Combinatorial Proofs of a Type of Binomial Identity

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Abstract The purpose of this article is to give combinatorial proofs of some binomial identities which were given by Z. Zhang.

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1. Introduction

Define

$$A_k = \sum_{j=0}^{k} \binom{n}{j},$$

and

$$S_p = A_0^p + \cdots + A_n^p, p \geq 1.$$

Hirschhorn[3] showed

$$S_1 = (n+2)2^{n-1}$$
$$S_2 = (n+2)2^{2n-1} - \tfrac{1}{2}n\binom{2n}{n}$$
$$S_3 = (n+2)2^{3n-1} - 3\times 2^{n-2}n\binom{2n}{n}.$$

In general, he obtained recurrence relations

$$\begin{aligned} S_{2p} &= \binom{p}{1}2^n S_{2p-1} - \binom{p}{2}2^{2n}S_{2p-2} + \cdots \\ &\quad +(-1)^{p-1}\binom{p}{p}2^{pn}S_p + (-1)^p P_p, \\ S_{2p+1} &= \binom{p}{1}2^n S_{2p} - \binom{p}{2}2^{2n}S_{2p-1} + \cdots \\ &\quad +(-1)^{p-1}\binom{p}{p}2^{pn}S_{p+1} + (-1)^p 2^{n-1}P_p, \end{aligned}$$

where $P_p = \sum_{m=0}^{n-1} A_m^p A_{n-1-m}^p$.

Earlier, Calkin[1] obtained the third identity in a somewhat indirect manner.

Zhang[6] discussed the following sum:

$$R_p = A_0^p - A_1^p + \cdots + (-1)^n A_n^p$$

and obtained

$$R_1 = (-1)^n 2^{n-1}, \tag{1}$$

$$R_2 = \begin{cases} 2^{2n-1} & \text{if } n \text{ is even,} \\ -2^{2n-1} - (-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \end{cases} \tag{2}$$

$$R_3 = (-1)^n 2^{3n-1} - 3\times 2^{n-1}(-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} \quad \text{if } n \text{ is odd.} \tag{3}$$

In general, he obtained recurrence relations

$$R_{2p} = \sum_{i=1}^{p}(-1)^{i-1}\binom{p}{i}2^{in}R_{2p-i} + (-1)^p Q_{p,p}, \tag{4}$$

$$R_{2p+1} = \sum_{i=1}^{p}(-1)^{i-1}\binom{p}{i}2^{in}R_{2p+1-i} + (-1)^p Q_{p+1,p}, \tag{5}$$

and in particular, if n is odd,

$$R_{2p+1} = \sum_{i=1}^{p}(-1)^{i-1}\binom{p}{i}2^{in}R_{2p+1-i} + (-1)^p 2^{n-1}Q_{p,p} \qquad (6)$$

where $Q_{p,q} = \sum_{m=0}^{n-1}(-1)^m A_m^p A_{n-1-m}^q$.

In this note, we shall give combinatorial proofs of (1)–(6).

2. Combinatorial Proofs

For any positive integers m and n with $m \leq n$, put $[m,n] = \{m, m+1, \ldots, n\}$. Let $\mathcal{B}_n$ be the Boolean algebra, i.e. the lattice of subsets of $[1,n]$ ordered by inclusion. For any $T \in \mathcal{B}_n$, T^c denotes the complement of T in $[1,n]$.

2.1 *Proof of (1)*

Let $\mathcal{R}_1$ be the collection of subsets of $[1,n]$ such that $|T|$ is even for all $T \in \mathcal{R}_1$, $\mathcal{R}_1^c$ the complement of $\mathcal{R}_1$ in $\mathcal{B}_n$, and $\mathcal{C}(n,k)$ the set of k-subsets of $[1,n]$. It is well known that $|\mathcal{R}_1| = |\mathcal{R}_1^c| = 2^{n-1}$, and $|\mathcal{C}(n,k)| = \binom{n}{k}$.

It is clear that $A_k - A_{k-1}$ is the cardinality of $\mathcal{C}(n,k)$.

If n is even, then

$$R_1 = \sum_{m=0}^{n}(-1)^m A_m = (A_n - A_{n-1}) + (A_{n-2} - A_{n-3}) + \cdots + (A_2 - A_1) + A_0$$

is the cardinality of $\mathcal{R}_1$.

If n is odd, then

$$-R_1 = \sum_{m=0}^{n}(-1)^{m+1}A_m = (A_n - A_{n-1}) + (A_{n-2} - A_{n-3}) + \cdots + (A_1 - A_0)$$

is the cardinality of $\mathcal{R}_1^c$.

Hence $R_1 = (-1)^n 2^{n-1}$. □

2.2 *Proof of (2)*

Consider the *Cartesian product*: $\mathcal{B}_n \times \mathcal{B}_n$. A pair $(U_1, U_2) \in \mathcal{B}_n \times \mathcal{B}_n$ is said to be even or odd according to the parity of $\max\{|U_1|, |U_2|\}$.

Let

$$\mathcal{R}_2 = \{(U_1, U_2) \in \mathcal{B}_n \times \mathcal{B}_n : (U_1, U_2) \text{ is even}\},$$

$$\mathcal{R}_2^c = \{(U_1, U_2) \in \mathcal{B}_n \times \mathcal{B}_n : (U_1, U_2) \text{ is odd}\}.$$

It is easy to see that $A_k^2 - A_{k-1}^2$ counts the number of pairs $(U_1, U_2) \in \mathcal{B}_n \times \mathcal{B}_n$ with $\max\{|U_1|, |U_2|\} = k$.

Hence

$$R_2 = \sum_{m=0}^{n} (-1)^m A_m^2 = \begin{cases} |\mathcal{R}_2| & \text{if } n \text{ is even,} \\ -|\mathcal{R}_2^c| & \text{if } n \text{ is odd.} \end{cases}$$

Now define a map

$$\begin{array}{rccc} \psi_2 : & \mathcal{B}_n \times \mathcal{B}_n & \longrightarrow & \mathcal{B}_n \times \mathcal{B}_n \\ & (U_1, U_2) & \longrightarrow & (T_1, T_2) \end{array}$$

where $T_i = \begin{cases} U_i \setminus \{n\} & \text{if } n \in U_i, \\ U_i \cup \{n\} & \text{if } n \notin U_i, \end{cases} \quad i = 1, 2.$

Then ψ_2 is an involution, i.e. ψ_2^2 is an identity map. It is easy to see that ψ_2 changes the parity of pairs $(U_1, U_2) \in \mathcal{B}_n \times \mathcal{B}_n$ unless $|U_1| = |U_2| + 1, n \in U_1, n \notin U_2$ or $|U_2| = |U_1| + 1, n \in U_2, n \notin U_1$. Let

$\mathcal{S} = \{(U_1, U_2) : |U_1| = |U_2| + 1, n \in U_1, n \notin U_2$ or $|U_2| = |U_1| + 1, n \in U_2, n \notin U_1\}$.

Then ψ_2 induces an 1–1 correspondence between $\mathcal{R}_2 \setminus \mathcal{R}_2 \cap \mathcal{S}$ and $\mathcal{R}_2^c \setminus \mathcal{R}_2^c \cap \mathcal{S}$ which means

$$|\mathcal{R}_2^c| - |\mathcal{R}_2| = |\mathcal{R}_2^c \cap \mathcal{S}| - |\mathcal{R}_2 \cap \mathcal{S}|. \tag{7}$$

Now we count the number of elements of $\mathcal{R}_2 \cap \mathcal{S}$ and $\mathcal{R}_2^c \cap \mathcal{S}$ respectively. In fact, $\mathcal{R}_2 \cap \mathcal{S}$ is the set of pairs (U_1, U_2) such that (U_1, U_2) is even and either $|U_1| = |U_2| + 1, n \in U_1, n \notin U_2$ or $|U_2| = |U_1| + 1, n \in U_2, n \notin U_1$. It is easy to establish a bijection from the set

$$\{(U_1, U_2) \in \mathcal{B}_n \times \mathcal{B}_n : |U_1| = |U_2| + 1, n \in U_1, n \notin U_2, |U_1| \text{ is even}\}$$

to the set

$$\{(U_1', U_2) \in \mathcal{B}_{n-1} \times \mathcal{B}_{n-1} : U_1' = U_1 \setminus \{n\}, |U_1'| = |U_2| \text{ is odd}\}.$$

Hence

$$\begin{aligned}
|\mathcal{R}_2 \cap \mathcal{S}| &= 2|\{(U_1, U_2) \in \mathcal{B}_n \times \mathcal{B}_n : \\
&\quad |U_1| = |U_2| + 1, n \in U_1, n \notin U_2, |U_1| \text{ is even}\}| \\
&= 2|\{(U_1', U_2) \in \mathcal{B}_{n-1} \times \mathcal{B}_{n-1} : \\
&\quad U_1' = U_1 \setminus \{n\}, |U_1'| = |U_2| \text{ is odd}\}| \\
&= \begin{cases} 2\left[\sum_{k=0}^{\frac{n-2}{2}} \binom{n-1}{2k+1}\right]^2 & \text{if } n \text{ is even,} \\ 2\left[\sum_{k=0}^{\frac{n-3}{2}} \binom{n-1}{2k+1}\right]^2 & \text{if } n \text{ is odd.} \end{cases} \qquad (8)
\end{aligned}$$

Just as we did above, we obtain

$$|\mathcal{R}_2^c \cap \mathcal{S}| = \begin{cases} 2\left[\sum_{k=0}^{\frac{n-2}{2}} \binom{n-1}{2k}\right]^2 & \text{if } n \text{ is even,} \\ 2\left[\sum_{k=0}^{\frac{n-1}{2}} \binom{n-1}{2k}\right]^2 & \text{if } n \text{ is odd.} \end{cases} \qquad (9)$$

Comparing (7),(8) and (9), we obtain

$$|\mathcal{R}_2^c| - |\mathcal{R}_2| = 2\left(\sum_{k=0}^{\frac{n-2}{2}} \binom{n-1}{2k}^2 - \sum_{k=0}^{\frac{n-2}{2}} \binom{n-1}{2k+1}^2\right) = 0$$

if n is even, and

$$|\mathcal{R}_2^c| - |\mathcal{R}_2| = 2\left[\sum_{k=0}^{\frac{n-1}{2}} \binom{n-1}{2k}^2 - \sum_{k=0}^{\frac{n-3}{2}} \binom{n-1}{2k+1}^2\right]$$

$$\begin{aligned}
&= 2\sum_{i=0}^{\frac{n-1}{2}}(-1)^i\binom{n-1}{i}^2 \\
&= 2 \times \text{the coefficient of } t^{n-1} \text{ in } (1-t)^{n-1}(1+t)^{n-1} \\
&= (1-t^2)^{n-1} \\
&= 2(-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} \qquad (10)
\end{aligned}$$

if n is odd. On the other hand,

$$|\mathcal{R}_2^c| + |\mathcal{R}_2| = |\mathcal{B}_n \times \mathcal{B}_n| = 2^{2n}.$$

Hence

$$R_2 = \begin{cases} |\mathcal{R}_2| = 2^{2n-1} & \text{if } n \text{ is even,} \\ -|\mathcal{R}_2^c| = -2^{2n-1} - (-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

□

2.3 *Proof of (3)*

We shall complete the proof in a similar way to that used in the section 2.2.

Consider the *Cartesian product*: $\mathcal{B}_n \times \mathcal{B}_n \times \mathcal{B}_n = \mathcal{B}_n^3$. A triple $(U_1, U_2, U_3) \in \mathcal{B}_n^3$ is said to be even or odd according to the parity of $\max\{|U_1|, |U_2|, |U_3|\}$.

Let n be odd,

$$\mathcal{R}_3 = \{(U_1, U_2, U_3) \in \mathcal{B}_n^3 : (U_1, U_2, U_3) \text{ is even}\},$$

$$\mathcal{R}_3^c = \{(U_1, U_2, U_3) \in \mathcal{B}_n^3 : (U_1, U_2, U_3) \text{ is odd}\}.$$

Then $-R_3 = A_n^3 - A_{n-1}^3 + \cdots + A_1^3 - A_0^3$ is the number of elements in $\mathcal{R}_3^c$.

Define a map

$$\begin{array}{rccc} \psi_3: & \mathcal{B}_n \times \mathcal{B}_n \times \mathcal{B}_n & \longrightarrow & \mathcal{B}_n \times \mathcal{B}_n \times \mathcal{B}_n \\ & (U_1, U_2, U_3) & \longrightarrow & (T_1, T_2, T_3) \end{array},$$

where $T_i = \begin{cases} U_i \setminus \{n\} & \text{if } n \in U_i, \\ U_i \cup \{n\} & \text{if } n \notin U_i, \end{cases} \quad i = 1, 2, 3.$

Then ψ_3 is an involution, i.e. ψ_3^2 is an identity map. Let $\mathcal{T}$ denote the set of triples (U_1, U_2, U_3) such that either $|U_i| = |U_j| + 1, |U_i| > |U_k|, n \in U_i, n \notin U_j$ or $|U_i| = |U_k| = |U_j| + 1, n \in U_i \cap U_k, n \notin U_j$, where i, j, k are distinct numbers from one to three. It is easy to see that ψ_3 changes the parity of (U_1, U_2, U_3) if and only if $(U_1, U_2, U_3) \notin \mathcal{T}$. Hence ψ_3 induces a bijection from $\mathcal{R}_3 \setminus \mathcal{R}_3 \cap \mathcal{T}$ to $\mathcal{R}_3^c \setminus \mathcal{R}_3^c \cap \mathcal{T}$ which implies that

$$|\mathcal{R}_3^c| - |\mathcal{R}_3| = |\mathcal{R}_3^c \cap \mathcal{T}| - |\mathcal{R}_3 \cap \mathcal{T}|. \tag{11}$$

Now we count the number of elements of $\mathcal{R}_3^c \cap \mathcal{T}$ and $\mathcal{R}_3 \cap \mathcal{T}$ respectively. In fact, $\mathcal{R}_3^c \cap \mathcal{T}$ is the set of (U_1, U_2, U_3) such that (U_1, U_2, U_3) is odd and either $|U_i| = |U_j| + 1, |U_i| > |U_k|, n \in U_i, n \notin U_j$ or $|U_i| = |U_k| = |U_j| + 1, n \in U_i \cap U_k, n \notin U_j$, where i, j, k are distinct numbers from one to three.

Let $\mathcal{W}_i = \{(U_1, U_2, U_3) \in \mathcal{R}_3^c \cap \mathcal{T} : |U_i| = \max\{|U_1|, |U_2|, |U_3|\}\}, i = 1, 2, 3$. Then

$$\mathcal{R}_3^c \cap \mathcal{T} = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3.$$

We count the number of the elements in $\mathcal{W}_1$.

Observe that

$$\mathcal{W}_1 = \mathcal{W}_{11} \cup \mathcal{W}_{12} \cup \mathcal{W}_{13} \cup \mathcal{W}_{14},$$

where

$\mathcal{W}_{11} = \{\mathbf{U} \in \mathcal{W}_1 : |U_1| = |U_2| + 1, |U_1| > |U_3|, n \in U_1, n \notin U_2\}$,
$\mathcal{W}_{12} = \{\mathbf{U} \in \mathcal{W}_1 : |U_1| = |U_3| = |U_2| + 1, n \in U_1 \cap U_3, n \notin U_2\}$,
$\mathcal{W}_{13} = \{\mathbf{U} \in \mathcal{W}_1 : |U_1| = |U_3| + 1, |U_1| > |U_2|, n \in U_1, n \notin U_3\}$,
$\mathcal{W}_{14} = \{\mathbf{U} \in \mathcal{W}_1 : |U_1| = |U_2| = |U_3| + 1, n \in U_1 \cap U_2, n \notin U_3\}$,
$\mathbf{U} = (U_1, U_2, U_3)$.

Through reasoning similar to that used in the section 2.2 for counting $|\mathcal{R}_2 \cap \mathcal{S}|$, we obtain

$$|\mathcal{W}_{11}| = |\mathcal{W}_{13}| = \left(\sum_{k=0}^{\frac{n-1}{2}} \binom{n-1}{2k}^2\right)\left(\sum_{j=0}^{2k} \binom{n}{j}\right),$$

$$|\mathcal{W}_{12}| = |\mathcal{W}_{14}| = \sum_{k=0}^{\frac{n-1}{2}} \binom{n-1}{2k}^3,$$

$$|\mathcal{W}_{11} \cap \mathcal{W}_{13}| = \sum_{k=0}^{\frac{n-1}{2}} \binom{n-1}{2k}^3,$$

$$|\mathcal{W}_{11} \cap \mathcal{W}_{12}| = |\mathcal{W}_{11} \cap \mathcal{W}_{14}| = |\mathcal{W}_{12} \cap \mathcal{W}_{13}|$$

$$= |\mathcal{W}_{12} \cap \mathcal{W}_{14}| = |\mathcal{W}_{13} \cap \mathcal{W}_{14}| = 0.$$

By the inclusion-exclusion principle,

$$|\mathcal{W}_1| = 2\left(\sum_{k=0}^{\frac{n-1}{2}} \binom{n-1}{2k}^2\right)\left(\sum_{j=0}^{2k} \binom{n}{j}\right) + \sum_{k=0}^{\frac{n-1}{2}} \binom{n-1}{2k}^3.$$

It can be seen that

$$|\mathcal{W}_1| = |\mathcal{W}_2| = |\mathcal{W}_3|,$$

$$|\mathcal{W}_1 \cap \mathcal{W}_2| = |\mathcal{W}_2 \cap \mathcal{W}_3| = |\mathcal{W}_3 \cap \mathcal{W}_1| = \sum_{k=0}^{\frac{n-1}{2}} \binom{n-1}{2k}^3,$$

and

$$|\mathcal{W}_1 \cap \mathcal{W}_2 \cap \mathcal{W}_3| = 0.$$

By the inclusion-exclusion principle,

$$|\mathcal{R}_3^c \cap \mathcal{T}| = |\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3| = 6\left(\sum_{k=0}^{\frac{n-1}{2}} \binom{n-1}{2k}^2\right)\left(\sum_{j=0}^{2k} \binom{n}{j}\right).$$

Similarly, we can obtain

$$|\mathcal{R}_3 \cap \mathcal{T}| = 6\left(\sum_{k=0}^{\frac{n-3}{2}} \binom{n-1}{2k+1}^2\right)\left(\sum_{j=0}^{2k+1}\binom{n}{j}\right).$$

Comparing (11) yields

$$\begin{aligned}
&|\mathcal{R}_3^c| - |\mathcal{R}_3| = |\mathcal{R}_3^c \cap \mathcal{T}| - |\mathcal{R}_3 \cap \mathcal{T}| \\
= \;& 6\left[\sum_{k=0}^{\frac{n-1}{2}} \binom{n-1}{2k}^2 \left(\sum_{j=0}^{2k}\binom{n}{j}\right)\right. \\
- \;& \left.\sum_{k=0}^{\frac{n-3}{2}} \binom{n-1}{2k+1}^2 \left(\sum_{j=0}^{2k+1}\binom{n}{j}\right)\right] \\
= \;& 6 \sum_{0\le j\le i\le n-1} (-1)^i \binom{n-1}{i}^2 \binom{n}{j} \\
= \;& 6 \sum_{0\le j\le i\le n-1} (-1)^i \binom{n-1}{i}^2 \left[\binom{n-1}{j-1} + \binom{n-1}{j}\right] \\
= \;& 6\left[\sum_{0\le j\le i\le n-1} (-1)^i \binom{n-1}{i}^2 \binom{n-1}{j-1}\right. \\
+ \;& \left.\sum_{0\le j\le i\le n-1} (-1)^i \binom{n-1}{i}^2 \binom{n-1}{j}\right] \\
= \;& 6\left[\sum_{i=0}^{n-1}(-1)^i \binom{n-1}{i}^2 \left[2^{n-1} - \sum_{j=i+1}^{n}\binom{n-1}{j-1}\right]\right. \\
+ \;& \left.\sum_{0\le j\le i\le n-1} (-1)^i \binom{n-1}{i}^2 \binom{n-1}{j}\right] \\
= \;& 6\left[2^{n-1}(-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} - \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}^2\right. \\
\times \;& \left.\sum_{j=i+1}^{n}\binom{n-1}{j-1} + \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}^2\sum_{j=0}^{i}\binom{n-1}{j}\right] \quad \text{(see (10))} \\
= \;& 6\left[2^{n-1}(-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} - \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{n-1-i}^2\right.
\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{j=i+1}^{n} \binom{n-1}{n-j} + \sum_{i=0}^{n-1}(-1)^i \binom{n-1}{i}^2 \sum_{j=0}^{i} \binom{n-1}{j} \Bigg] \\
&= 6\Bigg[2^{n-1}(-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} - \sum_{i=1}^{n}(-1)^{n-i}\binom{n-1}{i-1}^2 \\
&\times \sum_{j=0}^{i-1} \binom{n-1}{j} + \sum_{i=0}^{n-1}(-1)^i \binom{n-1}{i}^2 \sum_{j=0}^{i} \binom{n-1}{j} \Bigg] \\
&= 6\Bigg[2^{n-1}(-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}} - \sum_{i=0}^{n-1}(-1)^{n-i-1}\binom{n-1}{i}^2 \\
&\times \sum_{j=0}^{i} \binom{n-1}{j} + \sum_{i=0}^{n-1}(-1)^i \binom{n-1}{i}^2 \sum_{j=0}^{i} \binom{n-1}{j} \Bigg] \\
&= 6\left[2^{n-1}(-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}}\right] \quad (n \text{ is odd}).
\end{aligned}
$$

On the other hand,

$$|\mathcal{R}_3^c| + |\mathcal{R}_3| = |\mathcal{B}_n \times \mathcal{B}_n \times \mathcal{B}| = 2^{3n},$$

so

$$R_3 = -|\mathcal{R}_3^c| = (-1)^n 2^{3n-1} - 3 \times 2^{n-1}(-1)^{\frac{n-1}{2}}\binom{n-1}{\frac{n-1}{2}}.$$

□

2.4 *Proofs of (4)-(6)*

Let

$$
\begin{aligned}
E_p &= A_0^p + A_2^p + \cdots = \sum_{0\le m\le n,\ 2|m} A_m^p, \\
O_p &= A_1^p + A_3^p + \cdots = \sum_{0\le m\le n,\ 2\not| m} A_m^p, \\
W_{p,q} &= A_0^p A_{n-1}^q + A_2^p A_{n-3}^q + \cdots = \sum_{0\le m\le n-1,\ 2|m} A_m^p A_{n-1-m}^q, \\
V_{p,q} &= A_1^p A_{n-2}^q + A_3^p A_{n-4}^q + \cdots = \sum_{0\le m\le n-1,\ 2\not| m} A_m^p A_{n-1-m}^q.
\end{aligned}
$$

Then

$$R_p = E_p - O_p,$$

$$Q_{p,q} = W_{p,q} - V_{p,q}.$$

First , we consider the set

$$\mathcal{A} = \bigcup_{0\le m\le n, 2|m} \{(m, T_1, \ldots, T_p, T_{p+1}, \ldots, T_{2p}) : T_i\text{'s in } \mathcal{B}_n \text{ with } |T_j| \le m, j = 1, 2, \ldots, p\}$$

It is clear that

$$|\mathcal{A}| = \sum_{0\le m\le n, 2|m} A_m^p 2^{pn} = 2^{pn} E_p.$$

For $1 \le i \le p$, let

$$\mathcal{A}_i = \bigcup_{0\le m\le n, 2|m} \{(m, T_1, \ldots, T_p, T_{p+1}, \ldots, T_{2p}) \in \mathcal{A} : |T_{p+i}| \le m\}.$$

Then

$$|\mathcal{A}_i| = 2^{(p-1)n} E_{p+1}, \; i = 1, 2, \ldots, p,$$

$$|\mathcal{A}_i \cap \mathcal{A}_j| = 2^{(p-2)n} E_{p+2}, \; i \ne j; \; i, j = 1, 2, \ldots, p,$$

$$\cdots\cdots$$

$$|\mathcal{A}_1 \cap \cdots \cap \mathcal{A}_p| = E_{2p}.$$

Denote the complement of $\mathcal{A}_i$ in $\mathcal{A}$ by $\mathcal{A}_i^c, i = 1, 2, \ldots, p$. Then

$$\mathcal{A}_1^c \cap \cdots \cap \mathcal{A}_p^c = \bigcup_{0\le m\le n-1, 2|m} \{(m, T_1, \ldots, T_p, T_{p+1}, \ldots, T_{2p}) \in \mathcal{A} : |T_{p+i}| > m, i = 1, \ldots, p\}.$$

Hence

$$|\mathcal{A}_1^c \cap \mathcal{A}_2^c \cdots \cap \mathcal{A}_p^c| = \sum_{0\le m\le n-1, 2|m} A_m^p A_{n-1-m}^p = W_{p,p}.$$

By the inclusion-exclusion principle,

$$W_{p,p} = 2^{pn}E_p - \binom{p}{1} 2^{(p-1)n} E_{p+1} + \cdots + (-1)^p E_{2p},$$

so,

$$E_{2p} = \binom{p}{1} 2^n E_{2p-1} + \cdots + (-1)^{p-1} \binom{p}{p} 2^{pn} E_p + (-1)^p W_{p,p}.$$

Similarly,

$$O_{2p} = \binom{p}{1} 2^n O_{2p-1} + \cdots + (-1)^{p-1} \binom{p}{p} 2^{pn} O_p + (-1)^p V_{p,p}.$$

Observing that $R_p = E_p - O_p$ and $Q_{p,q} = W_{p,q} - V_{p,q}$, we obtain

$$R_{2p} = \binom{p}{1} 2^n R_{2p-1} + \cdots + (-1)^{p-1} \binom{p}{p} 2^{pn} R_p + (-1)^p Q_{p,p}.$$

Next, just as we did for R_{2p}, we consider the set

$$\mathcal{C} = \bigcup_{0 \le m \le n, 2|m} \{(m, T_1, \ldots, T_p, T_{p+1}, \ldots, T_{2p+1}) : T_i\text{'s in } \mathcal{B}_n, \text{ with } |T_{p+j}| \le m, j = 1, \ldots, p+1\}$$

which provides $|\mathcal{C}| = 2^{pn} E_{p+1}$. For $1 \le i \le p$, let

$$\mathcal{C}_i = \bigcup_{0 \le m \le n, 2|m} \{(m, T_1, \ldots, T_p, T_{p+1}, \ldots, T_{2p+1}) \in \mathcal{C} : |T_i| \le m\}.$$

Then

$$|\mathcal{C}_i| = 2^{(p-1)n} E_{p+2}, \quad i = 1, 2, \ldots, p,$$

$$|\mathcal{C}_i \cap \mathcal{C}_j| = 2^{(p-2)n} E_{p+3}, \quad i \ne j;\ i, j = 1, 2, \ldots, p,$$

$$\ldots\ldots$$

$$|\mathcal{C}_1 \cap \cdots \cap \mathcal{C}_p| = E_{2p+1}.$$

Denote the complement of C_i in C by $C_i^c, i = 1, 2, \ldots, p$. It is clear that

$$C_1^c \cap \cdots \cap C_p^c = \bigcup_{0 \le m \le n-1, 2|m} \{(m, T_1, \ldots, T_p, T_{p+1}, \ldots, T_{2p+1}) \in C : |T_i| > m, i = 1, \ldots, p\}.$$

Hence

$$|C_1^c \cap C_2^c \cdots \cap C_p^c| = \sum_{0 \le m \le n-1, 2|m} A_m^{p+1} A_{n-1-m}^p = W_{p+1,p}.$$

By the inclusion-exclusion principle,

$$W_{p+1,p} = 2^{pn} E_{p+1} - \binom{p}{1} 2^{(p-1)n} E_{p+2} + \cdots + (-1)^p \binom{p}{p} E_{2p+1},$$

so

$$\begin{aligned} E_{2p+1} &= \binom{p}{1} 2^n E_{2p} - \binom{p}{2} 2^{2n} E_{2p-1} + \cdots \\ &+ (-1)^{p-1} \binom{p}{p} 2^{pn} E_{p+1} + (-1)^p W_{p+1,p}. \end{aligned}$$

Similarly,

$$\begin{aligned} O_{2p+1} &= \binom{p}{1} 2^n O_{2p} - \binom{p}{2} 2^{2n} O_{2p-1} + \cdots \\ &+ (-1)^{p-1} \binom{p}{p} 2^{pn} O_{p+1} + (-1)^p V_{p+1,p}. \end{aligned}$$

Observing that $R_p = E_p - O_p$ and $Q_{p,q} = W_{p,q} - V_{p,q}$, we obtain

$$\begin{aligned} R_{2p+1} &= \binom{p}{1} 2^n R_{2p} - \binom{p}{2} 2^{2n} R_{2p-1} + \cdots \\ &+ (-1)^{p-1} \binom{p}{p} 2^{pn} R_{p+1} + (-1)^p Q_{p+1,p} \end{aligned}$$

which is (5).

In particular, if n is odd, then $m, n-1-m$ have the same parity,

$$\begin{aligned}
W_{p+1,p} &= \sum_{0\le m\le n-1, 2|m} A_m^{p+1} A_{n-1-m}^p \\
&= \frac{1}{2}\Big(\sum_{0\le m\le n-1, 2|m} A_m^{p+1} A_{n-1-m}^p + \sum_{0\le m\le n-1, 2|m} A_{n-1-m}^{p+1} A_m^p\Big) \\
&= \frac{1}{2}\Big[(A_0^{p+1}A_{n-1}^p + A_{n-1}^{p+1}A_0^p) + (A_2^{p+1}A_{n-3}^p + A_{n-3}^{p+1}A_2^p) + \cdots \\
&+ (A_{n-1}^{p+1}A_0^p + A_0^{p+1}A_{n-1}^p)\Big] \\
&= \frac{1}{2}\big[A_0^pA_{n-1}^p(A_0 + A_{n-1}) + A_2^pA_{n-3}^p(A_2 + A_{n-3}) + \cdots \\
&+ A_{n-1}^pA_0^p(A_{n-1} + A_0)\big] \\
&= 2^{n-1} \sum_{0\le m\le n-1, 2|m} A_m^p A_{n-1-m}^p \\
&= 2^{n-1} W_{p,p}.
\end{aligned}$$

Similarly, $V_{p+1,p} = 2^{n-1}V_{p,p}$. Hence

$$Q_{p+1,p} = W_{p+1,p} - V_{p+1,p} = 2^{n-1}(W_{p,p} - V_{p,p}) = 2^{n-1}Q_{p,p}$$

which implies

$$\begin{aligned}
R_{2p+1} &= \binom{p}{1} 2^n R_{2p} - \binom{p}{2} 2^{2n} R_{2p-1} + \cdots \\
&+ (-1)^{p-1}\binom{p}{p} 2^{pn} R_{p+1} + (-1)^p 2^{n-1} Q_{p,p}.
\end{aligned}$$

The proof of (6) is complete.

3. Conclusion

In 1994, Calkin obtained a curious identity of sums of 3-powers of the partial sum of binomial coefficients [1]. In 1996, Hirschhorn supplied the identities involving 1 and 2-powers of the partial sum of binomial coefficients and some recurrence relations [3]. In 1999, Zhang considered the alternative sums [6]. In view of the fundamentality of combinatorial interpretation for a combinatorial identity on enumerative combinatorics [4], recently, in [2] we show the combinatorial proofs of identities of Calkin and Hirschhorn. In this paper, we give combinatorial interpretations of some binomial identities involving the alternative sums. Finally, by applying our same method we expect to obtain the combinatorial proofs of this kind of curious identity was extended by J. Wang and Z. Zhang [5].

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