The Smallest Degree Sum that Yields Potentially $K_{2,s}$-graphic Sequences *

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Abstract. Let $\sigma(K_{r,s}, n)$ denote the smallest even integer such that every n-term graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with term sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n \geq \sigma(K_{r,s}, n)$ has a realization G containing $K_{r,s}$ as a subgraph, where $K_{r,s}$ is the $r \times s$ complete bipartite graph. In this paper, we determine $\sigma(K_{2,3}, n)$ for $n \geq 5$. In addition, we also determine the values $\sigma(K_{2,s}, n)$ for $s \geq 4$ and $n \geq 2\left[\frac{(s+3)^2}{4}\right] + 5$.

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1. Introduction

An n-term non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is refereed to as a *realization* of π. For a nonnegative integer sequence $\pi = (d_1, d_2, \ldots, d_n)$, denote $\sigma(\pi) = d_1+d_2+\cdots+d_n$. For a given graph H, a graphic sequence π is *potentially H-graphic* if there exists a realization of π containing H as a subgraph. Gould et al. [2] considered the following variation of the classical Turán-type

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extremal problems: determine the smallest even integer $\sigma(H, n)$ such that every n-term graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with $\sigma(\pi) \geq \sigma(H, n)$ has a realization G containing H as a subgraph. If $H = K_r$, the complete graph on r vertices, this problem was considered by Erdős et al. [1] where they showed that $\sigma(K_3, n) = 2n$ for $n \geq 6$ and conjectured that $\sigma(K_r, n) = (r-2)(2n-r+1)+2$ for sufficiently large n. Gould et al. [2] and Li and Song [4] proved independently that the conjecture holds for $r = 4$ and $n \geq 8$. Recently, Li et al. [5,6] showed that the conjecture is true for $r = 5$ and $n \geq 10$ and for $r \geq 6$ and $n \geq \binom{r-1}{2}+3$. For $H = K_{r,r}$, Gould et al. [2] proved that $\sigma(K_{2,2}, n) = 2\left[\frac{3n-1}{2}\right]$ for $n \geq 4$. Yin and Li [7] determined the values of $\sigma(K_{3,3}, n)$ for $n \geq 6$ and $\sigma(K_{4,4}, n)$ for $n \geq 8$. They [8] further determined the values $\sigma(K_{r,r}, n)$ for even $r(\geq 4)$ and $n \geq 4r^2 - r - 6$ and for odd $r(\geq 3)$ and $n \geq 4r^2 + 3r - 8$. A natural problem is to consider the general case, i.e., to determine the values $\sigma(K_{r,s}, n)$ for $s \geq r \geq 1$. In [9], Yin, Li and Chen determined the values $\sigma(K_{r,s}, n)$ for $s \geq r \geq 3$ and sufficiently large n. The purpose of the paper is to determine the values of $\sigma(K_{1,s}, n)$ for $n \geq s+1$, $\sigma(K_{2,3}, n)$ for $n \geq 5$ and $\sigma(K_{2,s}, n)$ for $s \geq 4$ and $n \geq 2\left[\frac{(s+3)^2}{4}\right] + 5$. In order to prove our main results, we also need the following notations and results.

Let $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integers, and for a given positive integer k, $1 \leq k \leq n$, let

$$\pi''_k = \begin{cases} (d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1, \ldots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n) & \text{if } d_k \geq \\ (d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n) & \text{if } d_k < \end{cases}$$

Denote $\pi'_k = (d'_1, d'_2, \ldots, d'_{n-1})$, where $d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ is the rearrangement of the $n-1$ terms in π''_k. Then π'_k is called the *residual sequence* obtained by laying off d_k from π. It is easy to see that if π'_k is graphic then so is π, since a realization G of π can be obtained from a realization G' of π'_k by adding a new vertex of degree d_k and joining it to the vertices whose degrees are reduced by one in going from π to π'_k. In fact more is true:

Theorem 1.1. [3] Let $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integers. Then π is graphic if and only if π'_k is graphic.

Theorem 1.2. [7,8] Let $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integers, where $d_1 \leq m$ and $\sigma(\pi)$ is even. If there exists an integer $n_1 \leq n$ such that $d_{n_1} \geq h \geq 1$ and $n_1 \geq \frac{1}{h}\left[\frac{(m+h+1)^2}{4}\right]$, then π is graphic.

Theorem 1.3. [6] If $r \geq 5$, then $\sigma(K_{r+1}, n) \leq 2n(r-2)+8$ for $2r+2 \leq n \leq \binom{r}{2}+3$ and $\sigma(K_{r+1}, n) = (r-1)(2n-r)+2$ for $n \geq \binom{r}{2}+3$.

Theorem 1.4. [7,8] If $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n)$ is a graphic sequence with $d_{r+s} \geq r+s-1$ and $d_n \geq r$, then π is potentially $K_{r,s}$-graphic.

Theorem 1.5. [7,8] Let $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n)$ be a graphic sequence with $d_r \geq r+s-1$, $d_{r+s} \leq r+s-2$ and $d_n \geq r$. If $n \geq (r+2)(s-1)$, then π is potentially $K_{r,s}$-graphic.

Throughout this paper, $G \cup H$ denotes the disjointed union of the graphs G and H, and $\pi(G)$ denotes the degree sequence of the graph G(in non-increasing order).

2. The values of $\sigma(K_{1,s}, n)$ for $n \geq s+1$ and $\sigma(K_{2,3}, n)$ for $n \geq 5$

We first determine the values $\sigma(K_{1,s}, n)$ for $n \geq s+1$. We have

Theorem 2.1. Let $s \geq 1$ and $n \geq s+1$. Then

$$\sigma(K_{1,s}, n) = \begin{cases} (s-1)n+2 & \text{if } s \text{ is odd or } n \text{ is even,} \\ (s-1)n+1 & \text{if } s \text{ is even and } n \text{ is odd.} \end{cases}$$

Proof. Assume that s is odd or n is even. Clearly, $\pi = ((s-1)^n)$ is the degree sequence of any $(s-1)$-regular graph G on n vertices, where the symbol x^y in a sequence stands for y consecutive terms x, and G contains no $K_{1,s}$ as a subgraph. Thus $\sigma(K_{1,s}, n) \geq \sigma(\pi)+2 = (s-1)n+2$. Now assume that $\pi = (d_1, d_2, \ldots, d_n)$ is a graphic sequence with $\sigma(\pi) \geq (s-1)n+2$. Clearly, $d_1 \geq s$. Hence any realization of π contains $K_{1,s}$ as a subgraph. In other words, π is potentially $K_{1,s}$-graphic. Thus $\sigma(K_{1,s}, n) \leq (s-1)n+2$. So $\sigma(K_{1,s}, n) = (s-1)n+2$.

Now assume that s is even and n is odd. Consider the sequence $\pi = ((s-1)^{n-1}, s-2)$. Since $\pi(\frac{n-1}{2}K_2 \cup K_1) = (1^{n-1}, 0)$, where pK_2 is the matching consisted of p edges, π is graphic for $s=2$. Assume that $s \geq 4$ is even and $n \geq s+1$ is odd. Then the graph H obtained from a $(s-1)$-regular graph on $n-1$ vertices by deleting a matching $\frac{s-2}{2}K_2$ has the degree sequence $\pi(H) = ((s-1)^{n-s+1}, (s-2)^{s-2})$. Hence π is the degree sequence of the graph G obtained from H by adding a new vertex x and joining x to $s-2$ vertices with degree $s-2$ in H. So π is graphic. It is easy to see that π is not potentially $K_{1,s}$-graphic since the largest term in π is $s-1$. Hence $\sigma(K_{1,s}, n) \geq \sigma(\pi)+2 = (s-1)n+1$. Now suppose that $\pi = (d_1, d_2, \ldots, d_n)$ is a graphic sequence with $\sigma(\pi) \geq (s-1)n+1$. Clearly, $d_1 \geq s$. Hence π is potentially $K_{1,s}$-graphic. In other words, $\sigma(K_{1,s}, n) \leq (s-1)n+1$. So $\sigma(K_{1,s}, n) = (s-1)n+1$. □

We now determine the values $\sigma(K_{2,3}, n)$ for $5 \leq n \leq 9$.

Theorem 2.2. (1) $\sigma(K_{2,3}, 5) = 16$;
(2) $\sigma(K_{2,3}, 6) = 22$;
(3) $\sigma(K_{2,3}, 7) = 26$;
(4) $\sigma(K_{2,3}, 8) = 28$;

(5) $\sigma(K_{2,3}, 9) = 30$.

Proof. (1) Let C_r be the cycle of length r. By adding a new vertex x to $C_3 \cup K_1$ and joining it to each vertex in $C_3 \cup K_1$, we obtain a graph G with degree sequence $\pi_1 = (4, 3^3, 1)$. In other words, π_1 is graphic. Moreover, it is easy to see that G is unique realization of π_1, and G contains no $K_{2,3}$ as a subgraph. Hence $\sigma(K_{2,3}, 5) \geq \sigma(\pi_1) + 2 = 16$. Now assume that $\pi = (d_1, d_2, d_3, d_4, d_5)$ is a graphic sequence with $\sigma(\pi) \geq 16$, and G is a realization of π. The edge number of G is denoted by $e(G)$. Then $8 \leq e(G) \leq e(K_5) = 10$. In other words, any realization G of π is obtained from K_5 by deleting at most two edges. Clearly, G contains $K_{2,3}$ as a subgraph. Hence π is potentially $K_{2,3}$-graphic. Thus $\sigma(K_{2,3}, 5) \leq 16$. So $\sigma(K_{2,3}, 5) = 16$.

(2) Let G be a graph on 6 vertices obtained by adding a new vertex x to C_5 and joining it to each vertex in C_5. Then the degree sequence of G is $\pi_2 = (5, 3^5)$. Hence π_2 is graphic. It is easy to see that G is unique realization of π_2 and contains no $K_{2,3}$ as a subgraph. In other words, π_2 is not potentially $K_{2,3}$-graphic. Thus $\sigma(K_{2,3}, 6) \geq \sigma(\pi_2) + 2 = 22$. Now assume that $\pi = (d_1, \ldots, d_6)$ is a graphic sequence with $\sigma(\pi) \geq 22$. If $d_6 \leq 3$, then the residual sequence $\pi'_6 = (d'_1, \ldots, d'_5)$ obtained by laying off d_6 from π satisfies $\sigma(\pi'_6) = \sigma(\pi) - 2d_6 \geq 22 - 2 \times 3 = 16$. By (1), π'_6 is potentially $K_{2,3}$-graphic, and hence so is π by Theorem 1.1. If $d_6 \geq 4$, then $\pi = (5^{6-t}, 4^t)$, where t is even and $0 \leq t \leq 6$. It is easy to see that the graph G obtained from K_6 by deleting a matching $\frac{t}{2}K_2$ is unique realization of π, and G contains $K_{2,3}$ as a subgraph. In other words, π is potentially $K_{2,3}$-graphic. Hence $\sigma(K_{2,3}, 6) \leq 22$. Thus $\sigma(K_{2,3}, 6) = 22$.

(3) Clearly, $\pi_3 = (6, 3^6)$ is the degree sequence of G obtained by adding a new vertex x to C_6 and joining x to each vertex of C_6. Hence π_3 is graphic. Assume that H is a realization of π_3, where the degree of vertex x is 6. Then $\pi(H - x) = (2^6)$. It is easy to check that $K_{1,3}$ and $K_{2,2}$ both are not subgraphs of $H - x$. Hence π_3 is not potentially $K_{2,3}$-graphic. Thus $\sigma(K_{2,3}, 7) \geq \sigma(\pi_3) + 2 = 26$. Now assume that $\pi = (d_1, \ldots, d_7)$ is a graphic sequence with $\sigma(\pi) \geq 26$, and π'_7 is the residual sequence obtained by laying off d_7 from π. If $d_7 \leq 2$, then $\sigma(\pi'_7) = \sigma(\pi) - 2d_7 \geq 22$. By (2), π'_7 is potentially $K_{2,3}$-graphic, and hence so is π. If $d_5 \geq 4$ and $d_7 \geq 3$, then by Theorem 1.4, π is potentially $K_{2,3}$-graphic. Now we may assume that $\pi = (d_1, d_2, d_3, d_4, 3^3)$, where $d_2 \geq 4$. If $d_1 = 6$ or $d_2 \geq 5$, then π is clearly potentially $K_{2,3}$-graphic since the vertex x with degree d_1 and the vertex y with degree d_2 in any realization G of π have at least three neighbours in common. Hence we may further assume that $\pi = (5, 4^3, 3^3)$. The following Figure 1 shows that $(5, 4^3, 3^3)$ is potentially $K_{2,3}$-graphic. Thus $\sigma(K_{2,3}, 7) \leq 26$. So $\sigma(K_{2,3}, 7) = 26$.

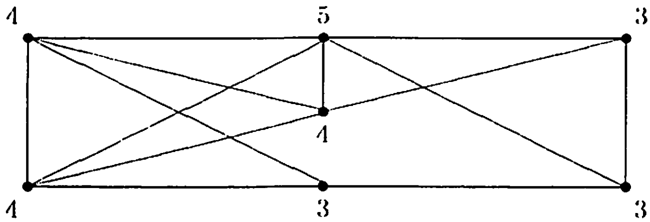

Figure 1

(4) Suppose that H is a graph obtained from the graph G in (3) by adding a new vertex y and joining y to x. Clearly, the degree sequence of H is $\pi_4 = (7, 3^6, 1)$. Hence π_4 is graphic. Clearly, the residual sequence of π_4 obtained by laying off $d_1 = 7$ is $(2^6, 0)$, and any realization of $(2^6, 0)$ contains no $K_{1,3}$ and $K_{2,2}$ as its subgraphs. So π_4 is not potentially $K_{2,3}$-graphic. Hence $\sigma(K_{2,3}, 8) \geq \sigma(\pi_4) + 2 = 28$. Now suppose that $\pi = (d_1, \ldots, d_8)$ is a graphic sequence with $\sigma(\pi) \geq 28$. If $d_8 \leq 1$, then the residual sequence π_8' obtained by laying off d_8 from π satisfies $\sigma(\pi_8') = \sigma(\pi) - 2d_8 \geq 26$. By (3), π_8' and π both are potentially $K_{2,3}$-graphic. If $d_2 \geq 4$ and $d_8 \geq 2$, then by Theorems 1.4 and 1.5, π is potentially $K_{2,3}$-graphic. So we may assume that $\pi = (7, 3^7)$. Clearly, the graph G obtained from $K_{2,2} \cup C_3$ by adding a new vertex x and joining it to each vertex of $K_{2,2} \cup C_3$ is a realization of π and contains $K_{2,3}$ as a subgraph. In other words, π is potentially $K_{2,3}$-graphic. Hence $\sigma(K_{2,3}, 8) \leq 28$. Thus $\sigma(K_{2,3}, 8) = 28$.

(5) Suppose that G is a realization of $\pi_4 = (7, 3^6, 1)$, where the degree of vertex x in G is 7. The graph obtained from G by adding a new vertex y and joining y to x is denoted by H. Clearly, the degree sequence of H is $\pi_5 = (8, 3^6, 1^2)$, in other words, π_5 is graphic. Since the residual sequence of π_5 obtained by laying off $d_1 = 8$ is $(2^6, 0^2)$, and each realization of $(2^6, 0^2)$ contains no $K_{1,3}$ and $K_{2,2}$ as its subgraphs, π_5 is not potentially $K_{2,3}$-graphic. Hence $\sigma(K_{2,3}, 9) \geq \sigma(\pi_5) + 2 = 30$. Now suppose that $\pi = (d_1, \ldots, d_9)$ is a graphic sequence with $\sigma(\pi) \geq 30$, and π_9' is the residual sequence obtained by laying off d_9 from π. If $d_9 \leq 1$, then $\sigma(\pi_9') = \sigma(\pi) - 2d_9 \geq 28$. It follows from (4) that π_9' is potentially $K_{2,3}$-graphic, and hence so is π. If $d_2 \geq 4$ and $d_9 \geq 2$, then π is potentially $K_{2,3}$-graphic by Theorems 1.4 and 1.5. If $d_2 \leq 3$ and $d_9 \geq 2$, then π is one of the following sequences:

$$(8, 3^6, 2^2), \quad (8, 3^8), \quad (7, 3^7, 2), \quad (6, 3^8).$$

Let G_1 be the graph obtained from $K_{2,2} \cup P_3$ by adding a new vertex that is adjacent to all vertices of $K_{2,2} \cup P_3$, where P_ℓ is the path of length ℓ, G_2 be the graph obtained from $K_{2,2} \cup K_{2,2}$ by adding a new vertex that is adjacent to all vertices of $K_{2,2} \cup K_{2,2}$, G_3 be the graph obtained from $K_{2,2} \cup K_{2,2}$ by adding a new vertex that is adjacent to seven vertices of $K_{2,2} \cup K_{2,2}$ and G_4 be the graph obtained from $K_{2,3} \cup C_3$ by adding a new vertex that is adjacent to all vertices of $K_{2,3} \cup C_3$ with degree 2. Clearly, $\pi(G_1) = (8, 3^6, 2^2)$, $\pi(G_2) = (8, 3^8)$, $\pi(G_3) = (7, 3^7, 2)$, $\pi(G_4) = (6, 3^8)$, and G_i contains $K_{2,3}$ as a subgraph for $1 \le i \le 4$. Hence π is potentially $K_{2,3}$-graphic. Thus $\sigma(K_{2,3}, 9) \le 30$. So $\sigma(K_{2,3}, 9) = 30$. □

Now we further determine the values $\sigma(K_{2,3}, n)$ for $n \ge 10$.

Theorem 2.3. If $n \ge 10$, then $\sigma(K_{2,3}, n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd,} \\ 3n+2 & \text{if } n \text{ is even.} \end{cases}$

In order to prove Theorem 2.3, we need the following Lemmas.

Lemma 2.4. If $n \ge 10$, then $\sigma(K_{2,3}, n) \ge \begin{cases} 3n+1 & \text{if } n \text{ is odd,} \\ 3n+2 & \text{if } n \text{ is even.} \end{cases}$

Proof. Case 1. n is odd. The graph obtained from $C_3 \cup \frac{n-5}{2} K_2 \cup K_1$ by adding a new vertex x and joining x to each vertex of $C_3 \cup \frac{n-5}{2} K_2 \cup K_1$ is denoted by G. Clearly, the degree sequence of G is $\pi = (n-1, 3^3, 2^{n-5}, 1)$. In other words, π is graphic. Since the residual sequence of π obtained by laying off $d_1 = n-1$ is $(2^3, 1^{n-5}, 0)$, and any realization of $(2^3, 1^{n-5}, 0)$ contains no $K_{1,3}$ and $K_{2,2}$ as its subgraphs, π is not potentially $K_{2,3}$-graphic. Hence $\sigma(K_{2,3}, n) \ge \sigma(\pi) + 2 = 3n+1$.

Case 2. n is even. Clearly, the degree sequence of the graph H obtained from $C_3 \cup \frac{n-4}{2} K_2$ by adding a new vertex y that is adjacent to all vertices of $C_3 \cup \frac{n-4}{2} K_2$ is $\pi = (n-1, 3^3, 2^{n-4})$. Hence π is graphic. Clearly, the residual sequence of π obtained by laying off $d_1 = n-1$ is $(2^3, 1^{n-4})$, and each realization of $(2^3, 1^{n-4})$ contains no $K_{1,3}$ and $K_{2,2}$ as its subgraphs. Hence π is not potentially $K_{2,3}$-graphic. This shows that $\sigma(K_{2,3}, n) \ge \sigma(\pi) + 2 = 3n+2$. □

Lemma 2.5. Let $n \ge 10$ and $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $d_n \ge 2$. If

$$\sigma(\pi) \ge \begin{cases} 3n+1 & \text{if } n \text{ is odd,} \\ 3n+2 & \text{if } n \text{ is even,} \end{cases}$$

then π is potentially $K_{2,3}$-graphic.

Proof. If $d_2 \ge 4$, then by Theorems 1.4 and 1.5, π is potentially $K_{2,3}$-graphic. Assume that $d_2 \le 3$. Then $\pi = (d_1, 3^t, 2^{n-t-1})$, where $d_1 \ge 4$ and $t \ge 4$. If $d_1 = 4$, then n is odd and $\pi = (4, 3^{n-1})$. Let H_1 be a 3-regular graph on $n-5$ vertices, and H_2 be the graph obtained from P_2 by adding two new vertices x and y and joining x and y to each vertex of P_2. Clearly, $H_1 \cup H_2$ is a realization of $\pi = (4, 3^{n-1})$ and contains $K_{2,3}$ as a subgraph. In other words, $(4, 3^{n-1})$ is potentially $K_{2,3}$-graphic. Now suppose that $d_1 \ge 5$, $\rho = (d_1 - 2, 3^{t-4}, 2^{n-t-1}, 1^2)$ and $\rho_1' = (d_1', \ldots, d_{n-3}')$ is the residual sequence obtained by laying off $d_1 - 2$ from ρ. It is easy to see that $\sigma(\rho)$ and $\sigma(\rho_1')$ both are even, $d_1' \le 3$ and $0 \le d_{n-3}' \le 1$. If $d_{n-3}' = 0$, then $d_1' \le 2$ and $d_{n-5}' \ge 1$. Clearly, ρ_1' is graphic, and hence so is ρ. If $d_{n-3}' = 1$, then by $\left\lceil \frac{(3+1+1)^2}{4} \right\rceil = 6 \le n-3$ and Theorem 1.2, ρ_1' is graphic, and hence so is ρ. Let H be a realization of ρ, and G be the graph obtained from H

by adding two new vertices x and y that both are adjacent to the vertex of H with degree $d_1 - 2$ and also to the two vertices of H with degree 1. Clearly, G is a realization of π and contains $K_{2,3}$ as a subgraph. Hence π is potentially $K_{2,3}$-graphic. □

Lemma 2.6. $\sigma(K_{2,3}, 10) = 32$ and $\sigma(K_{2,3}, 11) = 34$.

Proof. Assume that $\pi = (d_1, d_2, \ldots, d_{10})$ is a graphic sequence, where $\sigma(\pi) \geq 32$. If $d_{10} \geq 2$, then by Lemma 2.5, π is potentially $K_{2,3}$-graphic. If $d_{10} \leq 1$, then $\sigma(\pi'_{10}) = \sigma(\pi) - 2 \geq 30$, where π'_{10} is the residual sequence of π obtained by laying off d_{10}. By Theorem 2.2(5), π'_{10} is potentially $K_{2,3}$-graphic, and hence so is π. Thus $\sigma(K_{2,3}, 10) \leq 32$. By Lemma 2.4, $\sigma(K_{2,3}, 10) = 32$. Similarly, we can prove that $\sigma(K_{2,3}, 11) = 34$. □

Proof of Theorem 2.3. By Lemma 2.4, it is enough to prove that (*): if $\pi = (d_1, d_2, \ldots, d_n)$ is a graphic sequence with

$$\sigma(\pi) \geq \begin{cases} 3n+1 & \text{if } n \text{ is odd,} \\ 3n+2 & \text{if } n \text{ is even,} \end{cases}$$

then π is potentially $K_{2,3}$-graphic. We use induction on n. It is known from Lemma 2.6 that (*) holds for $n = 10$ and 11. Now assume that (*) holds for $n - 1 \geq 11$. We will prove that (*) holds for n. If $d_n \leq 1$, then the residual sequence π'_n obtained by laying off d_n from π satisfies

$$\sigma(\pi'_n) = \sigma(\pi) - 2d_n \geq \begin{cases} 3(n-1)+2 & \text{if } n \text{ is odd,} \\ 3(n-1)+3 & \text{if } n \text{ is even.} \end{cases}$$

By the induction hypothesis, π'_n is potentially $K_{2,3}$-graphic, and hence so is π. If $d_n \geq 2$, then π is also potentially $K_{2,3}$-graphic by Lemma 2.5. □

3. The values of $\sigma(K_{2,s}, n)$ for $s \geq 4$ and $n \geq 2\left[\frac{(s+3)^2}{4}\right] + 5$

For convenience, we denote $m = \left[\frac{(s+3)^2}{4}\right]$. The main result in this section is the following

Theorem 3.1. If $s \geq 4$ and $n \geq 2m + 5$, then

$$\sigma(K_{2,s}, n) = \begin{cases} s(n-1)+2 & \text{if } s \text{ is even or } n \text{ is odd,} \\ s(n-1)+3 & \text{if } s \text{ is odd and } n \text{ is even.} \end{cases}$$

In order to prove our main result, we need the following Lemmas.

Lemma 3.2. If $s \geq 4$ and $n \geq 2s + 1$, then

$$\sigma(K_{2,s}, n) \geq \begin{cases} s(n-1)+2 & \text{if } s \text{ is even or } n \text{ is odd,} \\ s(n-1)+3 & \text{if } s \text{ is odd and } n \text{ is even.} \end{cases}$$

Proof. We consider two cases as follows.

Case 1. s is even or n is odd. Assume that H is a $(s-2)$-regular graph on $n-s-2$ vertices and G is a $(s-3)$-regular graph on s vertices. Clearly, H and G exist, and $\pi(H \cup G) = ((s-2)^{n-s-2}, (s-3)^s)$. The graph obtained by adding a new vertex u to $H \cup G$ and adding edges from u to $s-1$ vertices of G is denoted by F. Then $\pi(F) = (s-1, (s-2)^{n-3}, s-3)$. The graph obtained by adding a new vertex v to F and joining v to each vertex of F is denoted by Q. Then $\pi(Q) = (n-1, s, (s-1)^{n-3}, s-2)$. Hence $\pi = (n-1, s, (s-1)^{n-3}, s-2)$ is graphic. The residual sequence obtained by laying off $d_1 = n-1$ from π is $\pi_1' = (d_1', d_2', \ldots, d_{n-1}') = (s-1, (s-2)^{n-3}, s-3)$. If π is potentially $K_{2,s}$-graphic, then π_1' is potentially $K_{1,s}$ or $K_{2,s-1}$-graphic. Hence $d_1' \geq s$ or $d_2' \geq s-1$, a contradiction. So π is not potentially $K_{2,s}$-graphic. Thus $\sigma(K_{2,s}, n) \geq \sigma(\pi) + 2 = s(n-1) + 2$.

Case 2. s is odd and n is even. Assume that H is a $(s-2)$-regular graph on $n-s-1$ vertices and G is a $(s-3)$-regular graph on $s-1$ vertices. Clearly, H and G exist, and $\pi(H \cup G) = ((s-2)^{n-s-1}, (s-3)^{s-1})$. By adding a new vertex u to $H \cup G$ and adding u to each vertex of G, the resulting graph is denoted by F. Again, by adding a new vertex v to F and joining v to each vertex of F, the resulting graph is denoted by Q. Then $\pi(Q) = (n-1, s, (s-1)^{n-2})$. Hence $\pi = (n-1, s, (s-1)^{n-2})$ is graphic. Since the residual sequence of π obtained by laying off $d_1 = n-1$ is $\pi_1' = (s-1, (s-2)^{n-2})$, and any realization of $(s-1, (s-2)^{n-2})$ contains no $K_{1,s}$ and $K_{2,s-1}$ as its subgraphs, π is not potentially $K_{2,s}$-graphic. Hence $\sigma(K_{2,s}, n) \geq \sigma(\pi) + 2 = s(n-1) + 3$. □

Lemma 3.3. If $s \geq 4$ and $n = m$, then

$$\sigma(K_{2,s}, n) \leq s(n-1) + 3 + (s-2)(m+5).$$

Proof. Clearly, $n = m = \left[\frac{(s+3)^2}{4}\right] \leq \binom{s+1}{2} + 3$. Hence by Theorem 1.3,

$$\begin{aligned}\sigma(K_{2,s}, n) &\leq \sigma(K_{s+2}, n) \leq 2n(s-1) + 8 \\ &= sn - s + 3 + (s-2)m + s + 5 \\ &\leq sn - s + 3 + (s-2)m + 5(s-2) \\ &= s(n-1) + 3 + (s-2)(m+5).\end{aligned}$$

Lemma 3.4. Let $s \geq 4$, $n \geq m$ and $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $d_n \geq 2$. If $\sigma(\pi) \geq s(n-1) + 3$, then π is potentially $K_{2,s}$-graphic.

Proof. If $d_{s+2} \geq s+1$, then π is potentially $K_{2,s}$-graphic by Theorem 1.4. If $d_{s+2} \leq s$ and $d_2 \geq s+1$, then by $n \geq m = \left[\frac{(s+3)^2}{4}\right] \geq 4(s-1)$ and Theorem 1.5, π is also potentially $K_{2,s}$-graphic. Now assume that $d_2 \leq s$. Then $\sigma(\pi) \geq s(n-1)+3$ implies that $\pi = (d_1, s^3, d_5, \ldots, d_{s+1}, d_{s+2}, d_{s+3}, \ldots, d_n$ where $d_{s+2} \geq 3$. Denote $\rho = (p_1, p_2, \ldots, p_{n-2})$, where $p_1 \geq p_2 \geq \cdots \geq p_{n-2}$

is the rearrangement of $d_1-2, s-2, d_5-2, \dots, d_{s+1}-2, d_{s+2}-2, d_{s+3}, \dots, d_n$, and let $\rho_1' = (p_1', p_2', \dots, p_{n-3}')$ be the residual sequence obtained by laying off p_1 from ρ. Clearly, $p_{n-2} = \min\{d_{s+2}-2, d_n\} \geq 1$.

Suppose that ρ is graphic and is realized by a graph H with vertex set $V(H) = \{v_1, v_2, \dots, v_{n-2}\}$ such that $d(v_1) = d_1 - 2$, $d(v_2) = s - 2$, $d(v_i) = d_{i+2} - 2$ for $3 \leq i \leq s$ and $d(v_i) = d_{i+2}$ for $s+1 \leq i \leq n-2$, where $d(v)$ denotes the degree of vertex v in H. Now form G from H by adding two new vertices x and y that both are adjacent to all of $v_1, v_2, \dots, v_s$. Clearly, G is a realization of π and contains $K_{2,s}$ as a subgraph, i.e., π is potentially $K_{2,s}$-graphic.

This proves the result provided that ρ is graphic. Clearly, $\sigma(\rho)$ and $\sigma(\rho_1')$ are even. If $d_1 \leq s+2$, then $p_1 = \max\{d_1 - 2, d_{s+3}\} \leq s$. By $\left[\frac{(s+1+1)^2}{4}\right] \leq \left[\frac{(s+3)^2}{4}\right] - 3 < n-2$ and Theorem 1.2, ρ is graphic. Assume $d_1 \geq s+3$. Then $p_1 = \max\{d_1 - 2, d_{s+3}\} = d_1 - 2$ and $p_1' \leq s$. If $d_{s+2} \geq 4$, then $p_{n-2} = \min\{d_{s+2} - 2, d_n\} \geq 2$. Clearly, $p_{n-3}' \geq 1$. Since $\left[\frac{(s+1+1)^2}{4}\right] \leq \left[\frac{(s+3)^2}{4}\right] - 3 \leq n-3$, ρ_1' is graphic by Theorem 1.2. Hence ρ is also graphic. If $d_{s+2} = 3$ and $s \geq 5$, then $\sigma(\pi) = d_1 + 3s + d_5 + \cdots + d_{s+1} + d_{s+2} + \cdots + d_n \leq n - 1 + s^2 + 3(n-s-1) = 4n + s^2 - 3s - 4 < sn - s + 3 \leq \sigma(\pi)$, a contradiction. If $d_{s+2} = 3$ and $s = 4$, then $n \geq 12$ and $\pi = (n-1, 4^4, 3^{n-5})$. Thus $\rho = (n-3, 3^{n-6}, 2^2, 1)$. Clearly, $\rho = (n-3, 3^{n-6}, 2^2, 1)$ is the degree sequence of the graph H' obtained from $P_{n-5} \cup K_1$ by adding a new vertex x that is adjacent to each vertex of $P_{n-5} \cup K_1$. Hence ρ is graphic. □

Lemma 3.5. Let $s \geq 4$ and $n = m + t$, where $0 \leq t \leq m+5$. Then

$$\sigma(K_{2,s}, n) \leq s(n-1) + 3 + (s-2)(m+5) - (s-2)t.$$

Proof. We use induction on t. It is known from Lemma 3.3 that the result holds for $t = 0$. Assume that the result holds for $t-1, 0 \leq t-1 \leq m+4$. Let $n = m+t$ and $\pi = (d_1, d_2, \dots, d_n)$ be a graphic sequence with $\sigma(\pi) \geq s(n-1) + 3 + (s-2)(m+5) - (s-2)t$. Clearly, $\sigma(\pi) \geq s(n-1) + 3$. If $d_n \geq 2$, then by Lemma 3.4, π is potentially $K_{2,s}$-graphic. If $d_n \leq 1$, then the residual sequence π_n' obtained by laying off d_n from π satisfies $\sigma(\pi_n') = \sigma(\pi) - 2d_n \geq s(n-1) + 3 + (s-2)(m+5) - (s-2)t - 2 = s(n-2) + 3 + (s-2)(m+5) - (s-2)(t-1)$. By the induction hypothesis, π_n' is potentially $K_{2,s}$-graphic, and hence so is π. Thus $\sigma(K_{2,s}, n) \leq s(n-1) + 3 + (s-2)(m+5) - (s-2)t$.□

Lemma 3.6. If $s \geq 4$ and $n \geq 2m+5$, then $\sigma(K_{2,s}, n) \leq s(n-1) + 3$.

Proof. Let $\pi = (d_1, d_2, \dots, d_n)$ be a graphic sequence with $\sigma(\pi) \geq s(n-1) + 3$. It is enough to prove that π is potentially $K_{2,s}$-graphic. If $n = 2m+5$, then by Lemma 3.5, π is potentially $K_{2,s}$-graphic. Now we use induction on $n (\geq 2m+6)$. If $d_n \geq 2$, then π is potentially $K_{2,s}$-graphic by Lemma 3.4. If $d_n \leq 1$, then the residual sequence π_n' obtained by laying off

d_n from π satisfies $\sigma(\pi'_n) = \sigma(\pi) - 2d_n \geq s(n-2) + 3$. By the induction hypothesis, π'_n is potentially $K_{2,s}$-graphic, and hence so is π. □

Proof of Theorem 3.1. It follows from $2m+5 \geq 2s+1$ and Lemmas 3.2 and 3.6 that $\sigma(K_{2,s}, n) = s(n-1)+3$ if s is odd and n is even, and $s(n-1)+2 \leq \sigma(K_{2,s}, n) \leq s(n-1)+3$ if s is even or n is odd. Since $\sigma(K_{2,s}, n)$ is even, we have $\sigma(K_{2,s}, n) = s(n-1)+2$ if s is even or n is odd.
□

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