An Extremal Problem on the Potentially $K_{r+1} - e$ Graphic Sequences *

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Abstract. Gould et al.(Combinatorics, Graph Theory and Algorithms, Vol. 1(1999), 387–400)considered a variation of the classical Turán-type extremal problems as follows: for a given graph H, determine the smallest even integer $\sigma(H,n)$ such that every n-term positive graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with term sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n \geq \sigma(H,n)$ has a realization G containing H as a subgraph. In particular, they pointed out that $3n-2 \leq \sigma(K_4 - e, n) \leq 4n-4$, where $K_{r+1} - e$ denotes the graph obtained by removing one edge from the complete graph K_{r+1} on $r+1$ vertices. Recently, Lai determined the values of $\sigma(K_4 - e, n)$ for $n \geq 4$. In this paper, we determine the values of $\sigma(K_{r+1} - e, n)$ for $r \geq 3$ and $r+1 \leq n \leq 2r$, and give a lower bound of $\sigma(K_{r+1} - e, n)$. In addition, we prove that $\sigma(K_5 - e, n) = 5n-6$ for even n and $n \geq 10$ and $\sigma(K_5 - e, n) = 5n-7$ for odd n and $n \geq 9$.
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1. Introduction

A non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is said to be *graphic* if it is the degree sequence of a simple graph on n vertices, and such graph is called a *realization* of π. If each term of a graphic sequence π is non-zero, then π is called *positive graphic*. For a graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$, define $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. For a given graph H, a graphic sequence π is *potentially H graphic* if there exists a realization of π containing H as a subgraph. Gould et al. [2] considered the following variation of the classical Turán-type extremal problems: determine the smallest even integer $\sigma(H, n)$ such that every n-term positive graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with $\sigma(\pi) \geq \sigma(H, n)$ has a realization G containing H as a subgraph, and proved that $\sigma(C_4, n) = 3n - 1$ for odd n and $\sigma(C_4, n) = 3n - 2$ for even n, where C_4 is a cycle of length 4. If $H = K_{r+1}$, this problem was considered by Erdős et al.[1] where they showed that $\sigma(K_3, n) = 2n$ for $n \geq 6$ and conjectured that $\sigma(K_{r+1}, n) = (r-1)(2n-r) + 2$ for sufficiently large n. Gould et al. [2] and Li and Song [5] also independently proved that $\sigma(K_4, n) = 4n - 4$ for $n \geq 8$, i.e., the conjecture holds for $r = 3$ and $n \geq 8$. Recently, Li et al. [6,7] showed that the conjecture is holds for $r = 4$ and $n \geq 10$ and for $r \geq 5$ and $n \geq \binom{r}{2} + 3$. In the end of [2], Gould et al. pointed out that $\sigma(C_4, n) \leq \sigma(K_4 - e, n) \leq \sigma(K_4, n)$, and hence it would be nice to see where in the range from $3n - 2$ to $4n - 4$, the value $\sigma(K_4 - e, n)$ lies. Recently, Lai [4] further determined the exact value of $\sigma(K_4 - e, n)$, i.e.,

Theorem 1.1. $\sigma(K_4 - e, n) = \begin{cases} 2\left[\frac{3n-1}{2}\right] & \text{if } n \geq 4 \text{ and } n \neq 6, \\ 20 & \text{if } n = 6, \end{cases}$

where $[x]$ denotes the integer part of x.

The purpose of this paper is to determine the values of $\sigma(K_{r+1} - e, n)$ for $r \geq 3$ and $r+1 \leq n \leq 2r$ and give a lower bound of $\sigma(K_{r+1} - e, n)$, and prove that $\sigma(K_5 - e, n) = 5n - 6$ for even n and $n \geq 10$ and $\sigma(K_5 - e, n) = 5n - 7$ for odd n and $n \geq 9$. In order to prove our results, the following notations and results are needed.

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The degree of v_i is denoted by d_i for $1 \leq i \leq n$. Then $\pi = (d_1, d_2, \ldots, d_n)$ is the degree sequence of G, where $d_1, d_2, \ldots, d_n$ may be not in non-increasing order. The degree sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be *potentially A_{r+1} graphic*(resp. *$A_{r+1} - e$ graphic*) if it has a realization $H = (V(H), E(H))$, where $V(H) = \{u_1, u_2, \ldots, u_n\}$ and the degree of u_i is d_i for $1 \leq i \leq n$, such that the subgraph induced by $\{u_1, u_2, \ldots, u_{r+1}\}$ is K_{r+1}(resp. contains $K_{r+1} - e$ as a subgraph).

Theorem 1.2. [2] If $\pi = (d_1, d_2, \ldots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization

G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π. In particular, A graphic sequence π is potentially K_{r+1} graphic(resp. $K_{r+1}-e$ graphic) if and only if it is potentially A_{r+1} graphic(resp. $A_{r+1}-e$ graphic).

Theorem 1.3. [7,8] Let $n \geq 2r+2$, and let $\pi=(d_1,d_2,\ldots,d_n)$ be a graphic sequence with $d_{r+1} \geq r$. If $n-2 \geq d_1 \geq \cdots \geq d_r = d_{r+1} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n \geq r-1$, then π is potentially A_{r+1} graphic.

Let $\pi=(d_1,d_2,\ldots,d_n)$ be a non-increasing sequence of nonnegative integers, and let

$$\pi_k'' = \begin{cases} (d_1-1,\ldots,d_{k-1}-1,d_{k+1}-1,\ldots,d_{d_k+1}-1,d_{d_k+2},\ldots,d_n) \\ \text{if } d_k \geq k, \\ (d_1-1,\ldots,d_{d_k}-1,d_{d_k+1},\ldots,d_{k-1},d_{k+1},\ldots,d_n) \\ \text{if } d_k < k. \end{cases}$$

Denote $\pi_k' = (d_1',d_2',\ldots,d_{n-1}')$, where $d_1' \geq d_2' \geq \cdots \geq d_{n-1}'$ is the rearrangement of the $n-1$ terms in π_k''. Then π_k' is called the *residual sequence* obtained by laying off d_k from π.

Theorem 1.4. [3] Let $\pi=(d_1,d_2,\ldots,d_n)$ be a non-increasing sequence of nonnegative integers. Then π is graphic if and only if π_k' is graphic. Moreover, one realization G of π can be obtained from any one realization G' of π_k' by adding a new vertex v_k' of degree d_k to G' and joining it to the vertices whose degrees are reduced by one in going from π to π_k'.

Theorem 1.5. [5] If $n \geq 8$, then $\sigma(K_4,n)=4n-4$.

2. The value $\sigma(K_{r+1}-e,n)$ for small n and a lower bound of $\sigma(K_{r+1}-e,n)$

Theorem 2.1. If $r \geq 3$ and $r+1 \leq n \leq 2r-2$, then

$$\sigma(K_{r+1}-e,n)=(r-1)(2n-r)+(n-r)(n-r-1).$$

Proof. Take $\pi=((n-1)^{2r-2-n},(n-2)^{2n-2r+2})$, where the symbol x^y stands for y consecutive terms x. It is easy to see that the only graph realizing π is $K_n-(n-r+1)K_2$, where the graph operation $-$ which is used in this paper only means deletion of edges and pK_2 denotes the union of p complete graphs K_2. Since $K_n-(n-r+1)K_2$ contains no $K_{r+1}-e$ as a subgraph, π is not potentially $K_{r+1}-e$ graphic. Hence $\sigma(K_{r+1}-e,n) \geq \sigma(\pi)+2=(r-1)(2n-r)+(n-r)(n-r-1)$.

Now suppose that $\pi=(d_1,d_2,\ldots,d_n)$ is a positive graphic sequence with $\sigma(\pi) \geq (r-1)(2n-r)+(n-r)(n-r-1)$. Moreover, suppose that G is a realization of π and G^c is the complementary graph of G . Then

$2|E(G)| = \sigma(\pi)$ and $2|E(G^c)| = n(n-1) - \sigma(\pi) \leq 2(n-r)$. Hence G^c is a graph on n vertices with at most $n-r$ edges. Assume that $B_1, B_2, \ldots, B_x$ are all nontrivial connected components of G^c, where $x \leq n-r \leq 2r-2-r = r-2$. Then $|V(B_i)| \leq |E(B_i)| + 1$ for $i = 1, \ldots, x$, and hence $|V(G^c) \setminus V(\cup_{i=1}^{x} B_i)| = n - \sum_{i=1}^{x} |V(B_i)| \geq n - \sum_{i=1}^{x} |E(B_i)| - x \geq n-(n-r)-x = r-x$. Now let $v_1, v_2, \ldots, v_{r-x}$ be $r-x$ vertices of $V(G^c) \setminus V(\cup_{i=1}^{x} B_i)$, and take $u_1, u \in B_1$ and $u_i \in B_i$ for $i = 2, \ldots, x$. Then the subgraph induced by $\{v_1, v_2, \ldots, v_{r-x}, u, u_1, u_2, \ldots, u_x\}$ in G^c is K_{r+1}^c or $K_{r-1}^c \cup K_2$. Hence G contains K_{r+1} or $K_{r+1}-e$ as a subgraph. Thus π is potentially $K_{r+1}-e$ graphic. In other words, $\sigma(K_{r+1}-e, n) \leq (r-1)(2n-r) + (n-r)(n-r-1)$.□

Theorem 2.2. If $r \geq 3$ and $2r-1 \leq n \leq 2r$, then

$$\sigma(K_{r+1}-e, n) = (r-1)(2n-r) + (n-r-1)(n-r-2).$$

Proof. If $n = 2r-1$, we consider $\pi = (2r-2, (2r-4)^{2r-2})$. Clearly, π is graphic and $\sigma(\pi) = (2r-2)(2r-3) = (r-1)(2n-r)+(n-r-1)(n-r-2)-2$. Assume that G is a realization of π. Then the degree sequence of G^c is $\pi^c = (2^{2r-2}, 0)$. Hence $G^c = K_1 \cup G_1$, where $|V(G_1)| = 2r-2$ and G_1 is the union of disjoint cycles. It is easy to see that the subgraph induced by any r vertices in G_1 contains at least two edges. Hence G contains no $K_{r+1}-e$ as a subgraph. Thus π is not potentially $K_{r+1}-e$ graphic. In other words, $\sigma(K_{r+1}-e, 2r-1) \geq \sigma(\pi) + 2 = (r-1)(2n-r) + (n-r-1)(n-r-2)$.

If $n = 2r$, we consider $\pi = ((2r-3)^{2r})$. Then π is graphic and $\sigma(\pi) = 2r(2r-3) = (r-1)(2n-r)+(n-r-1)(n-r-2)-2$. If G is a realization of π, then G^c is the union of disjoint cycles. Since the subgraph induced by any $r+1$ vertices in G^c contains at least two edges, π is not potentially $K_{r+1}-e$ graphic. So $\sigma(K_{r+1}-e, 2r) \geq \sigma(\pi)+2 = (r-1)(2n-r)+(n-r-1)(n-r-2)$.

In order to show that $(*)$: $\sigma(K_{r+1}-e, n) \leq (r-1)(2n-r) + (n-r-1)(n-r-2)$ for $2r-1 \leq n \leq 2r$, we use induction on $r(\geq 3)$. It follows from Theorem 1.1 that $(*)$ holds for $r = 3$. Now suppose that $(*)$ holds for $r-1 (r \geq 4)$, and let $\pi = (d_1, d_2, \ldots, d_n)$ be a positive graphic sequence with $\sigma(\pi) \geq (r-1)(2n-r) + (n-r-1)(n-r-2)$. It is enough to prove that π is potentially $K_{r+1}-e$ graphic. We consider the following two cases:

Case 1. $n = 2r-1$. Then $\pi = (d_1, d_2, \ldots, d_{2r-1})$ satisfies $\sigma(\pi) \geq (r-1)(2n-r) + (n-r-1)(n-r-2) = (2r-2)(2r-3)+2$, and hence $2r-2 \geq d_1 \geq 2r-3$. If $d_{2r-1} = 1$, then $\pi = (2r-2, (2r-3)^{2r-3}, 1)$. It is easy to see that the unique realization of π contains K_{2r-2} as a subgraph, and clearly contains $K_{r+1}-e$ as a subgraph. Hence π is potentially $K_{r+1}-e$ graphic. Now assume that $d_{2r-1} \geq 2$, and let $\pi_1' = (d_1', d_2', \ldots, d_{2r-2}')$ be the residual sequence obtained by laying off d_1 from π. Then π_1' is a positive graphic sequence and $\sigma(\pi_1') = \sigma(\pi) - 2d_1 \geq (2r-2)(2r-3)+2-2(2r-2) = \sigma(K_{(r-1)+1}-e, 2(r-1))$. By induction hypothesis and Theorem 1.2, π_1' is potentially $A_r - e$ graphic. If $d_1 = n-1 = 2r-2$, or $d_1 = 2r-3$ and there exists an integer t, $r+1 \le t \le d_1+1$ such that $d_t > d_{t+1}$, then $d_2-1,\dots,d_{r+1}-1$ are the r largest terms in π'_1. Thus π is potentially $A_{r+1}-e$ graphic. So we may assume that

$$2r-3 = d_1 \ge d_2 \ge \cdots \ge d_{r+1} = \cdots = d_{2r-1}.$$

If $d_{r+1} \le 2r-5$, then $\sigma(\pi) \le (2r-3)r + (2r-5)(r-1) = 4r^2-10r+5 < (2r-2)(2r-3)+2 \le \sigma(\pi)$, a contradiction. If $d_{r+1} = 2r-3$, then $\sigma(\pi) = (2r-1)(2r-3)$ is odd, and π is not graphic, which is impossible. Hence $d_{r+1} = 2r-4$, and so $\pi = ((2r-3)^t, (2r-4)^{2r-1-t})$, where t is even and $2 \le t \le r$. If $t=2$, then $\sigma(\pi) = 2(2r-3) + (2r-4)(2r-3) = (2r-2)(2r-3) < \sigma(\pi)$, a contradiction. Assume $4 \le t \le r$, and let $G = K_{2r-1} - ((\frac{t}{2}-1)K_2 \cup P_{2r-t})$, where P_{2r-t} is a path of length $2r-t$. Clearly, G is a realization of π and contains $K_{r+1}-e$ as a subgraph. Hence π is potentially $K_{r+1}-e$ graphic. In other words, $\sigma(K_{r+1}-e,n) = (r-1)(2n-r) + (n-r-1)(n-r-2)$ for $n = 2r-1$.

Case 2. $n = 2r$. Then $\pi = (d_1, d_2, \dots, d_{2r})$ satisfies $\sigma(\pi) \ge (r-1)(2n-r) + (n-r-1)(n-r-2) = 2r(2r-3)+2$. Let $\pi'_{2r} = (d'_1, d'_2, \dots, d'_{2r-1})$ be the residual sequence obtained by laying off d_{2r} from π. If $d_{2r} \le 2r-3$, then $\sigma(\pi'_{2r}) = \sigma(\pi) - 2d_{2r} \ge 2r(2r-3)+2-2(2r-3) = (2r-2)(2r-3)+2 = \sigma(K_{r+1}-e, 2r-1)$. By Case 1, π'_{2r} is potentially $K_{r+1}-e$ graphic, and hence π is potentially $K_{r+1}-e$ graphic. If $d_{2r} \ge 2r-2$, then $\pi = ((2r-1)^t, (2r-2)^{2r-t})$, where t is even. It is easy to see that $K_{2r} - (r-\frac{t}{2})K_2$ is the only realization of π, and contains $K_{r+1}-e$ as a subgraph. Hence π is also potentially $K_{r+1}-e$ graphic. This shows that $\sigma(K_{r+1}-e,n) = (r-1)(2n-r) + (n-r-1)(n-r-2)$ for $n = 2r$. □

Now we give a lower bound of $\sigma(K_{r+1}-e,n)$.

Theorem 2.3. If $r \ge 2$ and $n \ge r+1$, then

$$\sigma(K_{r+1}-e,n) \ge \begin{cases} (r-1)(2n-r)+2-(n-r) & \text{if } n-r \text{ is even,} \\ (r-1)(2n-r)+1-(n-r) & \text{if } n-r \text{ is odd.} \end{cases} \quad (1)$$

Proof. Let

$$G = \begin{cases} K_{r-2} + (\frac{n-r}{2}+1)K_2 & \text{if } n-r \text{ is even,} \\ K_{r-2} + (\frac{n-r+1}{2}K_2 \cup K_1) & \text{if } n-r \text{ is odd,} \end{cases}$$

where the join G_1+G_2 of the graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2. Then

$$\pi = \begin{cases} ((n-1)^{r-2}, (r-1)^{n-r+2}) & \text{if } n-r \text{ is even,} \\ ((n-1)^{r-2}, (r-1)^{n-r+1}, r-2) & \text{if } n-r \text{ is odd,} \end{cases}$$

is the degree sequence of G, and G is unique realization of π. Since π only contains $r-2$ terms $n-1(\ge r)$, G does not contain $K_{r+1}-e$ as a subgraph.

Hence π is not potentially $K_{r+1} - e$ graphic. Thus

$$\begin{aligned}\sigma(K_{r+1} - e, n) &\geq \sigma(\pi) + 2 \\ &= \begin{cases} (r-1)(2n-r) + 2 - (n-r) & \text{if } n-r \text{ is even,} \\ (r-1)(2n-r) + 1 - (n-r) & \text{if } n-r \text{ is odd.} \end{cases}\end{aligned}$$

□

3. The value $\sigma(K_5 - e, n)$

In order to determine the value $\sigma(K_5 - e, n)$, we need the following theorem.

Theorem 3.1. Let $n \geq 2r+2$, and let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $d_n \geq r-1$.

(1) If $d_{r+1} \geq r$, then π is potentially A_{r+1} graphic.

(2) If $d_{r-1} \geq r$, then π is potentially $A_{r+1} - e$ graphic.

Proof. (1) We use induction on r. The conclusion is evident for $r = 1$. Now assume that the conclusion holds for $r-1$, $r \geq 2$. We will prove that the conclusion holds for r. Let $\pi_1' = (d_1', d_2', \ldots, d_{n-1}')$ be the residual sequence obtained by laying off d_1 from π. Then π_1' satisfies $n-1 \geq 2r+1 \geq 2(r-1)+2$, $d'_{(r-1)+1} \geq r-1$ and $d'_{n-1} \geq r-2$. By induction hypothesis, π_1' is potentially A_r graphic. If $d_1 = n-1$, or there exists an integer t, $r+1 \leq t \leq d_1+1$ such that $d_t > d_{t+1}$, then $d_2 - 1, \ldots, d_{r+1} - 1$ are the r largest terms in π_1'. Thus π is potentially A_{r+1} graphic. So we may assume that

$$n-2 \geq d_1 \geq \cdots \geq d_r \geq d_{r+1} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n.$$

If $d_r > d_{r+1}$, then by laying off d_{r+1} from π, the residual sequence $\pi'_{r+1} = (d_1', d_2', \ldots, d_{n-1}')$ satisfies $n-1 \geq 2(r-1)+2$, $d'_{(r-1)+1} \geq r-1$ and $d'_{n-1} \geq r-2$. By induction hypothesis, π'_{r+1} is potentially A_r graphic. Since $d_1 - 1, \ldots, d_r - 1$ are the r largest terms in π'_{r+1}, π is potentially A_{r+1} graphic. So we may further assume that

$$n-2 \geq d_1 \geq \cdots \geq d_r = d_{r+1} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n.$$

By Theorem 1.3, π is potentially A_{r+1} graphic.

(2) If $d_{r+1} \geq r$, then by (1), π is potentially A_{r+1} graphic, and so π is potentially $A_{r+1} - e$ graphic. If $d_{r+1} \leq r-1$, then $d_{r+1} = \cdots = d_n = r-1$. Let $\pi'_{r+1} = (d_1', d_2', \ldots, d_{n-1}')$ be the residual sequence obtained by laying off d_{r+1} from π. Then π'_{r+1} satisfies $n-1 \geq 2(r-1)+2$, $d'_{(r-1)+1} \geq r-1$ and $d'_{n-1} \geq r-2$. By (1), π'_{r+1} is potentially A_r graphic. Since $\{d_1 - 1, d_2 - 1, \ldots, d_{r-1} - 1\} \subseteq \{d_1', d_2', \ldots, d_r'\}$, π is potentially $A_{r+1} - e$ graphic. □

It follows from Theorems 2.1 and 2.2 that $\sigma(K_5 - e, 5) = 18$, $\sigma(K_5 - e, 6) = 26$, $\sigma(K_5 - e, 7) = 32$ and $\sigma(K_5 - e, 8) = 42$. Now we compute the value $\sigma(K_5 - e, 9)$.

Theorem 3.2. $\sigma(K_5 - e, 9) = 38$.

Proof. As Theorem 2.3 shows that $\sigma(K_5 - e, 9) \geq (4-1)(2\times 9-4)+1-(9-4) = 38$, we only need to prove that, if $\pi = (d_1, d_2, \ldots, d_9)$ is a positive graphic sequence with $\sigma(\pi) \geq 38$, then π is potentially $K_5 - e$ graphic. Clearly, $d_1 \geq 5$. For any integer k, $1 \leq k \leq 9$, let $\pi'_k = (d'_1, d'_2, \ldots, d'_8)$ be the residual sequence obtained by laying off d_k from π. Then $\sigma(\pi'_k) = \sigma(\pi) - 2d_k \geq 38 - 2 \times 8 = 22$, and so π'_k has at least 6 non-zero terms. By Theorem 1.1, we have $\sigma(K_4 - e, 6) = \sigma(K_4 - e, 7) = 20$ and $\sigma(K_4 - e, 8) = 22$. Hence $\sigma(\pi'_k) \geq \max\{\sigma(K_4 - e, 6), \sigma(K_4 - e, 7), \sigma(K_4 - e, 8)\}$. Thus by Theorem 1.2, π'_k is potentially $A_4 - e$ graphic. If $d_1 = 8$, or there exists an integer t, $5 \leq t \leq d_1 + 1$ such that $d_t > d_{t+1}$, then $d_2 - 1, d_3 - 1, d_4 - 1, d_5 - 1$ are the four largest terms in π'_1. Hence π is potentially $A_5 - e$ graphic. If $d_4 > d_5$, then $d_1 - 1, d_2 - 1, d_3 - 1, d_4 - 1$ are the four largest terms in π'_5. Thus π is also potentially $A_5 - e$ graphic. So we may assume that

$$7 \geq d_1 \geq d_2 \geq d_3 \geq d_4 = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_9.$$

We consider the following two cases:

Case 1. $d_1 = 7$. Then $d_4 = \cdots = d_9 \geq 3$. If $d_4 = \cdots = d_9 = 3$, then $\pi = (7^2, 6, 3^6)$, and hence $\pi'_4 = (6^2, 5, 3^5)$. It follows from Theorem 3.1(1) that π'_4 is potentially A_4 graphic. Thus π is potentially $A_5 - e$ graphic. If $d_4 = \cdots = d_9 \geq 4$, then by Theorem 3.1(1), the residual sequence $\pi'_1 = (d'_1, d'_2, \ldots, d'_8)$ is potentially A_4 graphic. Since $\{d_2 - 1, d_3 - 1, d_4 - 1\} \subseteq \{d'_1, d'_2, d'_3, d'_4\}$, π is potentially $K_5 - e$ graphic.

Case 2. $d_1 \leq 6$. Then $d_4 = \cdots = d_{d_1+2} \geq 4$. If $d_3 > d_4$, then the residual sequence $\pi'_4 = (d'_1, d'_2, \ldots, d'_8)$ is a positive graphic sequence and $\sigma(\pi'_4) = \sigma(\pi) - 2d_4 \geq 38 - 2 \times 5 = 28 = \sigma(K_4, 8)$. By Theorem 1.5, π'_4 is potentially A_4 graphic. Since $d_1 - 1, d_2 - 1, d_3 - 1$ are the three largest terms in π'_4, π is potentially $K_5 - e$ graphic. So we may further assume that

$$6 \geq d_1 \geq d_2 \geq d_3 = d_4 = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_9.$$

The following two subcases are considered:

Subcase 2.1. $d_1 = 6$. Then $d_3 = \cdots = d_8 \geq 4$. If $d_8 > d_9 \geq 2$, then the residual sequence $\pi'_1 = (d'_1, d'_2, \ldots, d'_8)$ satisfies $d'_8 \geq 2$ and $d'_4 \geq 3$. By Theorem 3.1(1), π'_1 is potentially A_4 graphic. Since $\{d_2 - 1, d_3 - 1, d_4 - 1\} \subseteq \{d'_1, d'_2, d'_3, d'_4\}$, π is potentially $K_5 - e$ graphic. If $d_8 > d_9 = 1$, then $\pi = (6, 5^7, 1)$. If $d_8 = d_9$, then π is one of $(6^2, 4^7), (6, 4^8), (6^9)$ and $(6, 5^8)$. It is easy to check that the above five sequences are all potentially $K_5 - e$ graphic.

Subcase 2.2. $d_1 = 5$. Then $d_3 = \cdots = d_7 \geq 4$. If $d_7 > d_8 \geq d_9 \geq 2$, then by Theorem 3.1(1), the residual sequence $\pi'_1 = (d'_1, d'_2, \ldots, d'_8)$ is potentially A_4 graphic. It follows from $\{d_2-1, d_3-1, d_4-1\} \subseteq \{d'_1, d'_2, d'_3, d'_4\}$ that π is potentially $K_5 - e$ graphic. If $d_7 > d_8 \geq d_9 = 1$, then $\pi = (5^7, 4, 1)$ or $(5^7, 2, 1)$. If $d_7 = d_8$, then π is one of $(5^2, 4^7), (5^8, 4)$ and $(5^8, 2)$. It is easy to verify that the above five sequences are all potentially $K_5 - e$ graphic.□

Theorem 3.3. If $n \geq 9$, then $\sigma(K_5 - e, n) = \begin{cases} 5n-6 & \text{if } n \text{ is even}, \\ 5n-7 & \text{if } n \text{ is odd}. \end{cases}$

Proof. By Theorem 2.3,

$$\sigma(K_5 - e, n) \geq \begin{cases} 5n-6 & \text{if } n \text{ is even}, \\ 5n-7 & \text{if } n \text{ is odd}. \end{cases}$$

In order to prove

$$\sigma(K_5 - e, n) \leq \begin{cases} 5n-6 & \text{if } n \text{ is even}, \\ 5n-7 & \text{if } n \text{ is odd}, \end{cases}$$

it is enough to prove that, if $\pi = (d_1, d_2, \ldots, d_n)$ is a positive graphic sequence with

$$\sigma(\pi) \geq \begin{cases} 5n-6 & \text{if } n \text{ is even}, \\ 5n-7 & \text{if } n \text{ is odd}, \end{cases}$$

then π is potentially $K_5 - e$ graphic. Use induction on n. By Theorem 3.2, the conclusion holds for $n = 9$. Now suppose that $n \geq 10$. If $d_n \leq 2$, then the residual sequence $\pi'_n = (d'_1, d'_2, \ldots, d'_{n-1})$ is a positive graphic sequence and $\sigma(\pi'_n) = \sigma(\pi) - 2d_n \geq 5n - 7 - 2 \times 2 = 5n - 11 = 5(n-1) - 6 \geq \sigma(K_5 - e, n-1)$. By induction hypothesis, π'_n is potentially $K_5 - e$ graphic, and hence so is π. If $d_n \geq 3$, then by $d_3 \geq 4$ and Theorem 3.1(2), π is potentially $A_5 - e$ graphic. So the conclusion follows. □

4. Conclusion

It is easy to see that

$$\sigma(K_3 - e, n) = \begin{cases} n+1 & \text{if } n \text{ is odd}, \\ n+2 & \text{if } n \text{ is even}. \end{cases}$$

In other words, if $r = 2$, then the lower bound (1) for $r = 2$ just is the exact value $\sigma(K_3 - e, n)$. By Theorems 1.1 and 3.3, the equality in (1) holds for $r = 3$ and $n \geq 7$, and for $r = 4$ and $n \geq 9$. So we feel that the equality in (1) just is the exact value $\sigma(K_{r+1} - e, n)$ for sufficiently large n.

Conjecture: For sufficiently large n,

$$\sigma(K_{r+1} - e, n) = \begin{cases} (r-1)(2n-r) + 2 - (n-r) & \text{if } n-r \text{ is even}, \\ (r-1)(2n-r) + 1 - (n-r) & \text{if } n-r \text{ is odd}. \end{cases}$$

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