On Dense Triple-Loop Networks*

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Abstract

Multi-loop digraphs are widely studied mainly because of their symmetric properties and their applications to loop networks. A multi-loop digraph, $G = G(N; s_1, ..., s_\Delta)$ with $1 \leq s_1 < s_2 < ... < s_\Delta \leq N - 1$ and $\gcd(N, s_1, ..., s_\Delta) = 1$, has set of vertices $V = \mathbb{Z}_N$ and adjacencies given by $v \longrightarrow v + s_i \bmod N$, $i = 1, ..., \Delta$.

For every fixed N, an usual extremal problem is to find the minimum value

$$D_\Delta(N) = \min_{s_1,...,s_\Delta \in \mathbb{Z}_N} D(N; s_1, ..., s_\Delta),$$

where $D(N; s_1, ..., s_\Delta)$ is the diameter of G. A closely related problem is to find the maximum number of vertices for a fixed value of the diameter.

For $\Delta = 2$, all optimal families have been found by using a geometrical approach. For $\Delta = 3$, only some dense families are known.

In this work a new dense family is given for $\Delta = 3$ using a geometrical approach. This technique was already adopted in several papers for $\Delta = 2$ (see for instance [5, 7]). This family improves the dense families recently found by several authors.

Keywords: Triple-loop network, diameter, dense digraph, Smith normal form, tessellation, 3D tile.

AMS subject classification: 05C20, 05C90.

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1 Introduction

Multi-loop networks have been widely studied in the last years because of their relevance to the design of some interconnection or communication computer networks. Multi-loop digraphs model such networks, and are usually called circulant because their adjacency matrices are circulant.

Definition 1 *A multi-loop digraph,* $G = G(N; s_1, ..., s_\Delta)$ *with* $1 \leq s_1 < s_2 < ... < s_\Delta \leq N-1$ *and* $\gcd(N, s_1, ..., s_\Delta) = 1$, *has the set of vertices* $V = \mathbb{Z}_N$ *and the set of adjacencies* $A = \{(v, v + s_i \bmod N) | \ v \in V, i = 1, ..., \Delta\}$.

The integers $s_1, ..., s_\Delta$ are usually called the *steps* of G. These digraphs are regular of in-degree and out-degree Δ and vertex symmetric. They are strongly connected iff $\gcd(N, s_1, ..., s_\Delta) = 1$. For a survey about multi-loop networks and multi-loop digraphs see [4], for the case $\Delta = 2$ see [11].

Double-loop (case $\Delta = 2$) and triple-loop digraphs (case $\Delta = 3$) are the most studied cases of multi-loop digraphs. The minimization of the diameter in the digraph, denoted by $D(N; s_1, ..., s_\Delta)$, corresponds to the problem of minimizing the message transmission delay of the related network. This problem have been studied using several techniques: from number theoretical reasonings to geometrical approaches. In this work we use the latter method which is presented in the next section. For fixed N and Δ, the optimal diameter value which can be attained is denoted by $D_\Delta(N)$, that is

$$D_\Delta(N) = \min_{s_1, ..., s_\Delta \in \mathbb{Z}_N} D(N; s_1, ..., s_\Delta).$$

For $\Delta = 2$ a sharp lower bound, $lb(N)$, is known for this optimal value

$$D_2(N) \geq lb(N) = \left\lceil \sqrt{3N} \right\rceil - 2.$$

For $\Delta = 3$ no analog to $lb(N)$ is known, however the relation $D_3(N) = \Theta(\sqrt[3]{N})$ (or equivalently $N_3(D) = \Theta(D^3)$, where $N_3(D)$ is the maximum number of vertices reachable by a triple-loop digraph with fixed diameter D) holds.

1.1 Known dense families of triple-loop digraphs

From now on we denote a triple-loop digraph by TLN (the 'N' stands for network). Because of the poor knowledge of the integral function $D_3(N)$, families with good N/D^3 ratio are known as dense ones. Many works have been published about dense families of TLN using several different methods. These techniques range from geometrical approaches [7, 10, 2, 1] to asymptotic bases [3].

Some known families follow, ordered from lower to higher asymptotic density:

- Hsu & Jia in [10]:

$$N(D) = \frac{1}{16}D^3 + O(D^2) \approx 0.062D^3 + O(D^2).$$

- Aguiló, Fiol & Garcia in [2]:

$$N(D) = \frac{2}{27}D^3 + O(D^2) \approx 0.074D^3 + O(D^2).$$

- Chen & Gu in [3]:

$$N(D) = \frac{5}{64}D^3 + O(D^2) \approx 0.078D^3 + O(D^2).$$

- Aguiló in [1]:

$$N(D) = \frac{860}{22^3}D^3 + O(D^2) \approx 0.0807D^3 + O(D^2).$$

- Fiduccia, Forcade & Zito in [6]:

$$N(D) = \frac{100}{1195}(D+3)^3 \approx 0.0836D^3 + O(D^2).$$

An explicit expression of $D_3(N)$ (or equivalently $N_3(D)$) resists to be known, as $D_2(N)$ does too. However some bounds on $N_3(D)$ have been proposed by several authors using geometrical arguments. In [10], Hsu & Jia gave the following lower bound for $N_3(D)$

$$N_3(D) \geq \frac{1}{16}D^3 + \frac{3}{8}D^2 + O(D) \text{ as } D \to \infty.$$

Recently, Fiduccia, Forcade & Zito in [6] gave the following upper bound

$$N_3(D) \leq \frac{3}{25}(D+3)^3 = 0.12D^3 + O(D^2). \tag{1}$$

The densest known family is proposed in the same work and has an asymptotic density $N(D) \approx 0.0836D^3 + O(D^2)$. This fact seems to point out two possibilities: the bound given in (1) is too optimistic, or the techniques used to find dense TLN are not powerful enough.

Computational observations suggest a complex behaviour of $D_3(N)$ versus N. In fact, the same was pointed out for $D_2(N)$ in several works on double-loop networks ([7], [4]) and no closed expression is known for $D_2(N)$. For instance, there are integral values of N, say N_0, having a neighbourhood $\mathcal{N} \in \mathbb{N}$ where for all $N \in \mathcal{N} - \{N_0\}$ the value $D_3(N)$ is a unit more (or a unit less) than $D_3(N_0)$. Take for instance $D_3(623) = 18$ and $D_3(622) = D_3(624) = 17$ for one unit less, and $D_3(638) = 17$ and $D_3(637) = D_3(639) = 18$ for one unit more.

2 Geometrical technique

The main idea of the geometrical approach is to link a basic 3D tile to a triple-loop network. This tile contains all the metric information of the digraph, in fact the diameter of the tile can be computed from its physical dimensions and upperbounds the diameter of its related digraph. We will comment this fact later using an example.

2.1 Obtaining a related tile from a given TLN

Consider the three dimensional space divided into unit cubes. Given a TLN $G = G(N; s_1, s_2, s_3)$, each of its vertices is assigned to one unit cube. The vertex $as_1 + bs_2 + cs_3 \pmod N$ in G, $a, b, c \in \mathbb{N}$, is assigned to the unit cube centered at the point (a, b, c). Then all unit cubes in $\mathbb{Z}^3$ are labelled with an element of $\mathbb{Z}_N$. These labels are periodically repeated and a basic 3D tile which periodically tessellates the space can be derived from this representation as we will describe later.

Consider now the tile containing the unit cube $(0, 0, 0)$ which is labelled with '0'. Any other unit cube (a, b, c) belonging to this tile has distance value $a + b + c$ from the vertex zero $(0, 0, 0)$. The diameter of this basic tile is defined as the distance of the farthest vertex from the zero. This diameter can be computed from the dimensions of the tile.

Given a TLN we can link several 3D tiles to it, however we are interested in those tiles with minimum diameter which are also known as *Minimum Distance Diagrams*, or MDD for short. The following algorithm searches for one of these MDD tiles:

- Assign one vertex of $V = \mathbb{Z}_N$ to every unit cube with the following the rule: from a cube labelled with $i \in V$, centered at the point (x, y, z), we label by

$$i + s_1 \pmod N \text{ the unit cube centered at } (x+1, y, z),$$
$$i + s_2 \pmod N \text{ the unit cube centered at } (x, y+1, z),$$
$$i + s_3 \pmod N \text{ the unit cube centered at } (x, y, z+1).$$

  Now we have labelled each unit cube of the space $\mathbb{Z}^3$ with one label in V. The labels repeat periodically in the space.

- Let S be the set of labels in V. Choose a unit cube with label $(0,0,0)$ (this cube will be called the *zero* cube), mark it and set $S = S - \{0\}$. Mark the other cubes, one per label in S, closed to the zero cube and extract the corresponding label from S per each marked cube. This marking process can be done by considering those labelled cubes that minimize the norm $\|(x, y, z)\| = |x| + |y| + |z|$ (only those placed in the same direction and positive sense of the vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, and $e_3 = (0,0,1)$). This minimization guarantees that the resulting tile will be a MDD also.

- We stop when $S = \varnothing$ and all the N labels of V have been assigned. Then all marked cubes form a 3D-shaped tile which tessellates the space.

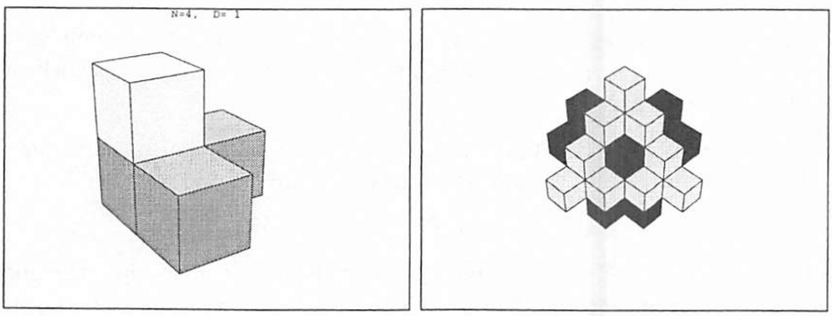

Figure 1: A MDD tile of volume 4 and its related tessellation

As an example, consider the TLN $G(4; 1, 2, 3)$. This algorithm gives the basic 3D tile depicted in Figure 1. Note that the diameter of this tile is $D = 1$, the same value as the diameter of the digraph $G(4; 1, 2, 3)$ because it is also a MDD. However there are other tessellating tiles with different diameter for the same distribution of labels in the space (first step of the above algorithm). For instance we have another tile with the same distribution of labels and diameter $D = 2$ in Figure 2.

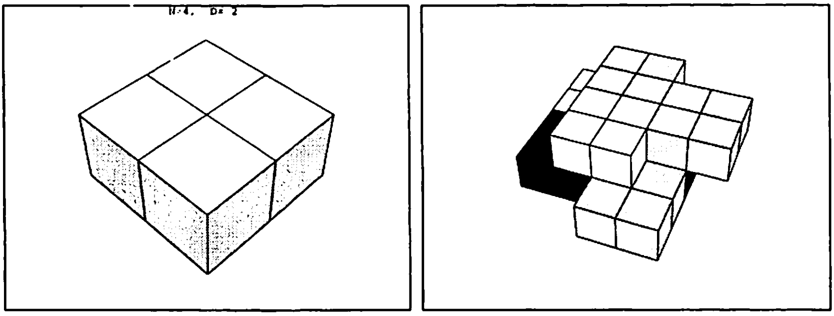

Figure 2: Another tile related to $G(4; 1, 2, 3)$ which is not a MDD

Now we can link a basic 3D tile to a TLN, however we also want to reverse this process. This reversion will allow us to study dense triple loop networks through dense 3D tiles. We want to obtain a TLN from a given basic 3D tile which tessellates the space. For this reason we must introduce a digraph which generalizes the triple loop network.

2.2 Triple commutative-step digraphs

A TLN $G(N; s_1, s_2, s_3)$ can be seen as the Cayley digraph of the cyclic group $\mathbb{Z}_N$ generated by $\{s_1, s_2, s_3\}$. This point of view of triple loop networks will allow us to generalize it.

Let M be an integral 3×3 matrix with $|\det(M)| = N$. Let us consider the (column) vectors $u, v \in \mathbb{Z}^3$ and the following equivalence relation

$$u \sim v \Leftrightarrow \exists \lambda \in \mathbb{Z}^3 : u - v = M\lambda.$$

Let be $\mathbb{Z}^3_M = \mathbb{Z}^3/M\mathbb{Z}^3$ the quotient group of the 3-vectors modulo the above relation. Note that the grid $M\mathbb{Z}^3 = \{M\lambda : \lambda \in \mathbb{Z}^3\}$ is a normal subgroup of $\mathbb{Z}^3$. This group has $|\det M|$ elements and generalizes the usual concept of congruence modulo N over $\mathbb{Z}^3$ as

$$u \equiv v \pmod{M} \Leftrightarrow u = v + M\lambda, \quad \text{for some } \lambda \in \mathbb{Z}^3.$$

See [8, 9] for more details.

Definition 2 *A triple commutative-step digraph $G(M)$ is the Cayley digraph of the additive group $\mathbb{Z}^3_M$ with generator set $S = \{e_1 = (1,0,0)^\top, e_2 = (0,1,0)^\top, e_3 = (0,0,1)^\top\}$.*

We can obtain a triple commutative-step digraph from a tessellation of the space. This technique has been used before in several works dealing with double and triple-loop networks. For instance see [7] for the 2D case and [2] for the 3D case. We remember this process using the basic 3D tile of Figure 1.

Consider the tessellation defined by the tile depicted in Figure 1. From the dimensions of this tile, we can compute three independent vectors defining the distribution of tiles in this tessellation. In this case we can take for instance $(1,0,1)^{\mathsf{T}}$, $(2,1,0)^{\mathsf{T}}$ and $(0,2,0)^{\mathsf{T}}$. Then we define the matrix M as the integral matrix with entries given by the above three (column) vectors, that is $M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$. So we can define the Cayley digraph $\mathrm{Cay}(\mathbb{Z}^3_M, S)$.

For any given tessellating tile T with related matrix M, let us denote by $D(\mathbb{Z}^3_M, S)$ the diameter of $\mathrm{Cay}(\mathbb{Z}^3_M, S)$ and let D_T be the diameter of T. Then we always have $D(\mathbb{Z}^3_M, S) \leq D_T$ and the identity is given when the tile T is also a MDD.

2.3 Obtaining a TLN from a given 3D tile

In the above section we have seen how to obtain a triple commutative-step digraph from a given tile T which periodically tessellates the space. Now we will see how to obtain a TLN from a given triple commutative-step digraph. In fact, we will use a digraph isomorphism through the Smith normal form of the 3×3 integral matrix, M, which defines the triple commutative-step digraph $G(M)$.

This isomorphism can be defined only when the group $\mathbb{Z}^3_M$ (defined in Section 2.2) is cyclic. Although a proof of this fact can be found in [8], we now introduce a breve explanation about this isomorphism and conditions on the matrix M to assure his existence.

The *Smith Normal Form*, $S(M)$, of M is a matrix $S(M) = UMV$, where U and V are unimodular integral matrices generated from the necessary elemental transformations to obtain $S(M)$ from M. The entries of $S(M)$ are given by $S(M) = \mathrm{diag}(d_1, d_2, d_3)$ with $d_1|d_2$ and $d_2|d_3$. These three *elementary divisors* can be obtained from the integral entries of M by $d_1 = \gcd(\text{integral entries of } M)$, $d_1 d_2 = \gcd(2 \times 2 \text{ minors of } M)$ and $d_1 d_2 d_3 = |\det M|$.

We have $\mathbb{Z}^3_M \cong \mathbb{Z}^3_{S(M)}$ and the cyclic case, the one we are interested in, is given iff

$$d_1 = d_2 = 1. \tag{2}$$

Then, in this case, we have $\mathbb{Z}^3_{S(M)} \cong \mathbb{Z}_N$ where $N = |\det M| = |\det S(M)|$, and the above mentioned isomorphism φ is given by

$$\begin{array}{rcl} \varphi : \mathbb{Z}^3_M & \longrightarrow & \mathbb{Z}_N \\ u & \longrightarrow & U_3 u \end{array}$$

where U_3 is the third row of the integral matrix U.

Note that $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ is a generator set of $\mathbb{Z}_N$ and so the wanted TLN is defined by the steps $s_k \equiv \varphi(e_k) \pmod N$ for $k = 1, 2, 3$. That is, in terms of digraph isomorphisms, $G(|\det M|; \varphi(e_1), \varphi(e_2), \varphi(e_3)) \cong G(M)$. Note that we also have $D(|\det M|; \varphi(e_1), \varphi(e_2), \varphi(e_3)) \leq D_T$.

3 A new dense family

We will use the geometrical approach from the above section to find a new dense family of TLN which improves all known dense families. Consider the basic 3D tile depicted in Figure 3 with n and m being positive integral values such that $n < m < 3n < 2m$. Let us denote by $T(m,n)$ such a tile, $N(m,n)$ its related volume and $D(m,n)$ its related diameter.

$T(m,n)$ is based on the tile $T(2,1)$ of $N(2,1) = 84$ unit cubes and diameter $D(2,1) = 7$.A portion of its related tessellation can be viewed in Figure 4.

Proposition 1 *We have*

$$\begin{array}{rcl} D(m,n) & = & \max\{m + 8n - 3, 3m + 4n - 3, 5m - 3\}, \\ N(m,n) & = & m^3 + 12m^2 n + 14mn^2. \end{array}$$

Proof:
We must compute the distance from the zero cube to the farthest unit cube/s. These possible farthest unit cube/s are marked with the bold letters **A**, **B**,...,**M** in Figure 3. Then we have:

$$\begin{array}{l} d(0,\mathbf{A}) = d(0,\mathbf{B}) = m + 8n - 3, \\ d(0,\mathbf{C}) = \ldots = d(0,\mathbf{K}) = 3m + 4n - 3, \\ d(0,\mathbf{L}) = d(0,\mathbf{M}) = 5m - 3. \end{array}$$

So the diameter is the maximum of these three values. The computation of the volume is obtained directly from Figure 3.□

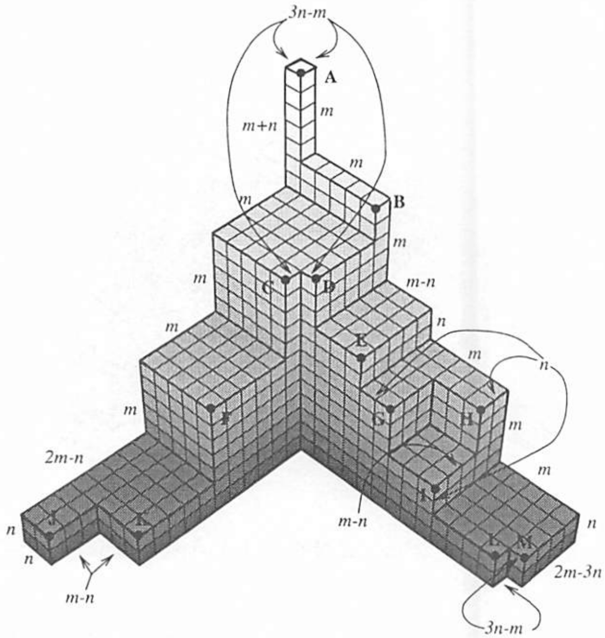

Figure 3: Basic 3D tile $T(m,n)$

A possible related 3×3 integral matrix, as it is described in Section 2, is

$$M(m,n) = \begin{pmatrix} n & m & -2m-2n \\ 3n+m & m & m+2n \\ 2n & -m & m+n \end{pmatrix}.$$

Note that $\det M(m,n) = N(m,n)$.

Theorem 1 *Let $G(N(t); s_1(t), s_2(t), s_3(t))$ be the family of TLN given by*

$$\begin{aligned} N(t) &= 2268t^3 + 666t^2 + 54t + 1, \\ s_1(t) &= -7938t^3 - 2142t^2 - 144t - 2 \pmod{N(t)}, \\ s_2(t) &= 54t^2(63t+8) \pmod{N(t)}, \\ s_3(t) &= -4536t^3 - 576t^2 + 1 \pmod{N(t)}. \end{aligned}$$

Let $D(t) = D(N(t); s_1(t), s_2(t), s_3(t))$, then $D(t) \le 30t + 2$ and its related assimptotic density is 0.084 at least.

Proof:

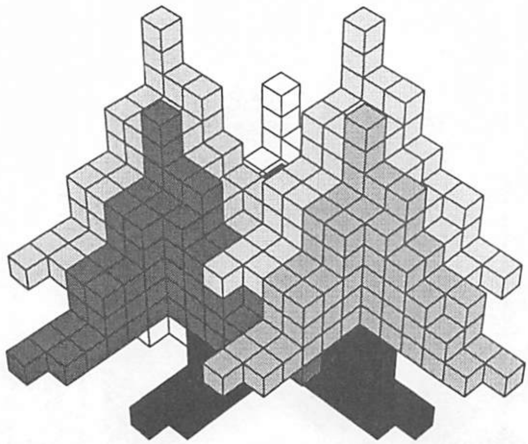

Figure 4: Part of the $T(2,1)$-tessellation

Let us consider the tile $T(6t+1,3t)$, with related integral matrix

$$M(6t+1,3t) = \begin{pmatrix} 3t & 6t+1 & -18t-2 \\ 15t+1 & 6t+1 & 12t+1 \\ 6t & -6t-1 & 9t+1 \end{pmatrix}.$$

This matrix fullfills conditions (2): From $\gcd(3t, 6t+1) = \gcd(3t, 1) = 1$, we have $d_1 = 1$. The 9 2×2 minors of $M(6t+1,3t)$ are $126t^2 + 33t + 2$, $63t^2 + 18t + 1$, $-126t^2 - 27t - 1$, $-54t^2 - 15t - 1$, $135t^2 + 15t$, $-54t^2 - 9t$, $180t^2 + 48t + 3$, $306t^2 + 51t + 2$ and $-72t^2 - 18t - 1$. From

$$\begin{aligned} &\gcd(126t^2 + 33t + 2, 63t^2 + 18t + 1, -126t^2 - 27t - 1, 135t^2 + 15t, \\ &-72t^2 - 18t - 1) = \\ &= \gcd(6t+1, -9t^2, -126t^2 - 27t - 1, 135t^2 + 15t, -72t^2 - 18t - 1) = \\ &= \gcd(6t+1, -9t^2, 9t^2 - 12t - 1, 135t^2 + 15t, -72t^2 - 18t - 1) = \\ &= \gcd(6t+1, -9t^2, -12t - 1, 135t^2 + 15t, -72t^2 - 18t - 1) = \\ &= \gcd(6t+1, -9t^2, 1, 135t^2 + 15t, -72t^2 - 18t - 1) = 1, \end{aligned}$$

we have $d_2 = 1$.

So we can define the isomorphism φ of Section 2.3 and $G(M(6t+1,3t))$ is isomorphic to some TLN. By Proposition 1 this TLN has $N(6t+1,3t) =$

$2268t^3 + 666t^2 + 54t + 1$ vertices and its diameter is at most

$$D(6t+1, 3t) = \max\{30t-2, 30t, 30t+2\} = 30t+2.$$

The corresponding Smith normal form of $M(6t+1, 3t)$ is

$$S(t) = \mathrm{diag}(1, 1, N(6t+1, 3t))$$

and its related unimodular matrices $V(t)$ and $U(t)$ are given by

$$\begin{pmatrix} 1 & 13+78\,t-378\,t^2 & 4+50\,t+30\,t^2-756\,t^3 \\ 0 & -5-63\,t & -2-31\,t-126\,t^2 \\ 0 & -3 & -6\,t-1 \end{pmatrix}$$

and

$$\begin{pmatrix} -3 & 1 & -1 \\ 1+7\,t & -3\,t & 4\,t \\ -2142\,t^2-144\,t-2-7938\,t^3 & 54\,t^2\,(63\,t+8) & 1-4536\,t^3-576\,t^2 \end{pmatrix}$$

respectively. So, the steps are the stated ones.

Finally, from $N(t) = 2268t^3 + O(t^2)$ and $D(t) \leq 30t + 2$, we have

$$\frac{N(t)}{D(t)^3} \geq \frac{2268t^2 + O(t^2)}{(30t+2)^3} = \frac{2268}{30^3}\frac{t^3}{(t+2/30)^3} + O(\frac{1}{t})$$

which tends to $\frac{2268}{30^3} = 0.084$ when $t \to \infty$.□

This family has better density than the one given by Fiduccia, Forcade and Zito in [6], however this density is far from the bound (1).

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