THE QUOTIENTS OF $G^{3,11,924}$

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The groups $G^{k,l,m}$ have been extensively studied by H. S. M. Coxeter. They are symmetric groups of the maps $\{k,l\}_m$ which are constructed from the tessellations $\{k,l\}$ of the hyperbolic plane by identifying two points, at a distance m apart, along a Petrie path. It is known that $PSL(2,q)$ is a quotient group of the Coxeter groups $G^{k,l,m}$ if -1 is a quadratic residue in the Galois field F_q, where q is a prime power. G. Higman has posed the question that for which values of k,l,m, all but finitely many alternating groups A_n and symmetric groups S_n are quotients of $G^{k,l,m}$. In this paper we have answered this question by showing that for $k=3, l=11$, all but finitely many A_n and S_n are quotients of $G^{3,11,m}$, where m has turned out to be 924.

1. Introduction

The group $G^{k,l,m}$ has been defined by H.S.M. Coxeter in his famous paper [2] as a group with presentation $< X,Y,Z : X^k = Y^l = Z^m = (XY)^2 = (YZ)^2 = (ZX)^2 = (XYZ)^2 = 1 >$. If we let $x = XY$, $y = X$ and $t = YZ$, then the group $G^{k,l,m}$ has the presentation $< x,y,t : x^2 = y^k = t^2 = (xy)^l = (xt)^2 = (yt)^2 = (xyt)^m = 1 >$. He has shown that $G^{k,l,m}$ is finite when $\frac{1}{k}+\frac{1}{l}+\frac{1}{m} > 1$, and infinite when $\frac{1}{k}+\frac{1}{l}+\frac{1}{m} \le 1$. The exceptions to this inequality being the spherical triangle groups, which are finite, or the Euclidean triangle groups, which are soluble.

$G^{k,l,m}$ groups are symmetric groups of the regular maps $\{k,l\}_m$. Let k and l be two points, at a distance m apart along a Petrie path. Then the map $\{k,l\}_m$ is constructed from the tessellation $\{k,l\}$ of the hyperbolic plane by identifying these points. Let q be a power of a prime p. Then H.S.M. Coxeter has shown also that $G^{k,l,m}$ is isomorphic to either $PGL(2,q)$ or $PSL(2,q)$ for some small values of k, l

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and m. S.E. Wilson [13] has proved that $PSL(2,q)$ is a quotient of $G^{k,l,m}$ if -1 is a quadratic residue in F_q and $PGL(2,q)$ is a quotient of $G^{k,l,m}$ otherwise.

G. Higman posed the question that for which values of k,l,m, all but finitely many alternating groups A_n and symmetric groups S_n are factor groups of $G^{k,l,m}$. He has shown that all but finitely many alternating groups A_n of finite degree are homomorphic images of the triangle group $\Delta(2,3,7)$. He has described in [3] that for $k=3, l=7$, and $m=19$, A_n is a homomorphic images of $G^{k,l,m}$. Note that $PSL(2,113)$ is another homomorphic image of $G^{3,7,19}$.

In [5,7,8,9], Higman's question has been answered for the triplets $(k,l,m) = (3,8,720)$, $(5,7,84)$, $(4,5,276)$, $(5,6,36)$. The authors of this paper have answered Higman's question for $k=4$, $l=5$ and $m=120$ in [1]. Recently, the Higman's question has been answered in [10] for the triplet $(k,l,m)=(5,5,24)$ where n is congruent to 2 or 11 modulus 20.

Information about $G^{3,11,8}$ is known in [2]. It seem, there is no information available for the groups $G^{3,11,m}$ where $m>8$. In this paper we have taken $k=3$, $l=11$ and answered Higman's question for minimum values of m by using a diagrammatic argument as in [6]. That is, we have shown that all but finitely many positive integers n, both A_n and S_n occur as homomorphic images of $G^{3,11,924}$.

If $\Delta(2,3,11) = \langle x,y : x^2 = y^3 = (xy)^{11} = 1 \rangle$ then it is of index 1 or 2 in $G^{3,11,m}$ and is isomorphic to the group $\Delta(2,3,11;s) = \langle x,y : x^2 = y^3 = (xy)^{11} = (x^{-1}y^{-1}xy)^s = 1 \rangle$ where $s=m$ if m is odd and $s=\frac{m}{2}$ if m is even. It is mentioned in [2] that $LF(2,23) \cong \Delta(2,3,11;4)$ which is of order 6072 and $G^{3,11,8}$ is isomorphic to $PGL(2,23)$ which is of order 12144.

2. Coset Diagrams for $G^{3,11,m}$ and their Composition

We shall use coset diagrams, attributed to G. Higman, to prove our result. These coset diagrams depict an action of $G = \langle x,y,t : x^2 = y^3 = t^2 = (xy)^{11} = (xt)^2 = (yt)^2 = 1 \rangle$ on a finite set (or space) and are defined as follows.

The 3-cycles of y are represented by triangles whose vertices are permuted counter-clockwise by y. Any two vertices which are interchanged by an involution x, is represented by an edge. Every vertex of the diagram is fixed by $(xy)^{11}$. The action of t is represented by reflection about a vertical line of axis. Fixed points of x and y are denoted by heavy dots. This graph can be interpreted as a coset diagram, with the vertices identified with the cosets of stab(v), the stabilizer of some vertex v of the graph, or as 1-skeleton of the cover of the fundamental complex of the presentation which corresponds to the subgroup stab(v). By $D(n)$ we shall mean a coset diagram with n vertices satisfying the relation $x^2 = y^3 = t^2 = (xy)^{11} = (xt)^2 = (yt)^2 = 1$.

For example, the following coset diagram depicts a transitive representation of the group $\langle x,y,t : x^2 = y^3 = t^2 = (xy)^8 = (xt)^2 = (yt)^2 = 1 \rangle$ of degree 16. Here

x acts as:(1 8)(2 5)(3 4)(6 12)(10 11)(13 14),

y acts as:(1 2 3)(5 6 7)(8 9 10)(11 12 13)(14 15 16), and

t acts as:(1 2)(5 8)(6 10)(7 9)(11 12)(15 16).

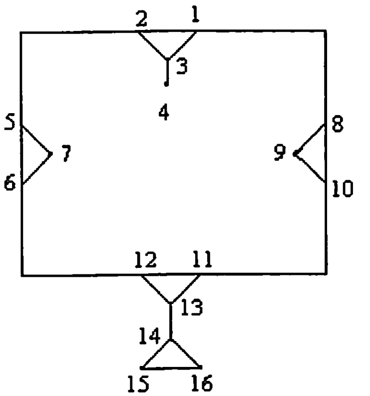

Fig-3

We will need to join together two, or more, coset diagrams. The technique of joining coset diagrams together has been given by W.W. Stothers [11]. Two diagrams can be joined together provided they contain a pair of a fragment of a special type called a 3-handle. By a 3-handle $3h_a^b$ in an arbitrary permutation representation of $G = < x,y,t : x^2 = y^3 = t^2 = (xy)^{11} = (xt)^2 = (yt)^2 = 1 >$, we mean a fragment of a coset diagram in which two vertices a, b are both fixed by x, and are interchanged by t, and also lie in the same cycle of $(xy)^3$. Diagrammatically it means:

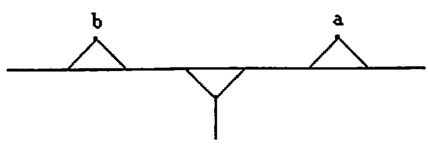

Fig-4

Given two coset diagrams P and Q with 3-handles $3h_a^b$ and $3h_{a'}^{b'}$ respectively, we can construct a new coset diagram $P+Q$ by placing the two diagrams on a common vertical axis of symmetry, one above the other, and joining a to a' and b to b' by x-edges as follows.

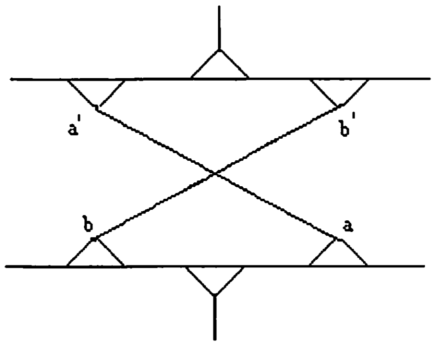

Fig-5

In this way, we can join any number of coset diagrams. The resulting diagram will again be a coset diagram for the action of G on a larger set. That is, the relations $x^2 = y^3 = (xy)^{11} = t^2 = (xt)^2 = (yt)^2 = 1$ are still satisfied. Also if $(\lambda, a_1, ..., a_{j-1}, \mu, a_j, ..., a_{q-2})$ and $(\sigma, b_1, ..., b_{j-1}, \tau, b_j, ..., b_{q-2})$ are the cycles of xy in the representation of G depicted by the two diagrams, then in the representation we see that $(\lambda, b_1, ..., b_{j-1}, \tau, a_j, ..., a_{q-2})$ and $(\sigma, a_1, ..., a_{j-1}, \mu, b_j, ..., b_{q-2})$ are two new cycles of the element xy in the resulting diagram. Other cycles of xy are unchanged, so xy still has order 11. Hence the new coset diagram is a coset diagram for G.

The required information from a coset diagram is written in a specific way. Each of the coset diagram is given a specification, consisting of the degree of the corresponding permutation representation of the group which is acting on a set of n elements, the number of 3-handles that will be used, the parity of the action of t, and the cycle structure of xyt and xy^2t.

We describe these as follows.

- By $D(n)$ we shall mean a diagram with n vertices satisfying the given properties, namely, $x^2 = y^3 = t^2 = (xy)^{11} = (xt)^2 = (yt)^2 = 1$.
- By $3h_a^b$, we mean the 3-handle with vertices a and b.
- By xyt :$(a \ \lambda)(b \ \mu)$, we mean that the vertices a and b lie in the cycles of xyt having lengths $\lambda + 1$ and $\mu + 1$ respectively.

3. Quotients of $G^{3,11,924}$

Theorem 3.1. (Theorem 13.9, page 39, [12]) Let p be a prime and G a primitive group of degree $n = p + k$ with $k \geq 3$. If G contains an element of degree and order p then G is either alternating or symmetric.

Now that the essential information, terminology, and mechanism has been set, we state and prove our main result.

Theorem 3.2. All but finitely many A_n and S_n are quotients of $G^{3,11,924}$.

Proof. We use coset diagrams for the group $G^{3,11,924}$ depicting a transitive permutation representation of $G^{3,11,924}$ of arbitrarily large degree.

The cycles of the permutation(s) induced by xy^2t are affected in the same sort of way: provided a' and b' lie in distinct cycles of xy^2t. The cycles ending in a and a' will be juxtaposed to form a single cycle, and those ending in b and b' will be similarly combined.

We will need two basic diagrams which we join together to form the required one. Each of these diagrams is given a specification discussed earlier.

Consider the vertices labelled α, β and γ in the diagram $D(24)$. Note that these three vertices lie in the same cycle of xy^2t having prime length.

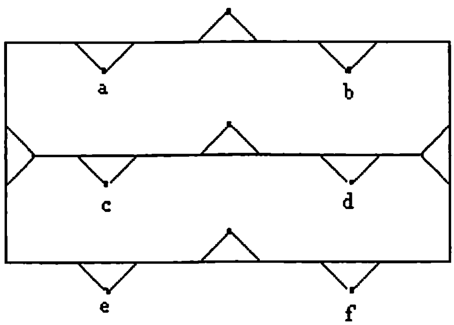

Fig-6

$D(33)$
Three (3) –handles, t is odd,
$xyt = (a\ 3)(b\ 6)(c\ 3)(d\ 6)(e\ 3)(f\ 6)$, and
$xy^2t = (a\ 6)(b\ 3)(c\ 6)(d\ 3)(e\ 6)(f\ 3)$.

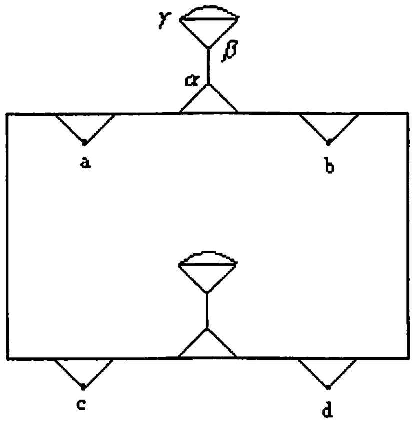

Fig-7

$D(24)$
Two (3) –handles, t is even,
$xyt = (a\ 6)(b\ 1\ d\ 1)(c\ 6)6$, and
$xy^2t = (a\ 1\ c\ 1)(b\ 6)(d\ 6)6$.

Take u numbers of $D(33)$ and v numbers of $D(24)$ diagrams and connect them together in a specific order: $D(33)u + D(24)v$, We cannot join $D(33)$ with $D(33)$ or $D(24)$ with $D(24)$. The resulting diagram $D(n)$ will have n vertices and it will be a diagram for the group $G^{3,11,924}$. The reflection t acts as an even or odd permutation, depending upon the values of n . For instance, if n is an odd number then t is odd and if n is an even number then t is even .

Also, the length of every cycle of xyt will be a divisor of 924 and so the diagram $D(n)$ will give a permutation representation of the group $G^{3,11,924}$.

Note that the cycles of xy^2t are all of length $d = 4,6,7$, or 11. With the exception of 7, d is a divisor of 132. Thus the element $(xy^2t)^{132}$ yields a power of the cycle, fixing the remaining vertices.

Next we show that the representation of $G^{3,11,924}$ is primitive on the n vertices of $D(n)$. Suppose that the representation is imprimitive. This means that the seven vertices of the cycle must lie in the same block, say B, of imprimitivity as $(xy^2t)^{132}$ fixes these vertices. Now α,β and γ belong to B and $\alpha x = \beta$, $\gamma y = \beta$ and $\beta t = \beta$. This means that B is preserved by the three generators x, y and t. This implies that B has n vertices or the representation is transitive. This contradicts the fact that $D(n)$ has n vertices. Thus the representation is primitive. Since the group $G^{3,11,924}$ is primitive on n vertices and there exists a cycle of prime length, namely 7-cycle, we can use theorem 1 to conclude that the permutations x, y and t generate A_n or S_n.

Since y and xy are of odd orders, they evolve even permutations and therefore so does x. The permutation t is even or odd depends upon the value of n. Therefore, if n is an odd number then t is an odd permutation and so yields S_n as a quotient of $G^{3,11,924}$. Similarly, if n is an even number then t is an even permutation and yields A_n as a quotient of $G^{3,11,924}$.

Note that if $N =< x,y >$, then N has index 1 or 2 in $G^{3,11,924}$ and is isomorphic to the group $\Delta(2,3,11;462) = < x,y : x^2 = y^3 = (xy)^{11} = (x^{-1}y^{-1}xy)^{462} = 1 >$. The triangle group $\Delta(2,3,11)$, of which $\Delta(2,3,11;462)$ is a quotient and acts as a discontinuous group of conformal homeomorphisms of a simple connected Riemann surface [4].

Corollary 3.3. *For all but finitely many positive integers n, A_n* has the presentation $< x, y : x^2 = y^3 = (xy)^{11} = (x^{-1}y^{-1}xy)^{462} = 1 >$.

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