CENTERS OF TENSOR PRODUCTS OF GRAPHS

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Abstract. Formulas for vertex eccentricity and radius for the tensor product $G \otimes H$ of two arbitrary graphs are derived. The center of $G \otimes H$ is characterized as the union of three vertex sets of form $A \times B$. This completes the work of Suh-Ryung Kim, who solved the case where one of the factors is bipartite. Kim's result becomes a corollary of ours.

1 Introduction

The *tensor product* of two simple graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph $G \otimes H$ whose vertex set is $V(G) \times V(H)$, and whose edge set is $\{(a, x)(b, y) | ab \in E(G) \text{ and } xy \in E(H)\}$. In the literature, the tensor product is also called the *Kronecker product*, the *categorical product* or the *direct product* (See Section 5.3 of [2]).

The *eccentricity* of a vertex a of G is the largest possible distance from a that a vertex b of G can have. The *radius* of G is the minimum of the eccentricities of all vertices of G. The *center* of G is the set of vertices whose eccentricity equals the radius of G. For a standard reference, see [1].

In this article, we derive formulas for the eccentricity of a vertex in $G \otimes H$, and for the radius of $G \otimes H$, where G and H are arbitrary simple graphs. We also present an explicit description of the center of $G \otimes H$. This completes the work of Suh-Ryung Kim [3], who treated the case in which one of the factors is bipartite. The general case that we address is considerably more complex.

In what follows, we recall some necessary definitions and terminology. In Section 2, we review the notion of distance in a graph. In order to obtain results on distance in a tensor product, it will be necessary to introduce a modified form of distance – called *upper distance* – on its factors. Section 3 reviews the graph-theoretic notions of eccentricity, radius and center. As a preliminary to deriving a formula for eccentricity in a tensor product, we introduce a modified form of eccentricity – called *upper eccentricity* – on its factors. Our main results are proved in Section 4, and in Section 5 we discuss how these results imply and generalize those of S. -R. Kim [3], and we present a counterexample to Kim's conjecture on the general case.

2 Distance in G and G $\otimes$ H

Here we review the notion of distance in a graph, and derive a few results concerning distance in $G \otimes H$. To achieve the latter goal, it will be necessary to present a variation of the usual distance in a graph G. The discussion is phrased in the language of walks.

Recall that a *walk* in G is a sequence of vertices $W = w_1 w_2 w_3 \cdots w_n$, where any two consecutive vertices are adjacent, and form an *edge* of the walk. A walk is regarded as a traversal of edges in a specified order. The *length* of W is $|W| = n - 1$, and is regarded as the number of edges in the walk (with the understanding that an edge may appear and be counted multiple times). A *trivial* walk consists of a single vertex, and has length 0. Two walks have the same *parity* if the difference of their lengths is even, and otherwise they have *opposite parity*. We also speak of a walk W and an integer k as having the same (or opposite) parity, meaning $|W| - k$ is even (or odd). An even (odd) walk is one whose length is even (odd).

If $W = w_1 w_2 w_3 ... w_n$ and $X = x_1 x_2 x_3 \cdots x_n$ are two walks of the same length in graphs G and H, respectively, we denote by $W \otimes X$ the walk $(w_1, x_1)(w_2, x_2)(w_3, x_3) \cdots (w_n, x_n)$ in $G \otimes H$. Notice that any walk in $G \otimes H$ can be written uniquely as $W \otimes X$, for appropriate walks W and X of the same length in G and H, respectively.

The *distance* between two vertices a and b of a graph G, denoted by $d_G(a, b)$, is the length of the shortest a-b walk in G, or ∞ if no such walk exists. The *upper distance* between a and b, denoted $D_G(a, b)$, is the length of the shortest a-b walk whose parity differs from that of $d_G(a, b)$. If G is bipartite or trivial, then no such walk exists, and we say $D_G(a, b) = \infty$. Note that if G is connected and contains an odd cycle, then $D_G(a, b)$ is always finite. For example, in Figure 1, $d_G(a, d) = 2$, $D_G(a, d) = 3$, $d_G(a, a) = 0$, and $D_G(a, a) = 5$.

An a-b walk W in a graph G is called *minimal* if $|W| = d_G(a, b)$, and it is called *slack* if $d_G(a, b) < |W| < D_G(a, b)$. It is called *critical* if $|W| =$

$D_G(a,b)$, and *ample* if $D_G(a,b) < |W|$. For example, if G is the 7-cycle *abcdefga*, the walk *abc* is minimal, and *cdefga* is critical. The walk *abcbc* is slack, and *abcbcbc* is ample. Notice that any minimal walk is necessarily a path. Observe also that any walk in a bipartite graph is either minimal or slack – it can be neither critical nor ample. The following observation, which follows from the above definitions, will be used frequently.

Observation 1: An a-b walk W is minimal if and only if there exist no shorter a-b walks. An a-b walk W is slack if and only if there exist shorter a-b walks, and all such shorter walks have the same parity as W. An a-b walk W is critical if and only if there exist shorter a-b walks, and all such walks have a parity different than that of W. An a-b walk W is ample if and only if there exist shorter a-b walks of both parities.

Next, we present two lemmas concerning distance in a tensor product. Although there exist more formulaic expressions for distance (see, for instance, [4, 3]) these lemmas will prevent our notation from getting out of hand.

Lemma 1: Suppose G has a nontrivial a-b walk W and H has a nontrivial x-y walk X. If W and X have the same parity, then $d_{G\otimes H}((a,x),(b,y)) \leq \max\{|W|,|X|\}$.

Proof. If $|W| = |X|$, then $W \otimes X$ is an (a,x)-(b,y) walk in $G \otimes H$ of length $|W| = \max\{|W|,|X|\}$, and the result follows. If $|W| < |X|$, we can extend W to an a-b walk $\widetilde{W}$ of length $|X|$ by appending to its end a walk $bcbcbc\cdots bcb$ of even length $|X| - |W|$ (which exists because X is nontrivial). Then $\widetilde{W} \otimes X$ is an (a,x)-(b,y) walk in $G \otimes H$ of length $|X| = \max\{|W|,|X|\}$. Thus, $d_{G\otimes H}((a,x),(b,y)) \leq \max\{|W|,|X|\}$. A symmetric construction works if $|W| > |X|$. ∎

Lemma 2: If there are no a-b and x-y walks of the same length in G and H, respectively, then $d_{G\otimes H}((a,x),(b,y)) = \infty$. Otherwise, $d_{G\otimes H}((a,x),(b,y)) = \min\{n \mid \exists\ a\text{-}b \text{ and } x\text{-}y \text{ walks of length } n \text{ in } G \text{ and } H, \text{ respectively}\}$.

Proof. If there are no a-b and x-y walks of the same length in G and H, respectively, then there can be no (a,x)-(b,y) walks in $G \otimes H$, for such a walk would be of form $W \otimes X$, where W and X are a-b and x-y walks of the same length in G and H, respectively. Hence $d_{G\otimes H}((a,x),(b,y)) = \infty$ in this case.

Now assume there are a-b and x-y walks of the same length in G and H. Set $M = \min\{n|\exists\ a\text{-}b \text{ and } x\text{-}y \text{ walks of length } n \text{ in } G \text{ and } H, \text{ respectively}\}$. By Lemma 1, it follows that $d_{G\otimes H}((a,x),(b,y)) \leq M$. On the other hand, any (a,x)-(b,y) walk in $G \otimes H$ can be written as $W \otimes X$ for some a-b walk W in G and x-y walk X in H, both of the same length n. It follows that $d_{G\otimes H}((a,x),(b,y)) \geq M$, and the proof is complete. ∎

The following result is our primary tool for constructing minimal walks in tensor products.

Proposition 1: Suppose W and X are walks of the same length in G and H, respectively. Then the walk $W \otimes X$ in $G \otimes H$ is minimal if and only if one of W or X is minimal, or if one is critical and the other is slack.

Proof. Let W be an a-b walk and let X be an x-y walk. If one of W or X is minimal, then $W \otimes X$ is minimal by Lemma 2. If W is critical and X is slack, then every a-b walk in G that is shorter than W has a parity that is opposite to that of W, but every x-y walk in H that is shorter than X has the same parity as X. Hence there is no integer $n < |W| = |X|$ for which there are a-b and x-y walks of length n, so $W \otimes X$ is minimal by Lemma 2. Reversing roles, if W is slack and X is critical, then $W \otimes X$ is minimal.

Conversely, suppose $W \otimes X$ is minimal. If one of W or X is minimal, there is nothing to prove. So suppose that neither is minimal, meaning that there are shorter a-b and x-y walks than W and X. Since $W \otimes X$ is minimal, $|W \otimes X| = d_{G\otimes H}((a,x),(b,y))$. Lemma 1 then implies that the a-b walks that are shorter than W have one parity, and the x-y walks that are shorter than X have the other parity. As $|W| = |X|$, Observation 1 shows that one of W and X is critical and the other is slack. ■

3 Eccentricity and Centers

The *eccentricity* of $a \in V(G)$ is $e_G(a) = \max\{d_G(a,b) | b \in V(G)\}$. The *upper eccentricity* of a is $E_G(a) = \max\{D_G(a,b) | b \in V(G)\}$. Moreover, we define the *even upper eccentricity* to be the even integer $E_G^0(a) = \max\{D_G(a,b) | b \in V(G), D_G(a,b) \text{ is even}\}$, and the *odd upper eccentricity* to be $E_G^1(a) = \max\{D_G(a,b) | b \in V(G), D_G(a,b) \text{ is odd}\}$. Therefore $E_G(a) = \max\{E_G^0(a), E_G^1(a)\}$. Notice that $E_G(a) = \infty$ if G is disconnected, bipartite, or trivial, and by convention we set $E_G^0(a) = \infty = E_G^1(a)$ in such cases. As an illustration of these ideas, each vertex v of the graph G in Figure 1 is labeled with a 4-tuple $(E_G^0(v), E_G^1(v), E_G(v), e_G(v))$.

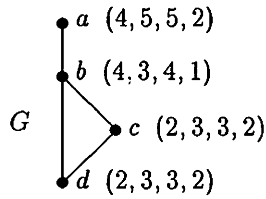

Figure 1. Each vertex v is labeled with $(E_G^0(v), E_G^1(v), E_G(v), e_G(v))$

The *radius* of G is $r(G) = \min\{e_G(a) | a \in V(G)\}$, and the *upper radius* is $R(G) = \min\{E_G(a) | a \in V(G)\}$. We also define the *even* and *odd upper radii* of G to be $R^0(G) = \min\{E_G^0(a) | a \in V(G)\}$ and $R^1(G) = \min\{E_G^1(a) | a \in V(G)\}$, respectively. For example, in Figure 1, $R^0(G) = 2$, $R^1(G) = 3$, $R(G) = 3$, and $r(G) = 1$.

Recall that the *center* of G is the subset of $V(G)$ consisting of all vertices a for which $e_G(a) = r(G)$. For example, the center of the graph G in Figure 1 consists of the single vertex b. Consideration of the even and odd upper eccentricity and radii in the factors of $G \otimes H$ will be instrumental in characterizing its center. We will need the following technical result.

Lemma 3: For any vertex a in a connected, nontrivial, nonbipartite graph G, the numbers $E_G^0(a)$ and $E_G^1(a)$ differ by 1. Also, if $a \in V(G)$ is such that $R(G) = E_G(a)$, then $R^0(G) = E_G^0(a)$ and $R^1(G) = E_G^1(a)$. (So $R^0(G)$ and $R^1(G)$ differ by 1, and $R(G) = \max\{R^0(G), R^1(G)\}$.)

Proof. Given a vertex a of G, $E_G(a) = \max\{E_G^0(a), E_G^1(a)\}$ is finite because G is connected, nontrivial and nonbipartite. Therefore, there exists a critical a-b walk W of length $E_G(a)$ in G. The statement will be proved if we can produce a critical a-c walk of length $E_G(a) - 1$ in G. We consider two exhaustive cases.

Case 1. The walk W is closed, that is $a = b$. Then, as $d_G(a, b) = 0$, W must have odd length. Let X be the a-c walk obtained by deleting the last edge ca of W. Then X has even length, and it must be critical, for were there a shorter even a-c walk Y, then Y with the edge ca appended to its end would be a shorter odd a-a walk than W, contradicting fact that W is critical. Hence $|X| = |W| - 1 = E_G(a) - 1$.

Case 2. The walk W is not closed, that is $a \neq b$. Let M be a minimal a-b walk. Let N be the minimal a-c walk obtained by deleting the last edge cb of M, and let X be a critical a-c walk. Now, X has parity opposite to N, which has parity opposite to M, which has parity opposite to W. Therefore X and W have opposite parities. Notice $|X| \geq |W| - 1$, for otherwise the walk obtained by appending the edge cb to the end of X would have the same parity as W, and be a shorter a-b walk than W, contradicting the fact that W is critical. On the other hand, $|X| \leq |W| - 1$, because W is a critical walk of maximal length beginning at a, and its parity differs from the critical walk X beginning at a. Thus $|X| = |W| - 1 = E_G(a) - 1$.

This completes the proof that $E_G^0(a)$ and $E_G^1(a)$ differ by 1. To prove the statements about R, R^0 and R^1, choose $a \in V(G)$ for which $R(G) = E_G(a)$.

Observe that $\min\{E_G^0(a), E_G^1(a)\} \leq R^0(G)$, for otherwise there is a $b \in V(G)$ with $R^0(G) = E_G^0(b) < \min\{E_G^0(a), E_G^1(a)\} < \max\{E_G^0(a), E_G^1(a)\} = E_G(a)$, whence $E_G^0(b) < E_G(a) - 1$. But, as $E_G^0(b)$ and $E_G^1(b)$ differ by 1, this yields $E_G(b) = \max\{E_G^0(b), E_G^1(b)\} < E_G(a)$, contradicting the choice of a. Thus, $\min\{E_G^0(a), E_G^1(a)\} \leq R^0(G)$, and similarly $\min\{E_G^0(a), E_G^1(a)\} \leq R^1(G)$.

But also, $R^0(G) \le E_G^0(a) \le \max\{E_G^0(a), E_G^1(a)\}$, and $R^1(G) \le E_G^1(a) \le \max\{E_G^0(a), E_G^1(a)\}$. Therefore we have

$$\min\{E_G^0(a), E_G^1(a)\} \le R^0(G), R^1(G) \le \max\{E_G^0(a), E_G^1(a)\}.$$

From this it follows that, as $E_G^0(a)$ and $E_G^1(a)$ differ by 1, $R^0(G) = E_G^0(a)$ and $R^1(G) = E_G^1(a)$. ■

4 Results

We are now in a position to compute the eccentricity of a vertex of $G \otimes H$, and also to find the radius and center of $G \otimes H$. This is done in Theorems 1, 2, and 3.

Theorem 1: If G and H are nontrivial graphs, and $(a,x) \in V(G \otimes H)$, then $e_{G\otimes H}(a,x) = \max\{e_G(a), e_H(x), \min\{E_G^0(a), E_H^1(x)\}, \min\{E_G^1(a), E_H^0(x)\}\}$.

Proof. For simplicity, throughout this proof M will be the integer $M = \max\{e_G(a), e_H(x), \min\{E_G^0(a), E_H^1(x)\}, \min\{E_G^1(a), E_H^0(x)\}\}$.

To begin, observe the theorem is true when $G \otimes H$ is disconnected: Recall (c.f. Theorem 5.29 of [2]) that $G \otimes H$ is disconnected if and only if one of G or H is disconnected, or if both G and H are bipartite. Then

$$e_{G\otimes H}(a,x) = \infty$$
$$\Updownarrow$$
$$G \otimes H \text{ is disconnected}$$
$$\Updownarrow$$
$$G \text{ or } H \text{ is disconnected, or both } G \text{ and } H \text{ are bipartite}$$
$$\Updownarrow$$
$$e_G(a) = \infty \text{ or } e_H(x) = \infty, \text{ or } E_G(a) = \infty = E_H(x)$$
$$\Updownarrow$$
$$e_G(a) = \infty \text{ or } e_H(x) = \infty, \text{ or } \min\{E_G^0(a), E_H^1(x)\} = \infty = \min\{E_G^1(a), E_H^0(x)\}$$
$$\Updownarrow$$
$$M = \infty.$$

It follows that the theorem is true when one of G or H is disconnected, or if both G and H are bipartite, and this happens if and only if $M = \infty$. Therefore, for the rest of the proof we may assume G and H are connected, and at least one is not bipartite, and that M is finite.

Now we show $e_{G\otimes H}(a,x) \le M$. For this, it suffices to show that any minimal walk in $G \otimes H$ which starts at (a,x) has length no greater than M. Thus, let $W \otimes X$ be a minimal (a,x)-(b,y) walk. By Proposition 1, one of W or X is minimal, or one is critical and the other is slack. We consider these possibilities one by one.

If W is a minimal a-b walk, $|W \otimes X| = |W| \leq e_G(a) \leq M$. If X is a minimal x-y walk, $|W \otimes X| = |X| \leq e_H(x) \leq M$.

Next, suppose W is critical and X is slack. (So G necessarily has an odd cycle.) First assume $|W \otimes X|$ is even, so $|W \otimes X| = |W| \leq E_G^0(a)$, while $|X| < E_H(x)$. But, as $|X|$ is slack and of even length, any critical x-y walk must have odd length, so $|X| < E_H^1(x)$. (Or, if H is bipartite, $|X| < E_H^1(x) = \infty$ automatically.) Therefore $|W \otimes X| = |X| < E_H^1(x)$. Now we have established $|W \otimes X| \leq E_G^0(a)$ and $|W \otimes X| \leq E_H^1(x)$, so $|W \otimes X| \leq \min\{E_G^0(a), E_H^1(x)\} \leq M$. On the other hand, if $|W \otimes X|$ is odd, then $|W \otimes X| = |W| \leq E_G^1(a)$, while $|X| < E_H(x)$. But, as $|X|$ is slack and of odd length, it follows that any critical x-y walk must have even length, so $|X| < E_H^0(x)$. Hence $|W \otimes X| = |X| < E_H^0(x)$. This shows $|W \otimes X| \leq E_G^1(a)$ and $|W \otimes X| \leq E_H^0(x)$, so $|W \otimes X| \leq \min\{E_G^1(a), E_H^0(x)\} \leq M$.

Finally, if W is slack and X is critical, an argument symmetric to that of the previous paragraph shows $|W \otimes X| \leq M$.

This completes the proof that $e_{G\otimes H}(a, x) \leq M$, and we now demonstrate $e_{G\otimes H}(a, x) \geq M$.

To prove $e_{G\otimes H}(a, x) \geq M$, it suffices to show that we can always find a minimal walk in $G \otimes H$, starting at (a, x), and having length at least M. To do this, it suffices to find four walks, each starting at (a, x), and having lengths no smaller than $e_G(a)$, $e_H(x)$, $\min\{E_G^0(a), E_H^1(x)\}$, and $\min\{E_G^1(a), E_H^0(x)\}$, respectively. The remainder of this proof is a construction of such walks.

Let W be a minimal walk of length $e_G(a)$ in G. Construct a walk X of length $|W|$ in H by defining $X = xyxyxy\cdots xyx$ if $|W|$ is even, or $X = xyxyxy\cdots xy$ if $|W|$ is odd (where xy is any edge of H that is incedent with x). Then $W \otimes X$ is minimal by Proposition 1, it starts at (a, x), and its length is $e_G(a)$. A symmetric construction produces a minimal walk starting at (a, x) and having length $e_H(x)$.

Now we make a minimal walk starting at (a, x), having length at least $\min\{E_G^0(a), E_H^1(x)\}$. We break into two cases.

Case 1. $E_G^0(a) > E_H^1(x)$. Since not more than one of G or H can be bipartite, it follows that not more than one of $E_G^0(a)$ or $E_H^1(x)$ can be infinite. Therefore $E_H^1(x)$ is finite and H is not bipartite. Thus there is a critical x-y walk X of length $E_H^1(x)$ in H. If G is bipartite, choose any edge $ac \in E(G)$, and let $W = acac\cdots ac$ be a walk of length $|X|$ in G, so W is either minimal or slack. Then $W \otimes X$ is minimal by Proposition 1, it starts at (a, x) and $|W \otimes X| = |X| = E_H^1(x) \leq \min\{E_G^0(a), E_H^1(x)\}$. On the other hand, if G is not bipartite, it has a critical a-b walk W of length $E_G^0(a)$. Let Z be a minimal a-b walk in G, so Z and X have the same parity. If $|Z| \geq |X|$, then X may be extended to an x-y walk $\widetilde{X}$ of length $|Z|$, by alternating back an forth along the last edge of X. Then, by Proposition 1, $Z \otimes \widetilde{X}$ is minimal, it starts at (a, x), and its length is $|\widetilde{X}| \geq |X| = E_H^1(x)$

$\geq \min\{E_G^0(a), E_H^1(x)\}$. If $|Z| < |X|$, then Z may be extended to an a-b walk $\widetilde{Z}$ of length $|X|$, by alternating back an forth along the last edge of Z. Notice that $\widetilde{Z}$ is slack because it is an a-b walk of length smaller than the critical a-b walk W. (Because $|\widetilde{Z}| = |X| = E_H^1(x) < E_G^0(a) = |W|$.) Thus, by Proposition 1, $\widetilde{Z} \otimes X$ is minimal, it starts at (a, x), and its length is $|X| = E_H^1(x) \geq \min\{E_G^0(a), E_H^1(x)\}$.
Case 2. $E_G^0(a) < E_H^1(x)$. As in the previous case, we reason that $E_G^0(a)$ is finite and G is not bipartite. Thus there is a critical a-b walk W of length $E_G^0(a)$ in G. If H is bipartite, let $X = xzxz\cdots zx$ be a walk of length $|W|$ in H, so X is either minimal or slack. Then $W \otimes X$ is minimal by Proposition 1, it starts at (a, x) and $|W \otimes X| = |W| = E_G^0(a) \leq \min\{E_G^0(a), E_H^1(x)\}$. On the other hand, if H is not bipartite, it has a critical x-y walk X of length $E_H^1(x)$. Let Y be a minimal x-y walk in H, so Y and W have the same parity. If $|Y| \geq |W|$, then W may be extended to an a-b walk $\widetilde{W}$ of length $|Y|$, by alternating back an forth along the last edge of W. Then, by Proposition 1, $\widetilde{W} \otimes Y$ is minimal, it starts at (a, x), and its length is $|\widetilde{W}| \geq |W| = E_G^0(a) \geq \min\{E_G^0(a), E_H^1(x)\}$. If $|Y| < |W|$, then Y may be extended to an x-y walk $\widetilde{Y}$ of length $|W|$, by alternating back an forth along the last edge of Y. Notice that $\widetilde{Y}$ is slack because it is an x-y walk of length smaller than the critical x-y walk X. (Because $|\widetilde{Y}| = |W| = E_G^0(a) < E_H^1(x) = |X|$.) Thus, by Proposition 1, $W \otimes \widetilde{Y}$ is minimal, it starts at (a, x), and its length is $|W| = E_G^0(a) \geq \min\{E_G^0(a), E_H^1(x)\}$.

Finally, the argument of the previous paragraph can be repeated, breaking into cases $E_G^1(a) > E_H^0(x)$ and $E_G^1(a) < E_H^0(x)$, and using W as critical a-b walk of length $E_G^1(a)$ and X as a critical x-y walk of length $E_H^0(x)$. We obtain a minimal walk in $G \otimes H$ which starts at (a, x) and has length at least $\min\{E_G^1(a), E_H^0(x)\}$. ■

Theorem 2: If G and H are nontrivial graphs, then the radius of $G \otimes H$ is $r(G \otimes H) = \max\{r(G), r(H), \min\{R^0(G), R^1(H)\}, \min\{R^1(G), R^0(H)\}\}$.

Proof. For simplicity, throughout this proof, K denotes the integer $K = \max\{r(G), r(H), \min\{R^0(G), R^1(H)\}, \min\{R^1(G), R^0(H)\}\}$.

Observe $r(G \otimes H) \geq K$: Choose a vertex (a, x) for which $r(G \otimes H) = e_{G \otimes H}(a, x)$. But Theorem 1 provides a formuls for $e_{G \otimes H}(a, x)$. We get $r(G \otimes H) = \max\{e_G(a), e_H(x), \min\{E_G^0(a), E_H^1(x)\}, \min\{E_G^1(a), E_H^0(x)\}\} \geq \max\{r(G), r(H), \min\{R^0(G), R^1(H)\}, \min\{R^1(G), R^0(H)\}\} = K$.

Next, we show $r(G \otimes H) \leq K$. For this, it suffices to show that some vertex (a, x) of $G \otimes H$ satisfies $e_{G \otimes H}(a, x) = K$. We break this into three cases.
Case 1: $R(G) > R(H)$. First, we simplify the expression for K. By Lemma 3, $\max\{R^0(G), R^1(G)\} > \max\{R^0(H), R^1(H)\}$, and another ap-

plication of Lemma 3 establishes $R^0(G) \geq R^1(H)$ and $R^1(G) \geq R^0(H)$. Then $K = \max\{r(G), r(H), R^1(H), R^0(H)\} = \max\{r(G), r(H), R(H)\} = \max\{r(G), R(H)\}$.

Now choose $a \in V(G)$ and $x \in V(H)$ for which $e_G(a) = r(G)$ and $E_H(x) = R(H)$. Then $\max\{E_G^0(a), E_G^1(a)\} = E_G(a) \geq R(G) > R(H) = E_H(x) = \max\{E_H^0(x), E_H^1(x)\}$, from which Lemma 3 yields $E_G^0(a) \geq E_H^1(x)$ and $E_G^1(a) \geq E_H^0(x)$. Putting all this information into the formula from Theorem 1 gives $e_{G\otimes H}(a,x) = \max\{e_G(a), e_H(x), E_H^1(x), E_H^0(x)\} = \max\{e_G(a), e_H(x), E_H(x)\} = \max\{e_G(a), E_H(x)\} = \max\{r(G), R(H)\} = K$.

Case 2: $R(G) < R(H)$. Reverse the roles of G and H in Case 1.

Case 3: $R(G) = R(H)$. By Lemma 3, this is $\max\{R^0(G), R^1(G)\} = \max\{R^0(H), R^1(H)\}$, and from this, Lemma 3 again implies $R^0(G) = R^0(H)$ and $R^1(G) = R^1(H)$. Thus in this case K simplifies as follows:

$$\begin{aligned} K &= \max\{r(G), r(H), \min\{R^0(G), R^1(G)\}, \min\{R^1(H), R^0(H)\}\} \\ &= \max\{r(G), r(H), R(G)-1, R(H)-1\} \\ &= \max\{R(G)-1, R(H)-1\} \\ &= R(G)-1 \end{aligned}$$

Choose $a \in V(G)$ and $x \in V(H)$ for which $E_G(a) = R(G)$ and $E_H(x) = R(H)$. Then $\max\{E_G^0(a), E_G^1(a)\} = E_G(a) = R(G) = R(H) = E_H(x) = \max\{E_H^0(x), E_H^1(x)\}$, from which Lemma 3 yields $E_G^0(a) = E_H^0(x)$ and $E_G^1(a) = E_H^1(x)$. Putting this into the formula from Theorem 1 gives $e_{G\otimes H}(a,x) = \max\{e_G(a), e_H(x), \min\{E_G^0(a), E_G^1(a)\}, \min\{E_H^1(x), E_H^0(x)\}\} = \max\{e_G(a), e_H(x), E_G(a)-1, E_H(x)-1\} = \max\{E_G(a)-1, E_H(x)-1\} = \max\{R(G)-1, R(H)-1\} = R(G)-1 = K$. ■

The next theorem is an explicit description of the center of $G \otimes H$. To set the stage, put $\rho = r(G \otimes H)$, and define the following nested sets.

$$A = \{a \in V(G) | E_G(a) \leq \rho\} \subseteq \bar{A} = \{a \in V(G) | E_G(a) \leq \rho + 1\} \subseteq \widetilde{A} = \{a \in V(G) | e_G(a) \leq \rho\}$$

$$B = \{x \in V(H) | E_H(x) \leq \rho\} \subseteq \bar{B} = \{x \in V(H) | E_H(x) \leq \rho + 1\} \subseteq \widetilde{B} = \{x \in V(H) | e_H(x) \leq \rho\}$$

Theorem 3: If G and H are nontrivial, then the center of $G \otimes H$ is the vertex set $(A \times \widetilde{B}) \cup (\bar{A} \times \bar{B}) \cup (\widetilde{A} \times B)$.

Proof. We begin by verifying that each of the sets $A \times \widetilde{B}$, $\bar{A} \times \bar{B}$, and $\widetilde{A} \times B$ is in the center of $G \otimes H$. It suffices to show that if (a,x) is in one of these sets, then $e_{G\otimes H}(a,x) \leq \rho$.

First, suppose $(a,x) \in \bar{A} \times \bar{B}$, so $E_G(a) \leq \rho + 1$ and $E_H(x) \leq \rho + 1$. Since $e_G(a) < E_G(a)$, it follows that $e_G(a) \leq \rho$, and similarly $e_H(x) \leq \rho$. Also, since $E_G(a) = \max\{E_G^0(a), E_G^1(a)\} \leq \rho + 1$, and $E_H(x) =$

$\max\{E_H^0(x), E_H^1(x)\} \leq \rho + 1$, it follows that each member of the multiset $\{E_G^0(a), E_G^1(a), E_H^0(x), E_H^1(x)\}$ is at most $\rho + 1$. But, as at most two members (of the same parity) can equal $\rho + 1$, we infer that the set contains two members of the same parity that are strictly smaller than $\rho + 1$. Hence, $\min\{E_G^0(a), E_H^1(x)\} \leq \rho$ and $\min\{E_G^1(a), E_H^0(x)\} \leq \rho$. So Theorem 1 implies $e_{G\otimes H}(a, x) \leq \rho$.

Next suppose $(a, x) \in \widetilde{A} \times B$. (A symmetric argument will work for the case $(a, x) \in A \times \widetilde{B}$). By definition of $\widetilde{A}$ and B, we get $e_G(a) \leq \rho$ and $e_H(x) < E_H(x) = \max\{E_H^0(x), E_H^1(x)\} \leq \rho$. Hence, $\min\{E_G^0(a), E_H^1(x)\} \leq \rho$ and $\min\{E_G^1(a), E_H^0(x)\} \leq \rho$. Theorem 1 now implies $e_{G\otimes H}(a, x) \leq \rho$.

Conversely, suppose (a, x) is in the center of $G \otimes H$, so $e_{G\otimes H}(a, x) = \rho$. By Theorem 1, we obtain the inequalities $e_G(a) \leq \rho$, $e_H(x) \leq \rho$, $\min\{E_G^0(a), E_H^1(x)\} \leq \rho$, and $\min\{E_G^1(a), E_H^0(x)\} \leq \rho$. The first two of these inequalities are enough to show $a \in \widetilde{A}$ and $x \in \widetilde{B}$. To finish the proof, we consider four mutually exclusive and exhaustive cases.
Case 1: $\min\{E_G^0(a), E_H^1(x)\} = E_G^0(a) \leq \rho$ and $\min\{E_G^1(a), E_H^0(x)\} = E_G^1(a) \leq \rho$. Then $E_G(a) = \max\{E_G^0(a), E_G^1(a)\} \leq \rho$, and $(a, x) \in A \times \widetilde{B}$.
Case 2: $\min\{E_G^0(a), E_H^1(x)\} = E_H^1(x) \leq \rho$ and $\min\{E_G^1(a), E_H^0(x)\} = E_H^0(x) \leq \rho$. Then $E_H(x) = \max\{E_H^0(x), E_H^1(x)\} \leq \rho$, and $(a, x) \in \widetilde{A} \times B$.
Case 3: $\min\{E_G^0(a), E_H^1(x)\} = E_H^1(x) \leq \rho$ and $\min\{E_G^1(a), E_H^0(x)\} = E_G^1(a) \leq \rho$. Then, by Lemma 3, $E_H(x) = \max\{E_H^0(x), E_H^1(x)\} \leq \rho + 1$, and $E_G(a) = \max\{E_G^0(a), E_G^1(a)\} \leq \rho + 1$, so $(a, x) \in \bar{A} \times \bar{B}$.
Case 4: $\min\{E_G^0(a), E_H^1(x)\} = E_G^0(a) \leq \rho$ and $\min\{E_G^1(a), E_H^0(x)\} = E_H^0(x) \leq \rho$. By the same argument as in Case 3, we get $(a, x) \in \bar{A} \times \bar{B}$.

These four cases show $(a, x) \in (A \times \widetilde{B}) \cup (\bar{A} \times \bar{B}) \cup (\widetilde{A} \times B)$. ∎

Theorems 1, 2, and 3 simplify greatly if one factor of the tensor product is bipartite. If G is a connected graph with an odd cycle, and H is bipartite, then $E_G(a)$, $E_G^0(a)$ and $E_G^1(a)$ are finite, whereas $E_H(x) = E_H^0(x) = E_H^1(x) = \infty$ for any vertex x in H. Theorem 1 becomes $e_{G\otimes H}(a, x) = \max\{e_G(a), e_H(x), E_G^0(a), E_G^1(a)\} = \max\{e_G(a), e_H(x), E_G(a)\} = \max\{E_G(a), e_H(x)\}$. Likewise, if H is bipartite, Theorem 2 reduces to $r(G \otimes H) = \max\{R(G), r(H)\}$, and in Theorem 3, $B = \bar{B} = \emptyset$. These observations prove the following.

Corollary 1: If G and H are connected, G contains an odd cycle, and H is bipartite, then $e_{G\otimes H}(a, x) = \max\{E_G(a), e_H(x)\}$ for any vertex (a, x). Moreover, $r(G \otimes H) = \max\{R(G), r(H)\}$, and the vertices in the center of $G \otimes H$ are exactly $A \times \widetilde{B} = \{a \in V(G) | E_G(a) \leq r(G \otimes H)\} \times \{x \in V(H) | e_H(x) \leq r(G \otimes H)\}$.

5 Relation to Kim's Theorems

In the article *Centers of a tensor composite graph* [3], Suh-Ryung Kim presents formulas for eccentricity and radius of $G \otimes H$, where G is a connected graph which contains an odd cycle, and H is connected and bipartite. The notation and approach differ slightly from ours. Here we recall the definitions and results, and indicate how the results of [3] are special cases of ours.

Kim defines $d_e(a,b)$ and $d_o(a,b)$ to be the lengths of the shortest a-b walks of even and odd lengths, respectively. The *double eccentricity* of a vertex a of G is defined to be $de_G(a) = \max\{d_e(a,b), d_o(a,b) | b \in V(G)\}$, and the *double radius* is defined to be $dr(G) = \min\{de_G(a) | a \in V(G)\}$. Kim proves that $e_{G\otimes H}(a,x) = \max\{de_G(a), e_H(x)\}$, and (a,x) is in the center of $G \otimes H$ if and only if $e_{G\otimes H}(a,x) = \max\{dr(G), r(H)\}$ (i.e. that $r(G\otimes H) = \max\{dr(G), r(H)\}$). Simply observe $de_G(a) = E_G(a)$, and $dr(G) = R(G)$, and these results are our Corollary 1.

Kim conjectures in [3] that in the general case the formula for eccentricity is $e_{G\otimes H}(a,x) = \max\{de_G(a), de_H(x)\}$, from which we would presume $r(G \otimes H) = \max\{dr(G), dr(H)\}$. In our notation, this is $e_{G\otimes H}(a,x) = \max\{E_G(a), E_H(x)\}$, and $r(G \otimes H) = \max\{R(G), R(H)\}$. Our Theorems 1 and 2 suggest that the truth is somewhat more intricate. In fact, we have the following counterexample to Kim's conjecture.

Let G be the graph in Figure 1, and consider $G \otimes G$. Our Theorem 1 gives $e_{G\otimes G}(a,a) = \max\{2, 2, \min\{4,5\}, \min\{4,5\}\} = 4$, whereas Kim's conjecture would yield $e_{G\otimes G}(a,a) = \max\{5,4\} = 5$. Also, our Theorem 2 gives $r(G\otimes G) = \max\{1, 1, \min\{2,3\}, \min\{2,3\}\} = 2$, but Kim's conjecture is $r(G \otimes G) = \max\{3,3\} = 3$.

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