Dominating sets with at most k components

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1 Abstract

A connected dominating set D of a graph G has the property that not only does D dominate the graph but the subgraph induced by the vertices of D is also connected. We generalize this concept by allowing the subgraph induced by D to contain at most k components and examine the minimum possible order of such a set. In the case of trees we provide lower and upper bounds and a characterization for those trees which achieve the former.

2 Introduction

Given a graph G, a *connected dominating set* [5] is a set D of vertices such that D is a dominating set, i.e., every vertex in $G - D$ has at least one neighbour in D and, furthermore, the subgraph induced by D is connected. Although, to facilitate communication, it may be desirable that all of the vertices of the dominating set form only one component, there could well be situations in which it might be desirable to relax this condition somewhat; variations are given in [1, 2]; here we focus on the condition that the number of components induced by the dominating set should not be too large.

Given a graph G with at most k components, we define an *at most k-component dominating set* to be a subset D of the vertices G with the property that every vertex of $G - D$ is adjacent to at least one vertex of D and, furthermore, the subgraph induced by the set D has at most k components. We call an at most k-component dominating set of minimum possible order a γ_c^k -set for the graph G and denote its cardinality by $\gamma_c^k(G)$ or simply γ_c^k if the graph is clear. Observe that for $k = 1$, $\gamma_c^1(G) = \gamma_c(G)$, the connected domination number.

In what follows we shall use the term *leaf* to refer to a vertex of degree one and *stem* to refer to a vertex that is adjacent to at least one leaf. We let $L(G) = L$ and $S(G) = S$ denote the leaves and stems in G, respectively. As usual $\gamma(G)$ will denote the order of a minimum dominating set of the graph G. A *strong matching* is a matching $M = \{e_1, e_2, \ldots, e_\ell\}$ where no end of e_i is adjacent to an end of $e_j, 1 \leq i \neq j \leq \ell$.

3 Some Bounds and Observations

We begin with two very easy bounds.

Lemma 1 *For any positive integer k and any graph G with at most k components, $\gamma(G) \leq \gamma_c^k(G)$.*

Proof: This follows immediately from the fact that any γ_c^k-set of a given graph G must be a dominating set (although not necessarily minimum). □

Lemma 2 *For any positive integer k and any connected graph G, $\gamma_c^k(G) \leq \gamma_c(G)$.*

Proof: Any γ_c-set dominates and has precisely one and thus at most k components. □

Although the following observation is immediate it will be useful and so we refer to it in the form of a lemma.

Lemma 3 *For any positive integer k and any connected graph G, a vertex whose deletion results in at least $k+1$ components must belong to every $\gamma_c^k(G)$-set.*

Proof: Any γ_c^k-set has at most k components and so the result follows directly. □

Corollary 3.1 *Let $k \geq 1$. All vertices of degree at least $k+1$ must be included in any γ_c^k-set of a tree.*

It is also straightforward to note the following bound.

Lemma 4 *For any positive integer k and a connected graph G we have $\gamma_c(G) - 2(k-1) \leq \gamma_c^k(G)$.*

Proof: Let D be a $\gamma_c^k(G)$-set. Since G is connected and each vertex in $G-D$ has a neighbour in D, it follows that a component of D contains at least one vertex that is within distance three from some other component. Thus one can reduce the number of components by adding at most two more vertices (along the path between these two components). Hence one can find, by continuing in this fashion $k-1$ times a superset $\overline{D}$ of D such that $\overline{D}$ is connected, dominates and has at most $2(k-1)$ additional vertices. □

Restricting our attention to trees T, we characterize those which achieve the lower bound of $\gamma_c - 2(k-1) = \gamma_c^k$. We remind the reader that in the case of a tree T, there is a unique connected dominating set $V(T) - L(T)$ consisting of every vertex that is not a leaf.

Theorem 1 *Let k be a positive integer. Consider a tree T and let I_2 be the subset of vertices of T that have degree two and have no leaf as a neighbour. The tree T has a γ_c^k-set of cardinality $\gamma_c(T) - 2(k-1)$ if and only if there exists a strong matching of $k-1$ edges in the subgraph induced by I_2.*

Proof: First assume there exists a strong matching, say M, of $k-1$ edges in the subgraph induced by I_2. Then $(V(T)-L(T))-V(M)$ dominates T, has k components and $\gamma_c - 2(k-1)$ vertices and hence is a γ_c^k-set. Conversely, consider the situation where T has a γ_c^k-set D of cardinality $\gamma_c(T)-2(k-1)$. We may assume, without loss of generality, that D contains all of the stems but none of the leaves, since a leaf in D could be replaced by its adjacent stem. Hence the $\gamma_c(T)-2(k-1)$ vertices in D are all chosen from the vertices that are not leaves. Let U be the $2(k-1)$ vertices deleted from the γ_c-set in order to form D. Since the vertices of U are neither stems nor leaves, $U \subseteqq V(T)-(L \cup S)$. Consider the subgraph H induced in T by the set U. Let m_1 be the number of components of order 1 in H and $m_{\geq 2}$ the number of components of order ≥ 2 in H. Since no vertex of H was a leaf or stem in T, there must be at least two edges from each singleton in H to D. Also observe that each vertex of any component of order two or more of H must have at least one edge to D since D dominates G. Hence there are at least $2m_1 + (2(k-1)-m_1)$ edges between H and D. Thus, since T is a tree, the number of H-components plus the number of D-components total at least $(2m_1 + 2(k-1) - m_1) + 1$. That is, there are at least $(m_1 + 2k - 1) - (m_1 + m_{\geq 2})$ components in D. As D has at most k components that implies $k - 1 \leq m_{\geq 2}$. Since each component that is counted in $m_{\geq 2}$ contains at least two vertices and there are only a total of $2(k-1)$ vertices in U, it follows that the only possibility is that U consists of exactly $k-1$ pairs of vertices which induce $k-1$ isolated edges. In addition the vertices of U must be of degree two, else their deletion would create more than k components. □

In certain instances we can improve the lower bound given by Lemma 4.

Theorem 2 *Let $k \geq 2$ be an integer and T a tree such that all vertices which are neither leaves nor stems have degree at least d where $d \geq 3$. Then $\gamma_c^k(T) \geq \gamma_c(T) - \frac{k-2}{d-2}$.*

Proof: Let D be a γ_c^k-set of T. As before we may assume that all stems of T belong to D but no leaves do. Let $U = V(T) - L(T) - D$ be the set of vertices, other than leaves, that do not belong to D. By the hypothesis, the minimum degree in T of the vertices of U is at least $d \geq 3$. Consider the subgraph H induced by the set U. Say there are r components in H and m_i vertices in component $i, 1 \leq i \leq r$. Since each component is a tree and each vertex is of degree at least d in the original tree T, there must be at least $m_i d - 2(m_i - 1)$ edges from the ith component to the vertices of D. Hence the deletion of U must create at least

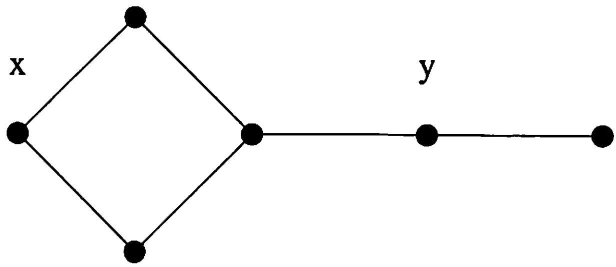

Figure 2: A graph where the γ_c^2-set is not contained in any γ_c-set.

In light of these remarks we note that the characterization of Theorem 1 gives us the means to easily determine γ_c^2 for a tree. Recall that a γ_c-set of a tree is simply all vertices other than the leaves. From Lemma 2 we have that $\gamma_c^2(T) \geq \gamma_c - 2(2-1) = \gamma_c - 2$. So the only possibilities are $\gamma_c, \gamma_c - 1$ and $\gamma_c - 2$. Hence first determine the degrees of all vertices of T. Let I be the set of vertices of T that are neither leaves nor stems. Let I_2 be those vertices in I that are of degree two. By Theorem 1, $\gamma_c^2(T) = \gamma_c(T) - 2$ if and only if there exists an edge in the subgraph induced by I_2. If $\gamma_c^2(T) \neq \gamma_c(T) - 2$ it follows (Corollary 3.1) that $\gamma_c^2(T) = \gamma_c(T) - 1$ only if I_2 is not empty.

We also observe that, with more work, one could check for γ_c^3 by examining the subgraph induced by those vertices, other than the stems, that are of degree 2 and 3.

5 An improved upper bound for trees

For trees one can improve the upper bound of Lemma 2 as follows.

Theorem 3 *For any positive integer k and a tree T with n vertices where $n \geq 2k+1$ we have $\gamma_c^k(T) \leq n - k - 1$.*

Proof: For $k = 1$, $\gamma_c^1(T) = \gamma_c(T)$ and since the γ_c-set for a tree consists of all vertices that are not leaves and since any tree has at least two leaves, we have $\gamma_c^1(T) \leq n-2$. Assume the result holds for all values less than or equal to $k-1$ and consider k. Let T be an arbitrary tree and let $v_1, v_2, \ldots, v_\alpha$ be a longest path in T. If the degree of v_2 is two let $T' = T - \{v_1, v_2\}$. Then by the induction hypothesis $\gamma_c^{k-1}(T') \leq (n-2) - (k-1) - 1 = n - k - 2$. But using the vertex v_2 as well in T we get $\gamma_c^k(T) \leq (n-k-2) + 1 = n - k - 1$.

$(m_1 d - 2(m_1 - 1)) + (m_2 d - 2(m_2 - 1) - 1) + (m_3 d - 2(m_3 - 1) - 1) + \ldots =$
$(\Sigma_{i=1}^r m_i)d - (2\Sigma_{i=1}^r m_i) + 2r - (r - 1) = (d - 2)\Sigma_{i=1}^r m_i + r + 1$
components in D. As $\Sigma_{i=1}^r m_i = |U|$ and $r \geq 1$ we obtain
$(d - 2)|U| + 2 \leq$ number of components in $D \leq k$
giving $|U| \leq \frac{k-2}{d-2}$ and thus
$|D| = \gamma_c^k(T) = |V(T)| - |L(T)| - |U| = \gamma_c(T) - |U| \geq \gamma_c(T) - \frac{k-2}{d-2}$. □

If $\frac{k-2}{d-2}$ is an integer, it is straightforward to verify that the trees T satisfying $\gamma_c^k(T) = \gamma_c(T) - \frac{k-2}{d-2}$ are precisely those with the following two properties: (i) the vertices, that are not leaves nor stems, are of degree at least d and (ii) there is at least one subtree with $\frac{k-2}{d-2}$ vertices, each of degree exactly d in T and each having at least one neighbour not in the subtree.

4 Some Algorithmic Considerations

Given the bound of Lemma 1 and the question of determining γ_c^k for a tree, one might be tempted to modify the linear algorithm for finding a γ-set on a tree as it would seem reasonable to suppose that one might be able to enlarge a γ-set to form a γ_c^k-set. That some care must be taken is shown by the example in Fig. 1 as the given tree has a γ_c^2-set that does not include any γ-set.

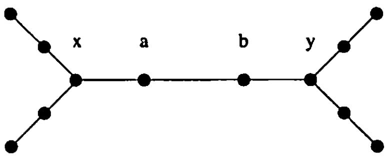

Figure 1: A tree in which the γ_c^2-set does not contain a γ-set.

Note that any γ-set of the tree in Fig. 1 must include either a or b as well as four more vertices (without loss of generality, the four stems) but any γ_c^2-set must include both x and y as well as the four stems.

On the other hand, in light of Lemma 2 one might try to delete vertices from a γ_c-set for a graph in order to determine a γ_c^k-set.While this is fine for a tree it is not necessarily possible even with just one cycle in the graph (Fig. 2). The only γ_c^2-set consists of the vertices x and y while no γ_c-set contains x.

If the degree of v_2 is three or more , v_2 is still a stem in $T' = T - \{v_1\}$ and hence may be assumed to be included in a γ_c^{k-1}-set of T'. Since T' has $n-1 \geq 2k \geq 2(k-1)+1$ vertices, we have by the induction hypothesis that $\gamma_c^{k-1}(T') \leq (n-1)-(k-1)-1 = n-k-1$. But the γ_c^{k-1}-set of T' is also a γ_c^{k-1}-set of T and, thus, a γ_c^{k}-set of T as required. Hence the result follows. □

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References

[1] Yair Caro, Douglas B. West, Raphael Yuster: "Connected domination and spanning trees with many leaves." SIAM J. Discrete Math. 13 (2000) 202-211.

[2] Yair Caro, Raphael Yuster: "Dominating a family of graphs with small connected subgraphs." Combin. Probab. Comput. 9 (2000) 309-313.

[3] B.L. Hartnell, P.D. Vestergaard: "Partitions and domination in graphs", pp 1-10. To appear in J. of Combin. Math. and Combin. Comput. Vol. 46, August 2003.

[4] Teresa W. Haynes, Stephen T. Hedetniemi, Peter J. Slater: "Fundamentals of domination in graphs". Marcel Dekker, New York 1998.

[5] E. Sampathkumar: "Domination parameters of a graph" in Teresa W. Haynes, Stephen T. Hedetniemi, Peter J. Slater, editors :"Domination in graphs, advanced topics". Marcel Dekker, New York 1998, pp. 271-299.

[6] S. Seager: "Partition domination of graphs of minimum degree two." Congr. Numer. 132 (1998), 85-91.

[7] Zsolt Tuza, P.D. Vestergaard: "Domination in partitioned graphs". Discussiones Mathematicae Graph Theory. 22 (2002), 199-210.