The Graphs $C_5^{(t)}$ are Graceful for $t \equiv 0, 3 \pmod 4$ *

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Abstract

Given $t(\geq 2)$ cycles C_n of length $n \geq 3$, each with a fixed vertex v_0^i, $i = 1, 2, ..., t$, let $C_n^{(t)}$ denote the graph obtained from the union of the t cycles by identifying the t fixed vertices($v_0^1 = v_0^2 = ... = v_0^t$). Koh et al. conjectured that $C_n^{(t)}$ is graceful if and only if $nt \equiv 0, 3 \pmod 4$. The conjecture has been shown true for $n = 3, 6, 4k$. In this paper, the conjecture is shown to be true for $n = 5$.
Keywords: *graceful graph, vertex labelling, edge labelling*

Let $G = (V, E)$ be a simple graph with $|V|$ vertices and $|E|$ edges. Let $f : V \to \{0, 1, \ldots, |E|\}$ be an injective mapping. Define an induced function $g : E \to \{1, 2, \ldots, |E|\}$ by setting $g(p, q) = |f(p) - f(q)|$ for all $(p, q) \in E$. If g maps E onto $\{1, 2, \ldots, |E|\}$, then f is said to be a graceful labelling of G. A graph is graceful if it has a graceful labelling.

A necessary condition for an Eulerian graph with m edges to be graceful is that $m \equiv 0$ or $3 \pmod 4$ [1]. Hence a necessary condition for $C_n^{(t)}$ to be graceful is that $nt \equiv 0$ or $3 \pmod 4$. Koh et al. conjectured that $C_n^{(t)}$ is graceful if and only if $nt \equiv 0, 3 \pmod 4$ [5]. The conjecture has been shown true for $n = 3$ [2, 3], $n = 4$ [6] and $n = 6, 4k$ [5]. For the literature

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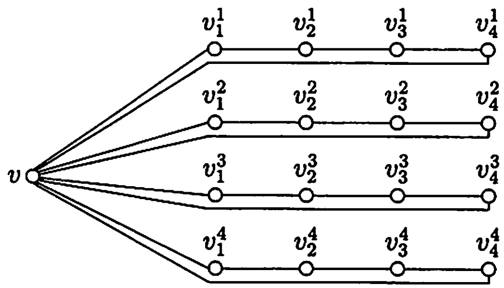

Figure 1: The graph $C_5^{(4)}$.

on graceful graphs, we refer to [4] and the relevant references given in it. In this paper, the conjecture is shown to be true for $n = 5$.

Let $v_0^i = v$ for all i. The graph of $C_5^{(4)}$ is shown in Figure 1.

Theorem 1 $C_5^{(t)}$ *is graceful if* $t \equiv 0, 3$ *(mod 4).*

Proof

Case 1. $t \equiv 0 \pmod 4$, say $t = 4k$.
We define a vertex labelling f as follows.

$$
\begin{aligned}
f(v) &= 0, \\
f(v_1^i) &= 5t + 1 - i, \qquad 1 \le i \le t, \\
f(v_2^i) &= \begin{cases} 2t + i, & 1 \le i \le t/2, \\ i, & t/2 + 1 \le i \le t, \end{cases} \\
f(v_3^i) &= 7t/2 + 2 - i, \qquad 1 \le i \le t, \\
f(v_4^i) &= \begin{cases} i, & 1 \le i \le t/2, \\ 3t + 1 + i, & t/2 + 1 \le i \le t \text{ and } i \; mod \; 2 = 1, \\ t - 1 + i, & t/2 + 1 \le i \le t \text{ and } i \; mod \; 2 = 0. \end{cases}
\end{aligned}
$$

Now we prove that f is a graceful labelling of $C_5^{(4k)}$.
Let

$$S_j = \{f(v_j^i) \mid 1 \le i \le t\}, \quad 0 \le j \le 4.$$

Then

$$\begin{aligned}
S_0 &= \{0\},\\
S_1 &= \{4t+1, 4t+2, \ldots, 5t\},\\
S_2 &= S_{21} \cup S_{22}\\
&= \{t/2+1, t/2+2, \ldots, t\} \cup \{2t+1, 2t+2, \ldots, 5t/2\},\\
S_3 &= \{5t/2+2, 5t/2+3, \ldots, 7t/2+1\},\\
S_4 &= S_{41} \cup S_{42} \cup S_{43}\\
&= \{1, 2, \ldots, t/2\} \cup \{3t/2+1, 3t/2+3, \ldots, 2t-1\}\\
&\quad \cup\{7t/2+2, 7t/2+4, \ldots, 4t\}.
\end{aligned}$$

Hence

$$\begin{aligned}
S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4 &= S_0 \cup S_1 \cup S_{21} \cup S_{22} \cup S_3 \cup S_{41} \cup S_{42} \cup S_{43}\\
&= (S_0 \cup S_{41} \cup S_{21}) \cup S_{42} \cup S_{22} \cup S_3 \cup S_{43} \cup S_1\\
&= \{0, 1, \ldots, t, \quad 3t/2+1, 3t/2+3, \ldots, 2t-1,\\
&\quad 2t+1, 2t+2, \ldots, 5t/2,\\
&\quad 5t/2+2, 5t/2+3, \ldots, 7t/2+1,\\
&\quad 7t/2+2, 7t/2+4, \ldots, 4t,\\
&\quad 4t+1, 4t+2, \ldots, 5t\}.
\end{aligned}$$

It is obvious that the labels of the vertices are different, and $Max\{f(v_j^i) | 1 \le i \le t\} = 5t = |E|$.

Let

$$D_j = \{g(v_j^i, v^i_{(j+1) \bmod 5}) |\ 1 \le i \le t\}, \quad 0 \le j \le 4; and$$

$$g(v_j^i, v^i_{(j+1) \bmod 5}) = |f(v^i_{(j+1) \bmod 5}) - f(v_j^i)|, \quad 1 \le i \le t, \quad 0 \le j \le 4.$$

Then

$$\begin{aligned}
D_0 &= \{4t+1, 4t+2, \ldots, 5t\},\\
D_1 &= \{2t+1, 2t+3, \ldots, 4t-1\},\\
D_2 &= \{t/2+2, t/2+4, \ldots, 5t/2\},\\
D_3 &= D_{31} \cup D_{32}\\
&= \{t/2+1, t/2+3, \ldots, 3t/2-1\} \cup \{5t/2+2, 5t/2+4, \ldots, 7t/2\},\\
D_4 &= D_{41} \cup D_{42} \cup D_{43}\\
&= \{1, 2, \ldots, t/2\}\\
&\quad \cup\{3t/2+1, 3t/2+3, \ldots, 2t-1\}\\
&\quad \cup\{7t/2+2, 7t/2+4, \ldots, 4t\}, and
\end{aligned}$$

$$\begin{aligned}
D_0 \cup D_1 \cup \ldots \cup D_4 &= D_0 \cup D_1 \cup D_2 \cup D_{31} \cup D_{32} \ D_{41} \cup D_{42} \cup D_{43}\\
&= D_{41} \cup (D_{31} \cup D_{42} \cup D_1) \cup (D_2 \cup D_{32} \cup D_{43}) \cup D_0\\
&= \{1, 2, \ldots, t/2\}\\
&\quad \cup\{t/2+1, t/2+3, \ldots, 4t-1\}\\
&\quad \cup\{t/2+2, t/2+4, \ldots, 4t\}\\
&\quad \cup\{4t+1, 4t+2, \ldots, 5t\}\\
&= \{1, 2, \ldots, 5t\}.
\end{aligned}$$

It is obvious that the labels of the edges are different. We thus conclude that $C_5^{(4k)}$ is graceful.

Case 2. $t \equiv 3 \pmod 4$, say $t = 4k - 1$.
We define a vertex labelling f as follows.

$$
\begin{aligned}
f(v) &= 0, \\
f(v_1^i) &= 4t + i, && 1 \le i \le t, \\
f(v_2^i) &= \begin{cases} 3t + 1 - i, & 1 \le i \le (t+1)/2, \\ (5t-1)/2 + 2 - i, & (t+3)/2 \le i \le t, \end{cases} \\
f(v_3^i) &= (t-1)/2 + i, && 1 \le i \le t, \\
f(v_4^i) &= \begin{cases} 4t + 1 - i, & 1 \le i \le (t+1)/2, \\ t + 1 - i, & (t+3)/2 \le i \le t \text{ and } i \ mod\ 2 = 1, \\ 3t + 1 - i, & (t+3)/2 \le i \le t \text{ and } i \ mod\ 2 = 0. \end{cases}
\end{aligned}
$$

Similar to the proof in Case 1, it can be shown that this assignment provides a graceful labelling of $C_5^{(4k-1)}$. □

In Figure 2 we show our graceful labellings for $C_5^{(8)}$ and $C_5^{(7)}$.

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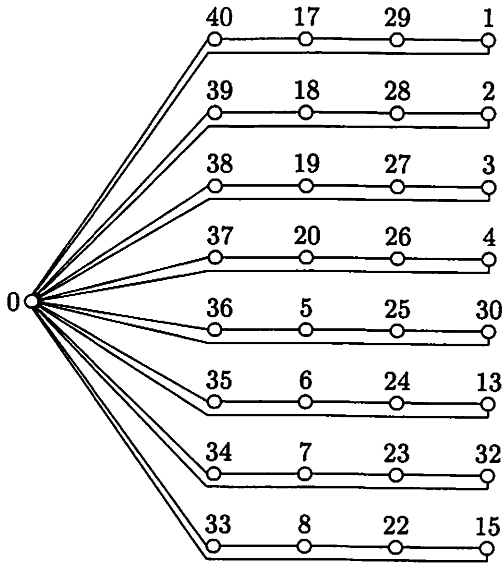
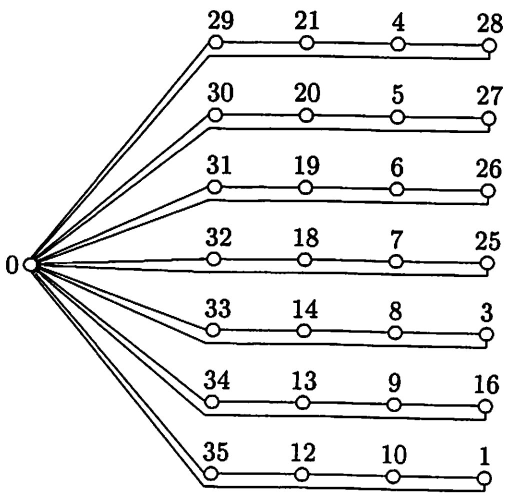


$C_5^{(8)}$



$C_5^{(7)}$

Figure 2: A graceful labelling of $C_5^{(8)}$ and $C_5^{(7)}$.

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