stackrel{\text{def}}{=} \mathscr{P}_C(u,v) - E_{nc}(G)$.

We conclude this introductory section with the following easy lemma, which we use in Sections 2 and 5.

Lemma 1.1. *Let G be a 3-connected graph of order at least 5, and let $C = v_0v_1 \cdots v_nv_0$ be a longest cycle of G. Let $0 \leq i, j, k, l \leq n$ and $i \neq j$, $k \neq l$ and $i \neq k$, and suppose that $\widetilde{\mathscr{P}_C}(v_i, v_j) \neq \phi$ and $\widetilde{\mathscr{P}_C}(v_k, v_l) \neq \phi$. Take $P \in \widetilde{\mathscr{P}_C}(v_i, v_j)$ and $Q \in \widetilde{\mathscr{P}_C}(v_k, v_l)$, and suppose that $(V(P) - \{v_i, v_j\}) \cap (V(Q) - \{v_k, v_l\}) \neq \phi$. Then the following hold.*

(i) $\widetilde{\mathscr{P}_C}(v_i, v_k) \neq \phi$.

(ii) $k \neq i + 1$.

Proof. Statement (i) is trivial. If $k = i + 1$, then since we can take $R \in \widetilde{\mathscr{P}_C}(v_i, v_{i+1})$ by (i), we get a cycle

$$C' = v_0v_1 \cdots v_{i-1}Rv_{i+2}v_{i+3} \cdots v_nv_0$$

such that $|E(C')| > |E(C)|$, which contradicts the maximality of the length of C.

2 Preliminaries

In this section, we prove fundamental results concerning noncontractible edges lying on a longest cycle of a 3-connected graph. Some of the assertions in lemmas of this section are already proved in Ota [7], but we include their proofs for the convenience of the reader.

Throughout this section, we let G denote a 3-connected graph of order at least 5, and let $C = v_0v_1 \cdots v_nv_0$ denote a longest cycle of G. Moreover, throughout this section, we assume that the edge v_nv_0 is noncontractible.

Lemma 2.1. *Let $u \in K(v_n, v_0)$. Then $u \in V(C)$ and, if we write $u = v_i$, then we have $2 \leq i \leq n-2$, and $\{v_k | 1 \leq k \leq i-1\}$ and $\{v_k | i+1 \leq k \leq n-1\}$ lie in distinct components of $G - \{v_n, v_0, v_i\}$.*

Proof. All assertions follow if we show that in $G - \{v_n, v_0, u\}$, $V(C) - \{v_n, v_0, u\}$ is not contained in a single component. By way of contradiction, suppose that $V(C) - \{v_n, v_0, u\}$ is contained in a single component of $G - \{v_n, v_0, u\}$. Then since $\{v_n, v_0, u\}$ is a cutset, there exists a component H of $G - \{v_n, v_0, u\}$ such that $V(H) \cap V(C) = \phi$. Since G is 3-connected, there exists a v_n-$v_0$path P such that $\phi \neq V(P) - \{v_n, v_0\} \subseteq V(H)$, and hence, we get a cycle

$$C' = v_0 v_1 \cdots v_{n-1} P.$$

Since $|E(C')| > |E(C)|$, this contradicts the maximality of the length of C.

Throughout the rest of this section, we fix $v_i \in K(v_n, v_0)$.

Lemma 2.2.

(i) $\mathscr{S}_C(v_k, v_l) = \phi$ *for any* k, l *with* $1 \leq k \leq i-1$ *and* $i+1 \leq l \leq n-1$.

(ii) *There exists* k *with* $1 \leq k \leq i-1$ *such that* $\mathscr{S}_C(v_n, v_k) \neq \phi$, *and there exists* p *with* $i+1 \leq p \leq n-1$ *such that* $\mathscr{S}_C(v_0, v_p) \neq \phi$.

Proof. Statement (i) follows immediately from Lemma 2.1. Since G is 3-connected, $G - \{v_0, v_i\}$ must be connected. Consequently, there exist k and l with $1 \leq k \leq i-1$ and $i+1 \leq l \leq n$ such that $\mathscr{S}_C(v_k, v_l) \neq \phi$, and we get $l = n$ by (i). Thus considering symmetry, (ii) is proved.

Lemma 2.3. *If i=2, then* $\mathscr{S}_C(v_n, v_1) = \{v_n v_1\}$.

Proof. By Lemma 2.2 (ii), $\mathscr{S}_C(v_n, v_1) \neq \phi$. Suppose that $\widetilde{\mathscr{S}_C}(v_n, v_1) \neq \phi$. Take $P \in \widetilde{\mathscr{S}_C}(v_n, v_1)$, and let H be the component of $G - V(C)$ containing $V(P) - \{v_n, v_1\}$. Since G is 3-connected, there is $v_p x \in E(V(C), V(H))$ with $v_p \neq v_n, v_1$. Then we have $\widetilde{\mathscr{S}_C}(v_1, v_p) \neq \phi$. Now if $3 \leq p \leq n-1$, then from $\widetilde{\mathscr{S}_C}(v_1, v_p) \neq \phi$, we get a contradiction to Lemma 2.2 (i); if $p = 0$ or 2, then $\widetilde{\mathscr{S}_C}(v_1, v_0) \neq \phi$ or $\widetilde{\mathscr{S}_C}(v_1, v_2) \neq \phi$, which contradicts the maximality of C. Thus $\widetilde{\mathscr{S}_C}(v_n, v_1) = \phi$. Since $\mathscr{S}_C(v_n, v_1) \neq \phi$, this forces $\mathscr{S}_C(v_n, v_1) = \{v_n v_1\}$.

Lemma 2.4. *Suppose that* $v_i v_{i+1}$ *is noncontractible, and let* $\{v_i, v_{i+1}, v_j\}$ *be a cutset associated with it. Then the following hold.*

(i) $i+3 \leq j \leq n$ *(and hence* $i \leq n-3$*).*

(ii) *If* $j = n$, *then* $\mathscr{S}_C(v_0, v_{i+1}) = \{v_0 v_{i+1}\}$ *and* $v_0 v_{i+1} \in E_c(G)$.

Proof. To prove (i), by way of contradiction, suppose that $0 \le j < i+3$. Then by Lemma 2.1, $0 \le j \le i-2$ and, applying Lemma 2.2 (ii) to $\{v_i, v_{i+1}, v_j\}$, we see that $\mathscr{S}_C(v_{i+1}, v_k) \ne \phi$ for some k with $j+1 \le k \le i-1$, which contradicts Lemma 2.2 (i). Thus (i) is proved. To prove (ii), suppose that $j = n$. Then applying Lemma 2.2 (ii) to $\{v_i, v_{i+1}, v_n\}$, we see that there exists l with $0 \le l \le i-1$ such that $\mathscr{S}_C(v_{i+1}, v_l) \ne \phi$ and, by Lemma 2.2 (i), we get $l = 0$, which implies $\mathscr{S}_C(v_0, v_{i+1}) \ne \phi$. To prove $\mathscr{S}_C(v_0, v_{i+1}) = \{v_0 v_{i+1}\}$, suppose that $\widetilde{\mathscr{S}_C}(v_0, v_{i+1}) \ne \phi$. Take $P \in \widetilde{\mathscr{S}_C}(v_0, v_{i+1})$, and let H be the component of $G - V(C)$ containing $V(P) - \{v_0, v_{i+1}\}$. Then there is $v_p x \in E(V(C), V(H))$ with $v_p \ne v_0, v_{i+1}$. By symmetry, we may assume $1 \le p \le i$. Now if $1 \le p \le i-1$, then since $\widetilde{\mathscr{S}_C}(v_p, v_{i+1}) \ne \phi$, we get a contradiction to Lemma 2.2 (i); if $p = i$, then $\widetilde{\mathscr{S}_C}(v_i, v_{i+1}) \ne \phi$, which contradicts the maximality of C. Thus $\widetilde{\mathscr{S}_C}(v_0, v_{i+1}) = \phi$, and hence $\mathscr{S}_C(v_0, v_{i+1}) = \{v_0 v_{i+1}\}$. It remains to show $v_0 v_{i+1} \in E_c(G)$. By way of contradiction, suppose that $v_0 v_{i+1} \in E_{nc}(G)$, and let

$$\{v_0, v_{i+1}, u\} \text{ be a cutset} \tag{2-1}$$

associated with it. From $\widetilde{\mathscr{S}_C}(v_0, v_{i+1}) = \phi$, we see that in $G - \{v_0, v_{i+1}, u\}$, there is no component which is disjoint from C. By Lemma 2.2 (ii), there exists k with $1 \le k \le i-1$ such that $\mathscr{S}_C(v_n, v_k) \ne \phi$. Further, applying Lemma 2.2 (ii) with $\{v_n, v_0, v_i\}$ replaced by $\{v_i, v_{i+1}, v_n\}$, we see that there exists m with $i+2 \le m \le n-1$ such that $\mathscr{S}_C(v_i, v_m) \ne \phi$. Take $P \in \mathscr{S}_C(v_n, v_k)$ and $Q \in \mathscr{S}_C(v_i, v_m)$. We now show that $u \in V(C)$. Suppose that $u \notin V(C)$. Then in $G - \{v_0, v_{i+1}, u\}$, $\{v_1, v_2, \dots, v_i\}$ and $\{v_{i+2}, v_{i+3}, \dots, v_n\}$ lie in distinct components. Since each of P and Q joins $v_1 v_2 \cdots v_i$ and $v_{i+2} v_{i+3} \cdots v_n$, this forces $u \in V(P) \cap V(Q)$. But then by Lemma 1.1 (i), $\mathscr{S}_C(v_k, v_m) \ne \phi$, which contradicts Lemma 2.2 (i). Thus $u \in V(C)$. Write $u = v_q$. By symmetry, we may assume $1 \le q \le i$. We show that $2 \le q \le i-1$. Suppose that $q = 1$. Then $\{v_0, v_1, v_{i+1}\}$ is a cutset by (2-1). Hence $v_0 v_1 \in E_{nc}(G)$ and $v_{i+1} \in K(v_0, v_1)$. Consequently, we get a contradiction by applying (i) with $\{v_n, v_0, v_i\}$ and $\{v_i, v_{i+1}, v_j\}$ replaced by $\{v_1, v_0, v_{i+1}\}$ and $\{v_{i+1}, v_i, v_j (= v_n)\}$, respectively. Similarly, if $q = i$, then by (2-1), $\{v_0, v_{i+1}, v_i\}$ is a cutset, which contradicts (i). Thus $2 \le q \le i-1$. Now since G is 3-connected, $G - \{v_0, v_q\}$ is connected, and hence there exist r, s with $r \in \{1, 2, \dots, q-1\}$ and $s \in \{q+1, q+2, \dots, n\}$ such that $\mathscr{S}_C(v_r, v_s) \ne \phi$. Note that $\{v_{q+1}, v_{q+2}, \dots, v_i\}$ and $\{v_{i+2}, v_{i+3}, \dots, v_n\}$ are contained in the same component of $G - \{v_0, v_{i+1}, v_q\}$ because Q joins $v_{q+1} v_{q+2} \cdots v_i$ and $v_{i+2} v_{i+3} \cdots v_m v_{m+1} \cdots v_n$. Now if $s \ne i+1$, then from $\mathscr{S}_C(v_r, v_s) \ne \phi$, we see that $\{v_1, v_2, \dots, v_{q-1}\}$, $\{v_{q+1}, v_{q+2}, \dots, v_i\}$ and $\{v_{i+2}, v_{i+3}, \dots, v_n\}$ are all contained in the same component of $G - \{v_0, v_{i+1}, v_q\}$, which contradicts the fact that there is no component of $G - \{v_0, v_{i+1}, v_q\}$ which is disjoint from C. Thus $s = i+1$, which implies

$\mathscr{S}_C(v_r, v_{i+1}) \neq \phi$. But this contradicts Lemma 2.2 (i), completing the proof.

Lemma 2.5. *If $i = 2$, then $v_1v_2 \in E_c(G)$.*

Proof. Considering symmetry, this follows immediately from Lemma 2.4 (i).

Lemma 2.6. *Suppose that v_0v_1 is noncontractible and $v_i \in K(v_0, v_1)$. Then $\widehat{\mathscr{S}_C}(v_n, v_1) \neq \phi$.*

Proof. It follows from Lemma 2.2 (ii) that there exists k with $1 \leq k \leq i-1$ such that $\mathscr{S}_C(v_n, v_k) \neq \phi$ and, applying Lemma 2.2 (i) to $\{v_0, v_1, v_i\}$, we get $k = 1$, which implies $\mathscr{S}_C(v_n, v_1) \neq \phi$. Now by way of contradiction, suppose that $\widehat{\mathscr{S}_C}(v_n, v_1) = \phi$. Then we have $v_nv_1 \in E_{nc}(G)$. Let

$$\{v_n, v_1, u\} \text{ be a cutset} \tag{2-2}$$

associated with it. Since the assumption $\widehat{\mathscr{S}_C}(v_n, v_1) = \phi$ implies $\widetilde{\mathscr{S}_C}(v_n, v_1) = \phi$, we see that in $G - \{v_n, v_1, u\}$, there is no component which is disjoint from C. Applying Lemma 2.2 (ii) to $\{v_n, v_0, v_i\}$ and $\{v_0, v_1, v_i\}$, we see that there exists m with $i + 1 \leq m \leq n - 1$ such that $\mathscr{S}_C(v_0, v_m) \neq \phi$, and there exists a with $2 \leq a \leq i - 1$ such that $\mathscr{S}_C(v_0, v_a) \neq \phi$. Take $P \in \mathscr{S}_C(v_0, v_m)$ and $Q \in \mathscr{S}_C(v_0, v_a)$. We now show that $u \in V(C)$. Suppose that $u \notin V(C)$. Then in $G - \{v_n, v_1, u\}$, v_0 and $\{v_2, v_3, \dots, v_{n-1}\}$ lie in distinct components. Since each of P and Q joins v_0 to $v_2v_3 \cdots v_{n-1}$, this forces $u \in V(P) \cap V(Q)$. But then by Lemma 1.1 (i), $\mathscr{S}_C(v_m, v_a) \neq \phi$, which contradicts Lemma 2.2 (i). Thus $u \in V(C)$. Write $u = v_q$. If $q = 0$, then we get a contradiction by Lemma 2.1. Thus $2 \leq q \leq n - 1$. By symmetry, we may assume $2 \leq q \leq i$. Suppose that $q = 2$. Then $\{v_n, v_1, v_2\}$ is a cutset by (2-2). Hence $v_1v_2 \in E_{nc}(G)$ and $v_n \in K(v_1, v_2)$. Consequently, we get a contradiction by applying Lemma 2.4 (i) with $\{v_n, v_0, v_i\}$ and $\{v_i, v_{i+1}, v_j\}$ replaced by $\{v_1, v_2, v_n\}$ and $\{v_n, v_0, v_i\}$, respectively. Thus $3 \leq q \leq i$. Now since G is 3-connected, $G - \{v_1, v_q\}$ is connected, and hence there exist r, s with $r \in \{2, 3, \dots, q - 1\}$ and $s \in \{0\} \cup \{q + 1, q + 2, \dots, n\}$ such that $\mathscr{S}_C(v_r, v_s) \neq \phi$. Note that v_0 and $\{v_{q+1}, v_{q+2}, \dots, v_{n-1}\}$ are contained in the same component of $G - \{v_n, v_1, v_q\}$ because P joins v_0 and $v_{q+1}v_{q+2} \cdots v_{n-1}$. Now if $s \neq n$, then from $\mathscr{S}_C(v_r, v_s) \neq \phi$, we see that v_0, $\{v_{q+1}, v_{q+2}, \dots, v_{n-1}\}$ and $\{v_2, v_3, \dots, v_{q-1}\}$ are contained in the same component of $G - \{v_n, v_1, v_q\}$, which contradicts the fact that there is no component of $G - \{v_n, v_1, v_q\}$ which is disjoint from C. Thus $s = n$, which implies $\mathscr{S}_C(v_r, v_n) \neq \phi$. But since $v_i \in K(v_0, v_1)$ by assumption, this contradicts Lemma 2.2 (i), completing the proof.

Lemma 2.7. *Let $1 \le j \le i-2$. Suppose that $v_j v_{j+1}$ is noncontractible, and let $\{v_j, v_{j+1}, v_l\}$ be a cutset associated with it, and suppose that $i+1 \le l \le n-1$. Then $l = i+1$, $v_i v_l \in E_c(G)$ and, unless $l = n-1$, we have $v_l \in K(v_n, v_0)$.*

Proof. Suppose that $l \ne i+1$. Then $\{v_k | i+1 \le k \le l-1\} \ne \phi$ and, by Lemma 2.2 (i), $\mathscr{S}_C(v_k, v_a) = \phi$ for any k, a with $i+1 \le k \le l-1$ and $1 \le a \le i-1$. Applying Lemma 2.2 (i) to $\{v_j, v_{j+1}, v_l\}$, we also see that $\mathscr{S}_C(v_k, v_b) = \phi$ for any k, b such that $i+1 \le k \le l-1$ and $l+1 \le b \le n$ or $0 \le b \le j-1$. Consequently, $\mathscr{S}_C(v_k, v_t) = \phi$ for any k, t such that $i+1 \le k \le l-1$ and $0 \le t \le i-1$ or $l+1 \le t \le n$, which means that $\{v_i, v_l\}$ is a cutset, a contradiction. Thus $l = i+1$. Suppose that $v_i v_l$ is noncontractible, and let $v_q \in K(v_i, v_l)$. Then $i+3 \le q \le n$ by Lemma 2.4 (i) and, applying Lemma 2.4 (i) to $\{v_{j+1}, v_j, v_l\}$ and $\{v_l, v_i, v_q\}$, we also get $j+1 \le q \le l-3$. But since $l-3 = i-2 < i+3$, this is impossible. Thus $v_i v_l$ is contractible. Now assume that $l \le n-2$. Then $\{v_k | l+1 \le k \le n-1\} \ne \phi$ and, by Lemma 2.2 (i), we see that $\mathscr{S}_C(v_k, v_a) = \phi$ for any k, a with $l+1 \le k \le n-1$ and $1 \le a \le i-1$, and that $\mathscr{S}_C(v_k, v_b) = \phi$ for any k, b with $l+1 \le k \le n-1$ and $j+2 \le b \le l-1$. Consequently, we get $\mathscr{S}_C(v_k, v_t) = \phi$ for any k, t with $l+1 \le k \le n-1$ and $1 \le t \le l-1$, which means $v_l \in K(v_n, v_0)$.

Lemma 2.8. *Suppose that $v_0 v_1$ is noncontractible, and let $\{v_0, v_1, v_j\}$ be a cutset associated with it, and suppose that $i+1 \le j \le n-2$. Then $v_j \in K(v_n, v_0)$.*

Proof. As in Lemma 2.7, applying Lemma 2.2 (i) to $\{v_n, v_0, v_i\}$ and $\{v_0, v_1, v_j\}$, we see that $\mathscr{S}_C(v_k, v_a) = \phi$ for any k, a with $j+1 \le k \le n-1$ and $1 \le a \le i-1$, and that $\mathscr{S}_C(v_k, v_b) = \phi$ for any k, b with $j+1 \le k \le n-1$ and $2 \le b \le j-1$, respectively. Consequently, $\mathscr{S}_C(v_k, v_t) = \phi$ for any k, t with $j+1 \le k \le n-1$ and $1 \le t \le j-1$, which means $v_j \in K(v_n, v_0)$.

3 Statement of Propositions

Let G be a 3-connected graph of order at least 5, and let C be a longest cycle of G such that no longest cycle has more contractible edges than C, i.e.,

$$\text{there is no longest cycle } C' \text{ of } G \text{ such that } |E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|. \quad (3\text{-}1)$$

If $E(C) \cap E_{nc}(G) = \phi$, then $E(C) \cap E_c(G) = E(C)$, and hence the Main

Theorem trivially holds. Thus we may assume that $E(C) \cap E_{nc}(G) \neq \phi$. Write

$$C = v_0 v_1 \cdots v_n v_0 \,, \text{ where } n \geq 4.$$

Without loss of generality, we may assume that $v_n v_0 \in E_{nc}(G)$. Let $\{v_n, v_0, v_k\}$ be a cutset associated with it (then $2 \leq k \leq n-2$ by Lemma 2.1). Set

$$P_1 = v_n v_0 v_1 \cdots v_k \text{ and } P_2 = v_k v_{k+1} \cdots v_n v_0 \,.$$

Then $|E(P_1)| + |E(P_2)| = |E(C)| + 1$, and $|E(P_1) \cap E_c(G)| + |E(P_2) \cap E_c(G)| = |E(C) \cap E_c(G)|$ because $v_n v_0 \in E_{nc}(G)$. Thus to prove the Main Theorem, it suffices to prove the following proposition:

Proposition 1. $|E(P_1) \cap E_c(G)| \geq \frac{1}{8}|E(P_1)| + \frac{4}{8}$ *and* $|E(P_2) \cap E_c(G)| \geq \frac{1}{8}|E(P_2)| + \frac{4}{8}$.

Now we define the following set $\mathscr{F}$:

Definition 3.1. $\mathscr{F} = \{ P \subsetneq C \mid$ there exist $x, y, z \in V(G)$ such that P is an x-z path, $xy \in E(P)$ and $z \in K(x,y) \}$.

Remark. Let $0 \leq i < j \leq n$. Then $v_i v_{i+1} \cdots v_j \in \mathscr{F}$ if and only if $v_j \in K(v_i, v_{i+1})$ or $v_i \in K(v_{j-1}, v_j)$.

Note that $P_1 \in \mathscr{F}$ and $P_2 \in \mathscr{F}$ because $v_k \in K(v_n, v_0)$. Hence Proposition 1 follows from the following proposition:

Proposition 2. *Let* $W \in \mathscr{F}$. *Then* $|E(W) \cap E_c(G)| \geq \frac{1}{8}|E(W)| + \frac{4}{8}$.

Thus we prove the Main Theorem by proving Proposition 2. The proof of Proposition 2 is given in Section 5. We conclude this section with two lemmas. The first lemma immediately follows from Lemmas 2.1 and 2.5.

Lemma 3.1. *Let* $P \in \mathscr{F}$. *Then* $|E(P)| \geq 3$. *Further if* $|E(P)| = 3$, *then* $|E(P) \cap E_c(G)| \geq 1$.

Lemma 3.2. *Let* $0 \leq i \leq n-3$, *and let* $A = v_i v_{i+1} v_{i+2} v_{i+3}$. *Suppose that* $A \in \mathscr{F}$. *Then one of the following holds:*

(i) $v_{i+3} \in K(v_i, v_{i+1})$ *and* $v_i v_{i+2} \in E(G)$; *or*

(ii) $v_i \in K(v_{i+2}, v_{i+3})$ *and* $v_{i+1} v_{i+3} \in E(G)$.

Proof. By the definition of $\mathscr{F}$, we have $v_{i+3} \in K(v_i, v_{i+1})$ or $v_i \in K(v_{i+2}, v_{i+3})$. If $v_{i+3} \in K(v_i, v_{i+1})$, then applying Lemma 2.3 with $\{v_n, v_0, v_i\}$ replaced by $\{v_i, v_{i+1}, v_{i+3}\}$, we obtain $v_i v_{i+2} \in E(G)$; if $v_i \in K(v_{i+2}, v_{i+3})$, then applying Lemma 2.3 with $\{v_n, v_0, v_i\}$ replaced by $\{v_{i+3}, v_{i+2}, v_i\}$, we obtain $v_{i+1}v_{i+3} \in E(G)$.

4 Admissible Partition

In [5], an *admissible partition* is defined under the condition that G is a 3-connected hamiltonian graph. But even if G is nonhamiltonian, an *admissible partition* can be defined in the identical way. In this section, we define an *admissible partition* for a 3-connected graph in general.

As in the preceding section, let G be a 3-connected graph of order at least 5, let C be a longest cycle of G satisfying (3-1), and write $C = v_0 v_1 \cdots v_n v_0$. Assume that $E(C) \cap E_{nc}(G) \neq \phi$, and let $\mathscr{F}$ be as in Definition 3.1. Note that for each of the lemmas in Sections 2 and 3 of [5], we have in this paper a corresponding lemma in Section 2 or 3. Thus the proof of the following lemma corresponds word for word to that of Lemma 4.29 in [5]:

Lemma 4.1. *Let $P \in \mathscr{F}$, and write $P = v_i v_{i+1} \cdots v_j$ (indices are to be read modulo $n+1$). Then there exists*

$$\mathscr{B} = \{ A_1, A_2, \ldots, A_t \}$$

such that

$$A_h \subsetneq P \text{ for all } 1 \le h \le t,$$

$$E(P) - \{v_i v_{i+1}\} = \bigcup_{h=1}^{t} E(A_h) \quad \text{(disjoint union)}$$

or

$$E(P) - \{v_{j-1} v_j\} = \bigcup_{h=1}^{t} E(A_h) \quad \text{(disjoint union)}$$

according as $v_j \in K(v_i, v_{i+1})$ or $v_i \in K(v_{j-1}, v_j)$,

$$|E(P) \cap E_c(G)| = \sum_{h=1}^{t} |E(A_h) \cap E_c(G)|,$$

and for each $1 \le h \le t$, one of the following holds:

(i) $A_h \in \mathscr{F}$ *(so $|E(A_h)| \ge 3$);*

(ii) $A_h \notin \mathscr{F}$, $|E(A_h)| = 5$ *and* $|E(A_h) \cap E_c(G)| \geq 2$;

(iii) $|E(A_h)| = 2$ *and* $|E(A_h) \cap E_c(G)| \geq 1$; *or*

(iv) $|E(A_h)| = |E(A_h) \cap E_c(G)| = 1$.

Having Lemma 4.1 in mind, we define an *admissible partition* in the same way as in Definition 4.5 of [5].

Definition 4.1 (an admissible partition). For $P \in \mathscr{F}$, a family $\mathscr{B} = \{ A_1, A_2, \ldots, A_l \}$ of paths satisfying the conditions stated in Lemma 4.1 is called an *admissible partition* of P. For $A_h \in \mathscr{B}$, A_h is said to be of *Type* R, *Type* F, *Type* S, or *Type* T according as A_h satisfies (i), (ii), (iii), or (iv) of Lemma 4.1.

5 Proof of Proposition 2

Recall that the Main Theorem follows from Proposition 2 as we saw in Section 3. Before proving Proposition 2, we prove the following lemma.

Lemma 5.1. *Let* $0 \leq i < j \leq n$. *Suppose that* $A = v_i v_{i+1} \cdots v_j$, $A \in \mathscr{F}$, $\mathscr{B}$ *is an admissible partition of* A, $|\mathscr{B}| = 1$, $\mathscr{B} = \{B\}$ *and* $B \in \mathscr{F}$. *Then one of the following two situations,* (I) *or* (II), *occurs.*

(I) $v_j \in K(v_i, v_{i+1})$, $B = v_{i+1} v_{i+2} \cdots v_j$ *and one of the following holds:*

- (i) $v_j \in K(v_{i+1}, v_{i+2})$ *and* $\widehat{\mathscr{P}_C}(v_i, v_{i+2}) \neq \phi$; *or*
- (ii) $v_{i+1} \in K(v_{j-1}, v_j)$ *and* $v_i v_{j-1} \in E_c(G)$.

(II) $v_i \in K(v_{j-1}, v_j)$, $B = v_i v_{i+1} \cdots v_{j-1}$ *and one of the following holds:*

- (i) $v_{j-1} \in K(v_i, v_{i+1})$ *and* $v_{i+1} v_j \in E_c(G)$; *or*
- (ii) $v_i \in K(v_{j-2}, v_{j-1})$ *and* $\widehat{\mathscr{P}_C}(v_{j-2}, v_j) \neq \phi$.

Proof. By the assumption that $A \in \mathscr{F}$, we have

$$v_j \in K(v_i, v_{i+1}), \tag{5-1}$$

or

$$v_i \in K(v_{j-1}, v_j). \tag{5-2}$$

First assume that (5-1) holds. Then it follows from Lemma 4.1 that $E(A) = E(B) \cup \{v_i v_{i+1}\}$ (disjoint union), which means $B = v_{i+1} v_{i+2} \cdots v_j$. Hence by the assumption that $B \in \mathscr{F}$,

$$v_j \in K(v_{i+1}, v_{i+2}), \tag{5-3}$$

or

$$v_{i+1} \in K(v_{j-1}, v_j). \tag{5-4}$$

If (5-3) holds, then by (5-1), we can apply Lemma 2.6 with $\{v_n, v_0, v_i\}$ and $\{v_0, v_1, v_i\}$ replaced by $\{v_i, v_{i+1}, v_j\}$ and $\{v_{i+1}, v_{i+2}, v_j\}$, to obtain $\widehat{\mathscr{F}_C}(v_i, v_{i+2}) \neq \phi$; if (5-4) holds, then by (5-1), we can apply Lemma 2.4 (ii) with $\{v_n, v_0, v_i\}$ and $\{v_i, v_{i+1}, v_j\}$ replaced by $\{v_{i+1}, v_i, v_j\}$ and $\{v_j, v_{j-1}, v_{i+1}\}$, to obtain $v_i v_{j-1} \in E_c(G)$. Thus (I) holds. By symmetry, we have (II) in the case where (5-2) holds.

Proof of Proposition 2. We proceed by induction on $|E(W)|$. By Lemma 3.1, $|E(W)| \geq 3$. If $|E(W)| = 3$, then $|E(W) \cap E_c(G)| \geq 1$ by Lemma 3.1, and hence $|E(W) \cap E_c(G)| \geq \frac{1}{8}|E(W)| + \frac{4}{8}$, as desired. Thus let

$$|E(W)| = w \geq 4, \tag{5-5}$$

and assume that

$$|E(P) \cap E_c(G)| \geq \frac{1}{8}|E(P)| + \frac{4}{8} \text{ for any } P \in \mathscr{F} \text{ such that } |E(P)| \leq w - 1. \tag{5-6}$$

Further by way of contradiction, suppose that

$$|E(W) \cap E_c(G)| < \frac{1}{8}|E(W)| + \frac{4}{8}. \tag{5-7}$$

Without loss of generality, we may assume that $W = v_0 v_1 \cdots v_w$. Since $W \in \mathscr{F}$, we may also assume that

$$v_w \in K(v_0, v_1), \tag{5-8}$$

i.e.,

$$v_0 v_1 \in E_{nc}(G). \tag{5-9}$$

We now prove the following claim:

Claim 5.1. *Let Q be a member of $\mathscr{F}$ such that $Q \subseteq W$, $|E(Q)| \geq 4$ and $|E(W)| - |E(Q)| \leq 3$, and let $\mathscr{B}$ be an admissible partition of Q. Then*

$$|\mathscr{B}| = 1,$$

and if we write $\mathscr{B} = \{B\}$, we have

$$|E(Q)| - |E(B)| = 1,$$

and

$$B \in \mathscr{F}.$$

Proof. Let $|\mathscr{B}| = t$ and $\mathscr{B} = \{A_1, A_2, \ldots, A_t\}$. We first prove the following subclaim:

Subclaim.

(i) $|E(A_h) \cap E_c(G)| \geq \frac{1}{8}|E(A_h)| + \frac{4}{8}$ *for all* $1 \leq h \leq t$.

(ii) $|E(Q)| = \sum_{h=1}^{t} |E(A_h)| + 1$.

(iii) $|E(Q) \cap E_c(G)| = \sum_{h=1}^{t} |E(A_h) \cap E_c(G)|$.

Proof. Statements (ii) and (iii) follow immediately from Lemma 4.1. We now prove (i). If A_h is of Type F, Type S or Type T, then the desired conclusion follows immediately from Lemma 4.1 (ii), (iii) and (iv). Thus we may assume that A_h is of Type R. Then it follows from Lemma 4.1 that $A_h \in \mathscr{F}$ and $E(A_h) \subsetneq E(Q)$, and hence $|E(A_h)| \leq |E(Q)| - 1 \leq w - 1$. Consequently, it follows from (5-6) that $|E(A_h) \cap E_c(G)| \geq \frac{1}{8}|E(A_h)| + \frac{4}{8}$, as desired.

Now since $|E(Q)| \geq |E(W)| - 3$ by assumption, it follows from Subclaim (ii) that $\sum_{h=1}^{t} |E(A_h)| \geq |E(W)| - 4$. By Subclaim (iii), we also have $|E(W) \cap E_c(G)| \geq \sum_{h=1}^{t} |E(A_h) \cap E_c(G)|$. Since $\sum_{h=1}^{t} |E(A_h) \cap E_c(G)| \geq \frac{1}{8}(\sum_{h=1}^{t} |E(A_h)|) + \frac{4}{8}t$ by Subclaim (i), we now obtain $|E(W) \cap E_c(G)| \geq \frac{1}{8}(|E(W)| - 4) + \frac{4}{8}t = \frac{1}{8}|E(W)| + \frac{4}{8}(t-1)$. If $t \geq 2$, then $|E(W) \cap E_c(G)| \geq \frac{1}{8}|E(W)| + \frac{4}{8}$, which contradicts (5-7). Thus $t = 1$. Hence by Subclaim (ii),

$$|E(Q)| - |E(A_1)| = 1. \tag{5-10}$$

It remains to show $A_1 \in \mathscr{F}$. Since $|E(Q)| \geq 4$ by assumption, $|E(A_1)| \geq 3$ by (5-10), and hence A_1 cannot be of Type S or T. Consequently, A_1 is of Type R or F by Lemma 4.1. Suppose that A_1 is of Type F. Then $|E(A_1) \cap E_c(G)| \geq 2$ and $|E(A_1)| = 5$, and hence

$$|E(W)| \leq |E(A_1)| + 4 = 9 \tag{5-11}$$

by (5-10) and the assumption that $|E(W)| \leq |E(Q)| + 3$. Consequently, $|E(W) \cap E_c(G)| \geq 2 \geq \frac{9}{8} + \frac{4}{8} \geq \frac{1}{8}|E(W)| + \frac{4}{8}$, which contradicts (5-7). Thus

A_1 is of Type R, and hence $A_1 \in \mathscr{F}$, as desired.

Returning to the proof of the proposition, let $\mathscr{B}$ be an admissible partition of $W \in \mathscr{F}$. Applying Claim 5.1 with $Q = W$, we see from (5-5) that $|\mathscr{B}| = 1$. Write $\mathscr{B} = \{B\}$. Then again by Claim 5.1,

$$|E(W)| - |E(B)| = 1 \tag{5-12}$$

and $B \in \mathscr{F}$. If $|E(B)| = 3$, then $|E(W)| = 4$ by (5-12) and $|E(W) \cap E_c(G)| \geq |E(B) \cap E_c(G)| \geq 1$ by Lemma 3.1, and hence $|E(W) \cap E_c(G)| \geq \frac{1}{8}|E(W)| + \frac{4}{8}$, which contradicts (5-7). Thus

$$|E(B)| \geq 4. \tag{5-13}$$

Let $\mathscr{D}$ be an admissible partition of $B \in \mathscr{F}$. By (5-12), (5-13) and Claim 5.1, $|\mathscr{D}| = 1$. Write $\mathscr{D} = \{D\}$. Then again by Claim 5.1, $D \in \mathscr{F}$ and $|E(B)| - |E(D)| = 1$, and hence

$$|E(W)| - |E(D)| = 2 \tag{5-14}$$

by (5-12). Now by (5-8), it follows from Lemma 5.1 (I) that $B = v_1 v_2 \cdots v_w$, and either

$$v_w \in K(v_1, v_2) \tag{5-15}$$

and

$$\widehat{\mathscr{F}_C}(v_0, v_2) \neq \phi, \tag{5-16}$$

or

$$v_1 \in K(v_{w-1}, v_w) \tag{5-17}$$

and

$$v_0 v_{w-1} \in E_c(G). \tag{5-18}$$

We now divide the proof into two cases, according as (5-15) and (5-16) hold, or (5-17) and (5-18) hold.

Case 1. (5-15) and (5-16) hold.
By (5-16), take

$$X_1 \in \widehat{\mathscr{F}_C}(v_0, v_2). \tag{5-19}$$

On the other hand, by (5-15), it follows from Lemma 5.1 (I) that $D = v_2 v_3 \cdots v_w$, and either

$$v_w \in K(v_2, v_3) \tag{5-20}$$

and

$$\widehat{\mathscr{F}_C}(v_1, v_3) \neq \phi, \tag{5-21}$$

or

$$v_2 \in K(v_{w-1}, v_w) \tag{5-22}$$

and

$$v_1 v_{w-1} \in E_c(G). \tag{5-23}$$

Case 1.1. (5-20) and (5-21) hold.
We can take $X_2 \in \widehat{\mathscr{S}_C}(v_1, v_3)$, and hence in view of (5-19), (5-9) and Lemma 1.1 (ii), we get a cycle

$$C' = X_1 X_2 v_4 v_5 \cdots v_w v_{w+1} \cdots v_n v_0$$

such that either $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$ or $|E(C')| > |E(C)|$, which contradicts (3-1) or the maximality of the length of C, respectively.

Case 1.2. (5-22) and (5-23) hold.
Note that (5-22) implies

$$v_{w-1} v_w \in E_{nc}(G). \tag{5-24}$$

Now if $|E(D)| = 3$, then by (5-22), it follows from Lemma 3.2 (ii) that $w = 5$ and $v_3 v_5 \in E(G)$, and hence in view of (5-19), (5-23), (5-9), (5-24), we obtain a cycle

$$C' = X_1 v_1 v_4 v_3 v_5 v_6 \cdots v_n v_0$$

such that either $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$ or $|E(C')| > |E(C)|$, which contradicts (3-1) or the maximality of the length of C, respectively. Thus $|E(D)| \geq 4$. Let $\mathscr{H}$ be an admissible partition of $D \in \mathscr{F}$. By (5-14) and Claim 5.1, $|\mathscr{H}| = 1$. Write $\mathscr{H} = \{H\}$. Then again by Claim 5.1, $H \in \mathscr{F}$ and $|E(D)| - |E(H)| = 1$, and hence

$$|E(W)| - |E(H)| = 3 \tag{5-25}$$

by (5-14). Moreover by (5-22), it follows from Lemma 5.1 (II) that $H = v_2 v_3 \cdots v_{w-1}$, and either

$$v_{w-1} \in K(v_2, v_3) \tag{5-26}$$

and

$$v_3 v_w \in E_c(G), \tag{5-27}$$

or

$$v_2 \in K(v_{w-2}, v_{w-1}) \tag{5-28}$$

and

$$\widehat{\mathscr{S}_C}(v_{w-2}, v_w) \neq \phi. \tag{5-29}$$

Case 1.2.1. (5-26) and (5-27) hold.
In view of (5-19), (5-23), (5-9), we get a cycle

$$C' = X_1 v_1 v_{w-1} v_{w-2} v_{w-3} \cdots v_4 v_3 v_w v_{w+1} v_{w+2} \cdots v_n v_0$$

such that either $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$ or $|E(C')| > |E(C)|$, which contradicts (3-1) or the maximality of the length of C, respectively.

Case 1.2.2. (5-28) and (5-29) hold.
Take

$$X_3 \in \widehat{\mathscr{S}_C}(v_{w-2}, v_w). \tag{5-30}$$

Claim 5.2. $|E(H)| \geq 4$.

Proof. By way of contradiction, suppose that $|E(H)| = 3$. Then by (5-28), it follows from Lemma 3.2 (ii) that $w = 6$ and $v_3 v_5 \in E(G)$. If $(V(X_1) - \{v_0, v_2\}) \cap (V(X_3) - \{v_4, v_6\}) = \phi$, then in view of (5-19), (5-23), (5-30), (5-24), (5-9), we obtain a cycle

$$C' = X_1 v_1 v_5 v_3 X_3 v_7 v_8 \cdots v_n v_0$$

such that either $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$ or $|E(C')| > |E(C)|$, which contradicts (3-1) or the maximality of the length of C, respectively. Thus $(V(X_1) - \{v_0, v_2\}) \cap (V(X_3) - \{v_4, v_6\}) \neq \phi$. Hence by (5-19) and (5-30), it follows from Lemma 1.1 (i) that $\widetilde{\mathscr{S}_C}(v_0, v_4) \neq \phi$. Take $X_4 \in \widetilde{\mathscr{S}_C}(v_0, v_4)$. Then in view of (5-23), we obtain a cycle

$$C' = X_4 v_3 v_2 v_1 v_5 v_6 v_7 \cdots v_n v_0$$

such that $|E(C')| > |E(C)|$, which contradicts the maximality of the length of C.

Returning to the proof of the proposition, let $\mathscr{M}$ be an admissible partition of $H \in \mathscr{F}$. Then by (5-25), Claim 5.2 and Claim 5.1, $|\mathscr{M}| = 1$. Write $\mathscr{M} = \{M\}$. Then again by Claim 5.1, $M \in \mathscr{F}$. Therefore by (5-28), it follows from Lemma 5.1 (II) that

$$\text{either } v_3 v_{w-1} \in E_c(G) \text{ or } \widehat{\mathscr{S}_C}(v_{w-3}, v_{w-1}) \neq \phi.$$

Case 1.2.2.1. $\widehat{\mathscr{S}_C}(v_{w-3}, v_{w-1}) \neq \phi$.
We can take $X_5 \in \widehat{\mathscr{S}_C}(v_{w-3}, v_{w-1})$, and hence in view of (5-30), (5-24) and Lemma 1.1 (ii), we get a cycle

$$C' = v_0 v_1 \cdots v_{w-4} X_5 X_3 v_{w+1} v_{w+2} \cdots v_n v_0$$

such that either $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$ or $|E(C')| > |E(C)|$, which contradicts (3-1) or the maximality of the length of C, respectively.

Case 1.2.2.2. $v_3 v_{w-1} \in E_c(G)$.
If $(V(X_1) - \{v_0, v_2\}) \cap (V(X_3) - \{v_{w-2}, v_w\}) = \phi$, then in view of (5-19), (5-23), (5-30), (5-9), we get a cycle

$$C' = X_1 v_1 v_{w-1} v_3 v_4 v_5 \cdots v_{w-3} X_3 v_{w+1} v_{w+2} \cdots v_n v_0$$

such that either $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$ or $|E(C')| > |E(C)|$, which contradicts (3-1) or the maximality of the length of C, respectively. Thus $(V(X_1) - \{v_0, v_2\}) \cap (V(X_3) - \{v_{w-2}, v_w\}) \neq \phi$. Hence by (5-19) and (5-30), it follows from Lemma 1.1 (i) that $\widetilde{\mathscr{S}_C}(v_0, v_{w-2}) \neq \phi$. Take $X_6 \in \widetilde{\mathscr{S}_C}(v_0, v_{w-2})$. Then in view of (5-23), we obtain a cycle

$$C' = X_6 v_{w-3} v_{w-4} \cdots v_3 v_2 v_1 v_{w-1} v_w v_{w+1} \cdots v_n v_0$$

such that $|E(C')| > |E(C)|$, which contradicts the maximality of the length of C. This concludes the discussion for Case 1.

Case 2. (5-17) and (5-18) hold.
Note that (5-17) implies

$$v_{w-1} v_w \in E_{nc}(G). \tag{5-31}$$

By (5-17), it also follows from Lemma 5.1 (II) that $D = v_1 v_2 \cdots v_{w-1}$, and either

$$v_{w-1} \in K(v_1, v_2) \tag{5-32}$$

and

$$v_2 v_w \in E_c(G), \tag{5-33}$$

or

$$v_1 \in K(v_{w-2}, v_{w-1}) \tag{5-34}$$

and

$$\widehat{\mathscr{S}_C}(v_{w-2}, v_w) \neq \phi. \tag{5-35}$$

We now divide the proof into two subcases, according as (5-32) and (5-33) hold, or (5-34) and (5-35) hold.

Case 2.1. (5-32) and (5-33) hold.
If $|E(D)| = 3$, then by (5-32), it follows from Lemma 3.2 (i) that $w = 5$ and

$v_1v_3 \in E(G)$, and hence in view of (5-18), (5-33), (5-9), (5-31), we obtain a cycle

$$C' = v_0v_4v_3v_1v_2v_5v_6v_7 \cdots v_nv_0$$

such that $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$, which contradicts (3-1). Thus $|E(D)| \geq 4$. Let $\mathscr{H}$ be an admissible partition of $D \in \mathscr{F}$. By (5-14) and Claim 5.1, $|\mathscr{H}| = 1$. Write $\mathscr{H} = \{H\}$. Then again by Claim 5.1, $H \in \mathscr{F}$ and $|E(D)| - |E(H)| = 1$, and hence

$$|E(W)| - |E(H)| = 3 \tag{5-36}$$

by (5-14). Moreover by (5-32), it follows from Lemma 5.1 (I) that $H = v_2v_3 \cdots v_{w-1}$, and either

$$v_{w-1} \in K(v_2, v_3) \tag{5-37}$$

and

$$\widehat{\mathscr{S}_C}(v_1, v_3) \neq \phi, \tag{5-38}$$

or

$$v_2 \in K(v_{w-2}, v_{w-1}) \tag{5-39}$$

and

$$v_1v_{w-2} \in E_c(G). \tag{5-40}$$

Case 2.1.1. (5-37) and (5-38) hold.
We can take $X_7 \in \widehat{\mathscr{S}_C}(v_1, v_3)$, and hence in view of (5-18), (5-33), (5-9), we get a cycle

$$C' = v_0v_{w-1}v_{w-2}v_{w-3} \cdots v_4\overline{X}_7v_2v_wv_{w+1}v_{w+2} \cdots v_nv_0$$

such that either $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$ or $|E(C')| > |E(C)|$, which contradicts (3-1) or the maximality of the length of C, respectively.

Case 2.1.2. (5-39) and (5-40) hold.
If $|E(H)| = 3$, then by (5-39), it follows from Lemma 3.2 (ii) that $w = 6$ and $v_3v_5 \in E(G)$, and hence in view of (5-18), (5-40), (5-33), (5-9), (5-31), we obtain a cycle

$$C' = v_0v_5v_3v_4v_1v_2v_6v_7v_8 \cdots v_nv_0$$

such that $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$, which contradicts (3-1). Thus $|E(H)| \geq 4$. Let $\mathscr{M}$ be an admissible partition of $H \in \mathscr{F}$. By (5-36) and Claim 5.1, $|\mathscr{M}| = 1$. Write $\mathscr{M} = \{M\}$. Then again by Claim 5.1, $M \in \mathscr{F}$. Therefore by (5-39), it follows from Lemma 5.1 (II) that

$$\text{either } v_3 v_{w-1} \in E_c(G) \text{ or } \widehat{\mathscr{F}_C}(v_{w-3}, v_{w-1}) \neq \phi.$$

Case 2.1.2.1. $\widehat{\mathscr{F}_C}(v_{w-3}, v_{w-1}) \neq \phi$.
We can take $X_8 \in \widehat{\mathscr{F}_C}(v_{w-3}, v_{w-1})$, and hence in view of (5-40), (5-33), (5-31), we get a cycle

$$C' = v_0 v_1 v_{w-2} \overline{X}_8 v_{w-4} v_{w-5} \cdots v_3 v_2 v_w v_{w+1} v_{w+2} \cdots v_n v_0$$

such that either $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$ or $|E(C')| > |E(C)|$, which contradicts (3-1) or the maximality of the length of C, respectively.

Case 2.1.2.2. $v_3 v_{w-1} \in E_c(G)$.
In view of (5-18), (5-40), (5-33), (5-31), we get a cycle

$$C' = v_0 v_{w-1} v_3 v_4 v_5 \cdots v_{w-3} v_{w-2} v_1 v_2 v_w v_{w+1} v_{w+2} \cdots v_n v_0$$

such that $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$, which contradicts (3-1).

Case 2.2. (5-34) and (5-35) hold.
By (5-35), take

$$X_9 \in \widehat{\mathscr{F}_C}(v_{w-2}, v_w). \tag{5-41}$$

We now prove the following claim:

Claim 5.3. $|E(D)| \geq 4$.

Proof. By way of contradiction, suppose that $|E(D)| = 3$. Then by (5-34), it follows from Lemma 3.2 (ii) that $w = 5$ and

$$v_2 v_4 \in E(G). \tag{5-42}$$

On the other hand, applying Lemma 2.5 with $\{v_n, v_0, v_i\}$ replaced by $\{v_4, v_3, v_1\}$, we see from (5-34) that $v_1 v_2 \in E_c(G)$. Hence if $v_2 v_3 \in E_c(G)$, then $|E(W) \cap E_c(G)| \geq 2$, which implies $|E(W) \cap E_c(G)| > \frac{9}{8} = \frac{1}{8}|E(W)| + \frac{4}{8}$, contradicting (5-7). Thus

$$v_2 v_3 \notin E_c(G). \tag{5-43}$$

Consequently in view of (5-42), (5-41), (5-31), (5-43), we obtain a cycle

$$C' = v_0 v_1 v_2 v_4 X_9 v_6 v_7 \cdots v_n v_0$$

such that either $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$ or $|E(C')| > |E(C)|$, which contradicts (3-1) or the maximality of the length of C, respectively.

Returning to the proof of the proposition, let $\mathscr{H}$ be an admissible partition of $D \in \mathscr{F}$. Then by (5-14), Claim 5.3 and Claim 5.1, $|\mathscr{H}| = 1$. Write $\mathscr{H} = \{H\}$. Then again by Claim 5.1, $H \in \mathscr{F}$ and $|E(D)| - |E(H)| = 1$, and hence

$$|E(W)| - |E(H)| = 3 \tag{5-44}$$

by (5-14). Moreover by (5-34), it follows from Lemma 5.1 (II) that $H = v_1 v_2 \cdots v_{w-2}$, and either

$$v_{w-2} \in K(v_1, v_2) \tag{5-45}$$

and

$$v_2 v_{w-1} \in E_c(G), \tag{5-46}$$

or

$$v_1 \in K(v_{w-3}, v_{w-2}) \tag{5-47}$$

and

$$\widehat{\mathscr{S}_C}(v_{w-3}, v_{w-1}) \neq \phi. \tag{5-48}$$

Case 2.2.1. (5-47) and (5-48) hold.
We can take $X_{10} \in \widehat{\mathscr{S}_C}(v_{w-3}, v_{w-1})$, and hence in view of (5-41), (5-31) and Lemma 1.1 (ii), we get a cycle

$$C' = v_0 v_1 \cdots v_{w-4} X_{10} X_9 v_{w+1} v_{w+2} \cdots v_n v_0$$

such that either $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$ or $|E(C')| > |E(C)|$, which contradicts (3-1) or the maximality of the length of C, respectively.

Case 2.2.2. (5-45) and (5-46) hold.
If $|E(H)| = 3$, then by (5-45), it follows from Lemma 3.2 (i) that $w = 6$ and $v_1 v_3 \in E(G)$, and hence in view of (5-18), (5-46), (5-41), (5-9), (5-31), we obtain a cycle

$$C' = v_0 v_5 v_2 v_1 v_3 X_9 v_7 v_8 \cdots v_n v_0$$

such that either $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$ or $|E(C')| > |E(C)|$, which contradicts (3-1) or the maximality of the length of C, respectively. Thus $|E(H)| \geq 4$. Let $\mathscr{M}$ be an admissible partition of $H \in \mathscr{F}$. Then by (5-44) and Claim 5.1, $|\mathscr{M}| = 1$. Write $\mathscr{M} = \{M\}$. Then again by Claim 5.1, $M \in \mathscr{F}$. Therefore by (5-45), it follows from Lemma 5.1 (I) that

$$\text{either } \widehat{\mathscr{P}_C}(v_1, v_3) \neq \phi \text{ or } v_1 v_{w-3} \in E_c(G).$$

Case 2.2.2.1. $v_1 v_{w-3} \in E_c(G)$.
In view of (5-46), (5-41), (5-31), we get a cycle

$$C' = v_0 v_1 v_{w-3} v_{w-4} v_{w-5} \cdots v_3 v_2 v_{w-1} X_9 v_{w+1} v_{w+2} \cdots v_n v_0$$

such that either $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$ or $|E(C')| > |E(C)|$, which contradicts (3-1) or the maximality of the length of C, respectively.

Case 2.2.2.2. $\widehat{\mathscr{P}_C}(v_1, v_3) \neq \phi$.
Take

$$X_{11} \in \widehat{\mathscr{P}_C}(v_1, v_3). \tag{5-49}$$

If $(V(X_{11}) - \{v_1, v_3\}) \cap (V(X_9) - \{v_{w-2}, v_w\}) = \phi$, then in view of (5-18), (5-46), (5-49), (5-41), (5-31), we get a cycle

$$C' = v_0 v_{w-1} v_2 X_{11} v_4 v_5 \cdots v_{w-3} X_9 v_{w+1} v_{w+2} \cdots v_n v_0$$

such that either $|E(C')| = |E(C)|$ and $|E(C') \cap E_c(G)| > |E(C) \cap E_c(G)|$ or $|E(C')| > |E(C)|$, which contradicts (3-1) or the maximality of the length of C, respectively. Thus $(V(X_{11}) - \{v_1, v_3\}) \cap (V(X_9) - \{v_{w-2}, v_w\}) \neq \phi$. Hence by (5-49) and (5-41), it follows from Lemma 1.1 (i) that $\widetilde{\mathscr{P}_C}(v_1, v_{w-2}) \neq \phi$. Take $X_{12} \in \widetilde{\mathscr{P}_C}(v_1, v_{w-2})$. Then in view of (5-46), we obtain a cycle

$$C' = v_0 X_{12} v_{w-3} v_{w-4} \cdots v_3 v_2 v_{w-1} v_w v_{w+1} \cdots v_n v_0$$

such that $|E(C')| > |E(C)|$, which contradicts the maximality of the length of C.

This completes the proof of Proposition 2, and hence also the proof of the Main Theorem.

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