A Few More RPMDs with $k = 4$

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Abstract

Let v, k, λ and n be positive integers. $(x_1, x_2, ..., x_k)$ is defined to be $\{(x_i, x_j) : i \neq j, i, j = 1, 2, ..., k\}$, in which the ordered pair (x_i, x_j) is called $(j-i)$-apart for $i < j$ and $(k+j-i)$-apart for $i > j$, and is called a cyclically ordered k-subset of $\{x_1, x_2, ..., x_k\}$.

A perfect Mendelsohn design, denoted by (v, k, λ)-PMD is a pair $(X, \mathcal{B})$, where X is a v-set (of points), and B is a collection of cyclically ordered k-subsets of X(called blocks), such that every ordered pair of points of X appears t-apart in exactly λ blocks of $\mathcal{B}$ for any t, where $1 \leq t \leq k-1$.

If the blocks of a (v, k, λ)-PMD for which $v \equiv 0 \pmod{k}$ can be partitioned into $\lambda(v-1)$ sets each containing v/k blocks which are pairwise disjoint, the $(v, k, 1)$-PMD is called resolvable, denoted by (v, k, λ)-RPMD.

In the paper [14], we have showed that a $(v, 4, 1)$-RPMD exists for all $v \equiv 0 \pmod{4}$ except for 4, 8 and with at most 49 possible exceptions of which the largest is 336.

In this article, we shall show that a $(v, 4, 1)$-RPMD for all $v \equiv 0 \pmod{4}$ except for 4, 8, 12 and with at most 27 possible exceptions of which the largest is 188.

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1 Introduction

Let v, k and λ be positive integers. $(x_1, x_2, ..., x_k)$ is defined to be $\{(x_i, x_j) : i \neq j, i, j = 1, 2, ..., k\}$, in which the ordered pair (x_i, x_j) is called $(j-i)$-apart for $i < j$ and $(k+j-i)$-apart for $i > j$, and is called a cyclically ordered k-subset of $\{x_1, x_2, ..., x_k\}$. A holey perfect Mendelsohn design, denoted by (v, k, λ)-HPMD (k-HPMD if $\lambda = 1$) is a triple $(X, \mathcal{G}, \mathcal{A})$

which satisfies the following properties:
(i) X is a v-set (of points);
(ii) $\mathcal{G}$ is a partition of X into groups;
(iii) $\mathcal{A}$ is a collection of cyclically ordered k-subsets of X (called blocks) each of which intersects each group in at most one point;
(iv) Every ordered pair (x, y) from distinct groups appears t-apart in exactly λ blocks of $\mathcal{A}$ for $t = 1, 2, ..., k-1$.

If $\mathcal{G} = \{G_i : 1 \leq i \leq h\}, |G_i| = g_i$, we say that $(g_1, g_2, ..., g_h)$ is the type of the HPMD.

A (v, n, k, λ)-IPMD can be viewed as an HPMD with the type of $(n, 1, 1, ..., 1)$ and a (v, k, λ)-PMD can be viewed as an IPMD with $n = 1$.

Definition 1.1. If the blocks of a (v, k, λ)-PMD for which $v \equiv 1 \pmod{k}$ can be partitioned into λv sets each containing $(v-1)/k$ blocks which are pairwise disjoint (as sets), we say that the (v, k, λ)-PMD is called resolvable, (briefly (v, k, λ)-RPMD) and each set of $(v-1)/k$ pairwise disjoint blocks will be called a parallel class.

Definition 1.2. If the blocks of a (v, k, λ)-PMD for which $v \equiv 0 \pmod{k}$ can be partitioned into $\lambda(v-1)$ sets each containing v/k blocks which are pairwise disjoint (as sets), we say that the (v, k, λ)-PMD is called resolvable, (briefly (v, k, λ)-RPMD) and each set of v/k pairwise disjoint blocks will be called a parallel class.

A resolvable PMD and parallel classes by Definition 1.1 are usually called an almost resolvable PMD and almost parallel classes. For convenience, we use Definition 1.1 in this article.

Definition 1.3. Suppose that X is a set of n players.

A *directed whist table*, denoted $(x, y; z, t,)$, is a set of four players with the pairs $\{x, y\}$ and $\{z, t\}$, known as partners and with the ordered pairs $(x, t), (t, y), (y, z), (z, x)$ known as opponents.

A *directed whist round* is a set of directed whist tables such that each player occurs at exactly one directed whist table, except possibly for one player not at any directed whist table.

A *directed whist tournament* for n players, denoted $DWh[n]$ is a set of directed whist rounds such that any two player are partners at exactly one directed whist table and any ordered pair of player are opponents at one directed whist table.

It is easy to see that the existence of a $(v,4,1)$-RPMD for $v = 4n$ is equivalent to that of a DWh[n].

For more details on the above terminology, the reader is referred to [4,7,15].

The following theorem was proved in [5,6].

Theorem 1.4 A $(v,3,1)$-RPMD exists if and only if $v \equiv 0,1 \pmod 3), v \neq 6$.

From Baker and Wilson [1], Bennett [2], and Lamken, Mills, and Wilson [9], we have

Theorem 1.5 A $(v,4,1)$-RPMD exists for all positive integer $v \equiv 1 \pmod 4$. There exists $(40,4,1)$-RPMD, and hence infinitely many $(v,4,1)$-RPMD for $v \equiv 0 \pmod 4$.

Furthermore, the author obtained the following results [14].

Theorem 1.6 A $(v,4,1)$-RPMD exists for all integers $v \geq 4$ where $v \equiv 0 \pmod 4$, except for $v = 4, 8$, and with 49 possible exceptions as follows :
$v \in$
$\{12,16,20,24,28,32,36,44,48,52,56,64,68,76,84,88,92,96,104,108,116, 120,124,132,136,148,152,156,172,184,188,204,212,216,228,232,236,244, 268,276,284,292,304,308,312,316,328,332,336.\}$

In this article, we will construct a $(28,4,1)$-RPMD using 7 base blocks under a non-abelian group, and furthermore show that A $(v,4,1)$-RPMD exists for all integers $v \geq 4$ where $v \equiv 0 \pmod 4$, except for $v = 4,8,12$, and with 27 possible exceptions as follows:
$v \in \{16,20,24,32,36,44,48,52,56,64,68,76,84,88,92, 96,104,108,116,124,132,148,152,156,172,184,188.\}$

We assume that the reader is familiar with the basic concepts in design theory, such as pairwise balanced design (PBD), group divisible design (GDD), transversal design (TD), and resolvable transversal design (RTD). For convenience, the reader can be referred to [7].

2 (28, 4, 1)-RPMD

A direct construction using groups below is a variation of the method using difference sets in the construction of BIBDs. Instead of listing all of the blocks of a design, it suffices to give the group acting on a set of base blocks.

Let G be a group, H a subgroup of G. Let $B = (b_1, b_2, b_3, b_4)$ be a block of G, and $B' = (\infty, b_2, b_3, b_4)$ a block of $G \cup \{\infty\}$. By developing the base block B or B' under H, we can obtain a set of blocks, that is,

dev $B = \{gB = (gb_1, gb_2, gb_3, gb_4) : g \in H\}$ or

dev $B' = \{gB' = (\infty, gb_2, gb_3, gb_4) : g \in H\}$.

When $H = G$, we define their t-apart difference sets $B(t)$, $B'(t)$ for $t = 1, 2$ as follows:

$$B(1) = (b_1^{-1}b_2, b_2^{-1}b_3, b_3^{-1}b_4, b_4^{-1}b_1),$$

$$B(2) = (b_1^{-1}b_3, b_2^{-1}b_4, b_3^{-1}b_1, b_4^{-1}b_2),$$

$$B'(1) = (b_2^{-1}b_3, b_3^{-1}b_4),$$

$$B'(2) = (b_2^{-1}b_4, b_4^{-1}b_2).$$

Lemma 2.1 There exists a (28, 4, 1)-RPMD.

Proof. Let G be the non-abelian group of order 27 in which $a^9 = b^3 = e$ and $ab = ba^4$. Develop the following 7 base blocks under G.

$$(\infty, e, a^8, a^6), (b^2a^7, a^2, ba^2, a^7), (ba^5, b^2a^6, a^4, b^2a^5), (ba^8, b^2a, a^3, b^2a^3),$$

$$(a, ba^6, ba^7, a^5), (ba^4, b^2a^4, b^2a^8, ba), (b, b^2, b^2a^2, ba^3).$$

It is readily checked that the union of the t-apart difference sets of the 7 base blocks is $G - \{e\}$ for $t = 1, 2$.

3 Nearly-IRPMD and Nearly-RPMD

The following definitions were first introduced in [14].

Definition 3.1. Let $(X, Y, \mathcal{A} \cup \mathcal{B})$ be a $(v, n, k, 1)$-IPMD for which $v \equiv 0 \pmod{k}$ and $n \equiv 1 \pmod{k}$. If $\mathcal{A}$ can be partitioned into $v - n$ parallel classes of X, and $\mathcal{B}$ can be partitioned into n partial parallel classes of $X \setminus Y$: $\mathcal{B}_j, j = 1, 2, ..., n$. The $(v, n, k, 1)$-IPMD is called nearly resolvable, denoted by $(v, n, k, 1)$-nearly-IRPMD.

Let B_j^c be the set of all points of $X \setminus Y$ not appearing in $\mathcal{B}_j$. Let b_j be the number of points of the set B_j^c. The vector $(b_1, b_2, ..., b_n)$ is called complement type of the $(v, n, k, 1)$-nearly-IRPMD. It is easy to see that $B_j^c, j = 1, 2, ..., n$ partition $X \setminus Y$, and hence $b_1 + b_2 + ... + b_n = v - n$.

Definition 3.2. Let $(X, \mathcal{A} \cup \mathcal{E})$ be a $(v, k, 1)$-PMD for which $v \equiv 0 \pmod{k}$. If $\mathcal{A}$ can be partitioned into $v - n$ parallel classes of X, and $\mathcal{E}$ can be partitioned into n partial parallel classes of X : $\mathcal{E}_j, j = 1, 2, ..., n$, then the $(v, k, 1)$-PMD is called nearly resolvable, denoted by $(v, k, 1)$-nearly-RPMD.

Let E_j^c be the set of all points of X not appearing in $\mathcal{E}_j$. Let e_j be the number of points of the set E_j^c. The vector $(e_1, e_2, ..., e_n)$ is called complement type of the $(v, k, 1)$-nearly-RPMD. It is easy to see that $E_j^c, j = 1, 2, ..., n$ partition X, and hence $e_1 + e_2 + ... + e_n = v, e_j \equiv 0 \pmod{k}, j = 1, 2, ..., n$.

By filling in holes, we have

Lemma 3.3 If there exist a $(n, k, 1)$-RPMD and a $(v, n, k, 1)$-nearly-IRPMD with complement type $(b_1, b_2, ..., b_n)$, then there exists a $(v, 4, 1)$-nearly-RPMD with complement type $(b_1 + 1, b_2 + 1, ..., b_n + 1)$, where $v \equiv 0 \pmod{k}$ and $n \equiv 1 \pmod{k}$.

Lemma 3.4 There exist $(v, n, 4, 1)$-nearly-IRPMDs for the following (v, n) with complement type $(b_1, b_2, ..., b_n)$.
(i) $(v, n) = (44, 5)$ with complement type $(7, 7, 7, 7, 11)$;
(ii) $(v, n) = (52, 9)$ with complement type $(3, 3, 3, 3, 3, 7, 7, 7, 7)$;
(iii)$(v, n) = (64, 9)$ with complement type $(3, 3, 7, 7, 7, 7, 7, 7, 7)$;
(iv) $(v, n) = (68, 9)$ with complement type $(3, 7, 7, 7, 7, 7, 7, 7, 7)$.

Proof. For each case, we take $Y = \{\infty_i : 1 \leq i \leq n\}$ and $X = Z_{v-n} \cup Y$, and present n partial parallel classes $\mathcal{B}_j$ and a set of base blocks $\mathcal{F}$ which is a parallel class, and develop the base blocks under Z_{v-n} to form $v - n$ parallel classes.

(i) $(v, n) = (44, 5)$ with complement type $(7, 7, 7, 7, 11)$;

$\mathcal{B}_1 = \{(0,1,3,2) + 4i : i = 0,1,...,7\}$

$\mathcal{B}_2 = \{(0,1,3,2) + 4i : i = 8,9,...15\}$

$\mathcal{B}_3 = \{(0,1,3,2) + 4i : i = 16,17,...,23\}$

$\mathcal{B}_4 = \{(0,1,3,2) + 4i : i = 24,25,...,31\}$

$\mathcal{B}_5 = \{(0,1,3,2) + 4i : i = 32,33,...,38\}$

$\mathcal{F}= \{(0,16,23,5),(4,17,22,26),(35,15,18,9),(19,7,27,33),$

$(11,8,1,32),(25,37,13,3),(\infty_1,34,30,14),(\infty_2,2,12,6),$

$(\infty_3,28,36,21),(\infty_4,38,10,24),(\infty_5,31,20,29)\}.$

(ii) $(v,n) = (52,9)$ with complement type $(3,3,3,3,3,7,7,7,7)$;

$\mathcal{B}_1 = \{(0,1,3,2) + i : i = 0,4,8,...,36\}$

$\mathcal{B}_2 = \{(0,1,3,2) + i : i = 40,1,5,...,33\}$

$\mathcal{B}_3 = \{(0,1,3,2) + i : i = 37,41,2,...,30\}$

$\mathcal{B}_4 = \{(0,1,3,2) + i : i = 34,38,42,...,27\}$

$\mathcal{B}_5 = \{(0,4,12,8) + 4j : j = 5,9,13,17,21,25,30,34,38,42\}$

$\mathcal{B}_6 = \{(0,4,12,8) + 4j : j = 3,7,16,20,24,28,32,36,40\}$

$\mathcal{B}_7 = \{(0,4,12,8) + 4j : j = 1,6,10,14,18,27,31,35,39\}$

$\mathcal{B}_8 = \{(0,4,12,8) + 4j : j = 2,8,12,19,23,29,33,37,41\}$

$\mathcal{B}_9 = \{(0,1,3,2) + i : i = 31,35,39\} \cup$

$\{(0,4,12,8) + 4j : j = 0,4,11,15,22,26\}$

$\mathcal{F}= \{(0,12,5,28),(1,11,14,20),(4,37,26,9),(2,7,38,17)$

$(\infty_1,22,33,24),(\infty_2,29,6,35),(\infty_3,32,3,21),(\infty_4,13,40,27),$

$(\infty_5, 25, 34, 8), (\infty_6, 41, 19, 16), (\infty_7, 42, 36, 18), (\infty_8, 10, 23, 30),$

$(\infty_9, 39, 15, 31)\}.$

(iii) $(v, n) = (64, 9)$ with complement type $(3, 3, 7, 7, 7, 7, 7, 7, 7)$;

$\mathcal{B}_1 = \{(0, 1, 3, 2) + i : i = 0, 4, 8, ..., 48\}$

$\mathcal{B}_2 = \{(0, 1, 3, 2) + i : i = 52, 1, 5, ..., 45\}$

$\mathcal{B}_3 = \{(0, 1, 3, 2) + i : i = 49, 53, 2, ..., 38\}$

$\mathcal{B}_4 = \{(0, 1, 3, 2) + i : i = 42, 46, 50, ..., 31\}$

$\mathcal{B}_5 = \{(0, 4, 12, 8) + 4j : j = 6, 10, 15, 19, 23, 27, 31, 35, 39, 44, 48, 52\}$

$\mathcal{B}_6 = \{(0, 4, 12, 8) + 4j : j = 8, 12, 16, 20, 24, 33, 37, 41, 45, 49, 53, 2\}$

$\mathcal{B}_7 = \{(0, 4, 12, 8) + 4j : j = 22, 26, 30, 34, 38, 43, 47, 51, 1, 5, 9, 13\}$

$\mathcal{B}_8 = \{(0, 4, 12, 8) + 4j : j = 3, 7, 11, 17, 21, 25, 29, 36, 40, 46, 50, 54\}$

$\mathcal{B}_9 = \{(0, 1, 3, 2) + i : i = 35, 39, 43, 47, 51\} \cup$

$\{(0, 4, 12, 8) + 4j : j = 0, 4, 14, 18, 28, 32, 42\}$

$\mathcal{F} = \{(0, 12, 5, 20), (3, 13, 19, 22), (1, 46, 35, 6), (2, 7, 39, 27)$

$(50, 37, 23, 43), (14, 28, 44, 17), (31, 25, 9, 49)$

$(\infty_1, 30, 41, 4), (\infty_2, 16, 29, 48), (\infty_3, 45, 54, 26), (\infty_4, 53, 36, 15),$

$(\infty_5, 24, 47, 38), (\infty_6, 52, 18, 42), (\infty_7, 10, 32, 8), (\infty_8, 34, 51, 21),$

$(\infty_9, 33, 11, 40)\}.$

(iv) $(v, n) = (68, 9)$ with complement type $(3, 7, 7, 7, 7, 7, 7, 7, 7)$;

$\mathcal{B}_1 = \{(0, 1, 3, 2) + i : i = 0, 4, 8, ..., 52\}$

$\mathcal{B}_2 = \{(0,1,3,2) + i : i = 56, 1, 5, ..., 45\}$

$\mathcal{B}_3 = \{(0,1,3,2) + i : i = 49, 53, 57, ..., 38\}$

$\mathcal{B}_4 = \{(0,1,3,2) + i : i = 42, 46, 50, ..., 31\}$

$\mathcal{B}_5 = \{(0,4,12,8) + 4j : j = 6, 10, 15, 19, 23, 27, 31, 35, 39, 44, 48, 52, 56\}$

$\mathcal{B}_6 = \{(0,4,12,8) + 4j : j = 8, 12, 16, 20, 24, 33, 37, 41, 45, 49, 53, 57, 2\}$

$\mathcal{B}_7 = \{(0,4,12,8) + 4j : j = 22, 26, 30, 34, 38, 43, 47, 51, 55, 1, 5, 9, 13\}$

$\mathcal{B}_8 = \{(0,4,12,8) + 4j : j = 3, 7, 11, 17, 21, 25, 29, 36, 40, 46, 50, 54, 58\}$

$\mathcal{B}_9 = \{(0,1,3,2) + i : i = 35, 39, 43, 47, 51, 55\} \cup$

$\{(0,4,12,8) + 4j : j = 0, 4, 15, 19, 30, 34, 45\}$

$\mathcal{F} = \{(0,32,46,37), (2,38,18,21), (1,28,7,54), (5,39,49,8),$

$(30,42,3,19), (4,57,33,20), (13,44,53,24), (48,15,10,23)$

$(\infty_1,16,6,50), (\infty_2,12,17,36), (\infty_3,11,35,52), (\infty_4,26,47,40),$

$(\infty_5,14,43,25), (\infty_6,9,51,58), (\infty_7,22,45,31), (\infty_8,27,55,29),$

$(\infty_9,41,56,34)\}.$

The complement type $(b_1, b_2, ..., b_n)$ for a $(v,k,n,1)$-nearly-IRPMD is called standard if $(b_1 \le b_2 \le ... \le b_n)$ and $b_n - b_1 \le k$. Similarly, the complement type $(e_1, e_2, ..., e_n)$ for a $(v,k,1)$-nearly-RPMD is called standard if $(e_1 \le e_2 \le ... \le e_n)$ and $e_n - e_1 \le k$.

Similarly, we have

Lemma 3.5 If there exist a $(n,k,1)$-RPMD and a $(v,n,k,1)$-nearly-IRPMD with standard complement type $(b_1, b_2, ..., b_n)$, then there exist a $(v,4,1)$-nearly-RPMD with standard complement type $(b_1+1, b_2+1, ..., b_n+1)$,where $v \equiv 0 \pmod{k}$ and $n \equiv 1 \pmod{k}$.

From Lemma 3.4, and Lemmas 4.1, 4.2 and 4.3 in [14], we have

Theorem 3.6 There exist $(v, n, 4, 1)$-nearly-IRPMDs with standard complement type for $(v, n) \in \{(20,5), (24,5), (28,5), (32,5), (36,5), (40,5), (44,5), (60,5), (36,9), (40,9), (44,9), (52,9), (64,9), (68,9), (72,9), (52,13), (56,13), (60,13), (68,17), (132,33)\}$.

From Lemma 3.5 and Theorem 1.5 we have

Theorem 3.7 There exist $(v, 4, 1)$-nearly-RPMDs with standard complement type $(e_1, e_2, ..., e_n)$ for
$(v, n) \in \{(20,5), (24,5), (28,5), (32,5), (36,5), (40,5), (44,5), (60,5), (36,9), (40,9), (44,9), (52,9), (64,9), (68,9), (72,9), (52,13), (56,13), (60,13), (68,17), (132,33)\}$.

4 k-HRPMD of type h^n and $h^{n-1}m^1$

If the blocks of a k−HPMD of type h^n, where $hn \equiv 0 \pmod k$, can be partitioned into $h(n-1)$ parallel classes, the k-HPMD is called resolvable, denoted by k-HRPMD.

If the blocks of a k-HPMD of type $h^{n-1}m^1$, where $h(n-1) \equiv 0 \pmod k$, $m \equiv 0 \pmod k$, $m \geq h$, can be partitioned into $h(n-1)$ parallel classes and $m-h$ partial parallel classes each covering every group of size h and not intersecting the group of size m, the k−HPMD is also called resolvable, denoted by k-HRPMD.

Theorem 4.1 There exist 4-RHPMDs of type 4^n for $n = 4, 5, 6, 7, 8$.

Proof. A 4-RHPMD of type 4^4 comes from Lemma 3.16 in [13], and a 4-RHPMD of type 4^8 comes from Lemma 4.5 in [14]. For each case of $n = 5, 6, 7$, let $Y = \{\infty_1, \infty_2, \infty_3, \infty_4.\}$, $X = Z_{4(n-1)} \cup Y$, and $G_0 = Y, G_{i+1} = \{i + (n-1)j : j = 0, 1, 2, 3\}, i = 0, 1, ..., n-2$. We present a set of base blocks which is a parallel class, and develop the base blocks under $Z_{4(n-1)}$ to form the blocks of the 4-RHPMD.

(i) $n = 5$,

$(0, 1, 15, 10)(\infty_1, 13, 12, 3), (\infty_2, 2, 7, 4), (\infty_3, 5, 14, 8), (\infty_4, 6, 9, 11)$.

(i) $n = 6$,

$(0, 1, 19, 7), (9, 12, 11, 15), (\infty_1, 14, 3, 10), (\infty_2, 16, 18, 4), (\infty_3, 8, 5, 17)$,

$(\infty_4, 6, 2, 13)$.

(i) $n = 7$,

$(0,1,9,4), (12,15,22,20), (19,21,6,5)$,

$(\infty_1, 11, 8, 13), (\infty_2, 10, 23, 3), (\infty_3, 14, 7, 18), (\infty_4, 16, 2, 17)$

Theorem 4.2 There exist 4-HRPMDs of type $4^n 8^1$ for $n = 6, 7$.

Proof. Let $Y = \{\infty_1, \infty_2, ..., \infty_8\}$, $X = Z_{4n} \cup Y$, and
$G_0 = Y, G_{i+1} = i + nj : j = 0, 1, 2, 3, i = 0, 1, ..., n-1$.
We present a set of base blocks (a parallel class) which forms $4n$ parallel classes of X, and present 4 parallel classes of $X \setminus Y$.

(i) $n = 6$,

$(\infty_1, 0, 8, 1), (\infty_2, 3, 18, 23), (\infty_3, 10, 14, 12), (\infty_4, 2, 11, 21)$,

$(\infty_5, 20, 16, 13), (\infty_6, 15, 4, 7), (\infty_7, 19, 6, 5), (\infty_8, 22, 17, 9)$,

$\{(0, 1, 3, 10) + 4j + i : j = 0, 1, .., 5\}, i = 0, 1, 2, 3$.

(i) $n = 7$,

$(0, 2, 24, 22)(\infty_1, 6, 16, 7), (\infty_2, 10, 9, 12), (\infty_3, 26, 18, 1), (\infty_4, 8, 25, 13)$,

$(\infty_5, 14, 11, 23), (\infty_6, 21, 17, 4), (\infty_7, 19, 27, 3), (\infty_8, 20, 15, 5)$,

$\{(0, 1, 6, 19) + 4j + i : j = 0, 1, .., 6\}, i = 0, 1, 2, 3$.

5 Construction methods

When we start with an $(h, k, 1)$-PMD and replace each block with a TD(k, m) we can obtain an $(hm, k, 1)$-HPMD of type m^h, this is the idea of Theorem 2.2 in [12]; similarly, when we start with a TD(h, m) and replace each block with an $(h, k, 1)$-PMD, we also can obtain an $(hm, k, 1)$-HPMD of type m^h, this is the idea of Theorem 2.4 in [12]. Based on these ideas and the Wilson's fundamental constructions for GDD (see [7]), we have the following constructions.

Let s be a function from X to non-negative integers with $s(x)$. If $A = \{x_1, x_2, ..., x_r\} \subseteq X$, then we denote $s_A = s(x_1) + s(x_2) + ... + s(x_r)$.

Let S be a mapping from X to the set of all subsets of $\{(x,i) : x \in X, i \geq 0\}$ by $S(x) = \{(x,i) : 1 \leq i \leq s(x)\}$. If $A = \{x_1, x_2, ..., x_r\} \subseteq X$, then we denote $S_A = s(x_1) \cup s(x_2) \cup ... \cup s(x_r)$.

Construction 5.1 Suppose $(X, \mathcal{G}, \mathcal{A})$ is a GDD with $\lambda = 1$ and for every block $A \in \mathcal{A}$, we have a $(s_A, k, 1)$-HPMD of type $\{s(x) : x \in A\}$. Then $(S_X, \{S_G : G \in \mathcal{G}\}, \{S_A : A \in \mathcal{A}\})$ is a $(s_X, k, 1)$-HPMD of type $\{s_G : G \in \mathcal{G}\}$.

Construction 5.2 Suppose $(X, \mathcal{G}, \mathcal{A})$ is a $(v, k, 1)$-HPMD, and for every block $A \in \mathcal{A}$, we have a $(s_A, k, 1)$-GDD of type $\{s(x) : x \in A\}$. Then $(S_X, \{S_G : G \in \mathcal{G}\}, \{S_A : A \in \mathcal{A}\})$ is a $(s_X, k, 1)$-HPMD of type $\{s_G : G \in \mathcal{G}\}$.

Theorem 5.3 (Filling in groups). Suppose that the following designs exist:
(i) a $(v, k, 1)$-HPMD of type $(g_1, g_2, ..., g_h)$;
(ii) a $(g_i, k, 1)$-PMD, for $1 \leq i \leq h$.
Then there exists a $(v, k, 1)$-PMD.

Theorem 5.4 (Filling in groups). Suppose that the following designs exist:
(i) a $(v, k, 1)$-HPMD of type $(g_1, g_2, ..., g_h)$;
(ii) a $(g_i + a, a, k, 1)$-IPMD, for $1 \leq i \leq h - 1$;
(iii) a $(g_h + a, k, 1)$-PMD.
Then there exists a $(v + a, k, 1)$-PMD.

The following is Theorem 2.5 in [14].

Theorem 5.5 Let $u \equiv 0 \pmod{k}, n, p \equiv 1 \pmod{k}$. Suppose that
(i) there exists a $(u, k, 1)$-RPMD;
(ii) there exists a $(u, n, k, 1)$-nearly-IRPMD;
(iii) there exists a $(p, k, 1)$-RPMD;
(iv) there exists a $RTD(k, u - n)$.
Then there exists a $((u - n)p + n, k, 1)$-RPMD.

Lemma 5.6 There exist $(v, 4, 1)$-RPMDs for $v = 120, 136, 212, 244, 304$.

Proof. Apply Theorem 5.5 with $u = 28, n = 5, k = 4, p = 5, 9, 13$ to obtain $(v, 4, 1)$-RPMDs for $v = 120, 212, 304$; and apply Theorem 5.5 with $u = 28, n =$ [illegible], $k = 4, p = 5, 9$ to obtain $(v, 4, 1)$-RPMDs for $v = 136, 244$.

Theorem 5.7 Let $h \leq m \leq 2h$. Suppose that
(i) there exists a RTD $(n+1, h)$;
(ii) there exists a k–HRPMD of type k^{n+1} if $m < 2h$;
(iii) there exists a k–HRPMD of type $k^n(2k)^1$ if $m > h$.
Then there exists a k–HRPMD of type $(kh)^n(km)^1$.

Proof. Let $\{x_1, x_2, ..., x_h\}$ be one group of the RTD$(n+1, h)$, and take $s(x) = 2k$ for $x \in \{x_1, x_2, ..., x_{m-h}\}$ and $s(x) = k$ for other points. Applying Construction 5.1 we obtain a big k-HPMD. It is easy to see that a parallel class of the RTD$(n+1, h)$ yields kn parallel classes of the big k–HPMD; all blocks of the RTD containing $x_i, 1 \leq i \leq m-h$, yield k partial parallel classes of the big k–HPMD each covering every groups of size kh and not intersecting the group of size km. Hence we obtain a k–HRPMD of type $(kh)^n(km)^1$.

Theorem 5.8 Let $w \equiv 1 \pmod{k}, h \leq m \leq 2h$. Suppose that
(i) there exists a $(kh, k, 1)$-nearly-RPMD with standard complement type $(c_1, c_2, ..., c_w)$;
(ii) there exists a $(km, k, 1)$-nearly-RPMD with standard complement type $(f_1, f_2, ..., f_w)$;
(iii) there exists a $(v, k, 1)$-HRPMD of type $(kh)^n(km)^1$ with the following property;
(iv) there is a parallel class which can be partitioned w parts such that the jth part cover f_j points of the group of size km and c_j points of each group of size kh.
Then there exists a $(k(nh+m), k, 1)$-RPMD.

Proof. By Theorem 5.3 we can obtain a $(k(nh+m), k, 1)$-PMD. The $k(m-h)$ partial parallel classes of the k-HRPMD and $k(m-h)$ parallel classes of the group of size km yield $k(m-h)$ parallel classes of the PMD; $kh - w$ parallel classes of each group yield $kh - w$ parallel classes of the PMD; $knh - 1$ parallel classes of the k-HRPMD are $knh - 1$ parallel classes of the PMD; finally w partial parallel classes of each group and a parallel class of the k-HRPMD with the property (iv) form w parallel classes of the PMD. Hence the PMD is resolvable.

Theorem 5.9 Let $w \equiv 1 \pmod{k}, h \leq m \leq 2h$. Suppose that
(i) there exists a $(kh, k, 1)$-nearly-RPMD with standard complement type $(c_1, c_2, ..., c_w)$;
(ii) there exists a $(km, k, 1)$-RPMD;

(iii) there exists a $(v, k, 1)$-HRPMD of type $(kn)^h(km)^1$ with the following property;
(iv) there is a partial parallel class which can be partitioned w parts, such that the jth part covers c_j points of each group of size kh.
Then there exists a $(k(nh + m), k, 1)$-RPMD.

Proof. The proof is similar to that of Theorem 5.8

From Theorem 1.5, Theorem 3.7 and Theorems 4.1 and 4.2, we can show the following results.

Lemma 5.10 There exist $(v, 4, 1)$-RPMDs for
$v = 204, 216, 228, 232, 236, 268, 276, 284, 292, 308, 312, 316, 328, 332, 336$.

Proof. Apply Theorem 5.7 with $k = 4$ and the following (n, h, m) to obtain a 4-HRPMD of type $(4h)^n(4m)^1$. Since each parallel class of the 4-HRPMD is composed of $m - h$ parts each being a parallel class of a 4-HRPMD of type 4^n8^1 and $2h - m$ parts each being a parallel class of a 4-HRPMD of type 4^{n+1}, it is easy to see the 4-HRPMD satisfy the property (iv) of Theorem 5.8 for the following (n, h, m, w). Therefore we can apply Theorem 5.8 to obtain the required results.
(i) $n = 5, m = h = 9, w = 5$ to $v = 216$;
(ii) $n = 5, m = h = 13, w = 9$ to $v = 312$;
(iii) $n = 7, h = 9, m = 10, w = 5$ to $v = 292$;
(iv) $n = 6, h = 7, m = 9, w = 5$ to $v = 204$;
(v) $n = 6, h = 8, m = 9, 10, 11, w = 5$ to $v = 228, 232, 236$;
(vi) $n = 6, h = 9, m = 13, 17, w = 9$ to $v = 268, 284$;
(vii) $n = 6, h = 11, m = 11, 13, 16, 17, 18, w = 9$ to
$v = 308, 316, 328, 332, 336$.
Apply Theorem 5.7 with $k = 4, n = 6, h = 9, m = 15$ to obtain a 4-HRPMD of type $(36)^6(60)^1$. Since each partial parallel class of the 4-HRPMD is composed of 9 parts each being a partial parallel class of a 4-HRPMD of type 4^68^1, it is easy to see the 4-HRPMD satisfy the property (iv) of Theorem 5.9 for $w = 9$. Therefore we can apply Theorem 5.9 to obtain a $(276, 4, 1)$-RPMD.

It is easy to see that the existence of a $(v, 4, 1)$-RPMD implies the existence of a $(v, 4, 3)$-RBIBD. Harri Haanpaa and Patric Ostergard [8] proved the non-existence of $(12, 4, 1)$-RPMD by using the results of [11].

Hence we have the following theorem.

Theorem 5.11 A $(v, 4, 1)$-RPMD exists for all integers $v \geq 4$ where $v \equiv 0 \pmod 4$, except for $v = 4, 8, 12$, and with 27 possible exceptions as follows: $v \in \{16, 20, 24, 32, 36, 44, 48, 52, 56, 64, 68, 76, 84, 88, 92, 96, 104, 108, 116, 124, 132, 148, 152, 156, 172, 184, 188.\}$

References

1. R.D.Baker and R.M.Wilson *The Whist-Tournament problem of E.H.Moore*, preprint.

2. F.E.Bennett, *Conjugate orthogonal Latin squares and Mendelsohn designs,* Ars Combinatoria 19(1985), 51-62.

3. F.E.Bennett, Xuebin Zhang and Lie Zhu *Perfect Mendelsohn designs with block size four,* Ars Combinatoria 29(1990), 65-72.

4. F.E.Bennett and Xuebin Zhang *Resolvable Mendelsohn designs with block size 4,* Aequationes Mathematicae 40(1990), 248-260.

5. F.E.Bennett and D.Sotteau *Almost resolvable decomposition of* K_n^* , J. Combin. Theory Ser.B 30(1981), 228-232.

6. J.C.Bermond, A.Germa and D.Sotteau *Almost resolvable decomposition of* K_n^* , J. Combin. Theory Ser.A 26(1979), 179-185.

7. J.H.Dinitz and D.R.Stinson, *Contemporary Design Theory, a Collection of Surveys* , New York, Toronto, John Wiley and Sons, Inc. 1992.

8. H.Haanpaa and P.Ostergard, *Classification of whist tournaments with up to twelve players* , preprint.

9. E.R.Lamken, W.H.Mills and R.M.Wilson, *Four pairwise balanced designs* , Designs, Codes and Crypt 1(1991), 63-68.

10. N.S.Mendelsohn *Perfect cyclic designs,* Discrete Math.20(1977), 63-68.

11. Luis.B.Morales and Carlos Velarde, *A complete classification of (12,4,3)-RBIBDs* , J. Combin. Designs, 9 (2001), 385-400.

12. Xuebin Zhang, *On the existence of* $(v, 4, 1) - PMD,$ Ars Combinatoria 29(1990), 3-12.

13. Xuebin Zhang, *Constructions for resolvable Mendelsohn designs,* Ars Combinatoria 34(1992), 225-250.

14. Xuebin Zhang, *On the existence of* $(v, 4, 1)$*-RPMD,* Ars Combinatoria 42(1996), 3-31.

15. L. Zhu, *Perfect Mendelsohn designs,* Journal of Combinatorial Mathematics and Combinatorial Computing,5(1989), 43-54.