On the lower bounds of partial signed domination number of graphs *

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Abstract: Let $G = (V, E)$ be a simple graph. For any real valued function $f : V \to \mathbf{R}$ and $S \subseteq V$, let $f(S) = \sum_{v \in S} f(v)$. Let c, d be positive integers such that $gcd(c, d) = 1$ and $0 < \frac{c}{d} \leq 1$. A $\frac{c}{d}$−dominating function(partial signed dominating function) is a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \geq 1$ for at least $\frac{c}{d}$ of the vertices $v \in V$. The $\frac{c}{d}$−domination number(partial signed domination number) of G is $\gamma_{\frac{c}{d}}(G) = \min\ \{f(V) | f$ is a $\frac{c}{d}$−dominating function on $G\}$. In this paper, we obtain a few lower bounds of $\gamma_{\frac{c}{d}}(G)$.

Key words: Partial signed domination number, Partial signed dominating function.

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1. Introduction

Let $G = (V, E)$ be a simple graph and v be a vertex in V. The open neighborhood of v, denoted by $N(v)$, is the set of vertices adjacent to v, i.e., $N(v) = \{u \in V | uv \in E\}$. The closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. The degree of v in G is $d_G(v) = |N(v)|$, a vertex v is called odd vertex if $d_G(v)$ is odd. $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of the vertices

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of G. When no ambiguity can occur, we often simply write $d(v)$, δ, Δ instead of $d_G(v)$, $\delta(G)$ and $\Delta(G)$, respectively.

For any real-valued function $f : V \to \mathbf{R}$ and $S \subseteq V$, let $f(S) = \sum_{v \in S} f(v)$. The weight of f is defined as $f(V)$.

A *signed dominating function* is defined in [2] as a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \geq 1$ for every $v \in V$. The *signed domination number* of a graph G is $\gamma_s(G) = \min\ \{f(V) |\ f$ is a signed domination function on $G\}$.

A *majority dominating function* function is defined in [3] as a function $f : V \to \{-1, 1\}$ such that for at least half the vertices $v \in V$, $f(N[v]) \geq 1$.The *majority domination number* of a graph G is $\gamma_{maj}(G) = \min\ \{f(V) |\ f$ is a majority dominating function on $G\}$.

Let c, d be positive integers such that $gcd(c, d) = 1$ and $0 < \frac{c}{d} \leq 1$. A $\frac{c}{d}$*−dominating function(partial signed dominating function)* is defined in [1] as a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \geq 1$ for at least $\frac{c}{d}$ of the vertices of $v \in V$. The $\frac{c}{d}$*−domination number(partial signed domination number)* of G is $\gamma_{\frac{c}{d}}(G) = \min\ \{f(V) | f$ is a $\frac{c}{d}$−dominating function on $G\}$. In the special cases where $\frac{c}{d} = 1$ and $\frac{c}{d} = \frac{1}{2}$, $\gamma_{\frac{c}{d}}(G)$ is the signed domination number and the majority domination number, respectively.

2. Lower bounds on partial signed domination number $\gamma_{\frac{c}{d}}(G)$

Theorem 1 *Let c, d be positive integers such that $gcd(c, d) = 1$ and $0 < q = \frac{c}{d} \leq 1$. For any graph G of order n and maximum Δ, minimum degree δ,*

$$\gamma_q(G) \geq \frac{\delta - 3\Delta - 2 + 2(\Delta + 2)q}{\Delta + \delta + 2} n.$$

Proof. Let $f : V \to \{-1, 1\}$ be any q−dominating function on G for which $f(V) = \gamma_q(G)$. Let P and M be the sets of vertices in G that are assigned the values +1 and −1 under f, respectively. Then $|P| + |M| = n$. Let $P = P_\Delta \cup P_\delta \cup P_\Theta$ where P_Δ and P_δ are sets of all vertices of P with degree equal to Δ and δ, respectively, and P_Θ contains all other vertices in P, if any. Let $M = M_\Delta \cup M_\delta \cup M_\Theta$ where P_Δ, P_δ and P_Θ are defined similarly. Further, for $i \in \{\Delta, \delta, \Theta\}$, let V_i be defined by $V_i = P_i \cup M_i$. Thus $n = |V_\Delta| + |V_\delta| + |V_\Theta|$. Since for at least q of the vertices $v \in V$, $f(N[v]) \geq 1$, we have

$$\sum_{v \in V} f(N[v]) \geq qn - (\Delta + 1)(1 - q)n = q(\Delta + 2)n - (\Delta + 1)n.$$

The sum $\sum_{v\in V} f(N[v])$ counts the value $f(v)$ exactly $d(v)+1$ times for each vertex $v \in V$, i.e., $\sum_{v\in V} f(N[v]) = \sum_{v\in V} f(v)(d(v)+1)$. Thus, $\sum_{v\in V} f(v)(d(v)+1) \geq q(\Delta+2)n - (\Delta+1)n$. Breaking the sum up into the six summations and replacing $f(v)$ with the corresponding value of 1 or -1 yields $\sum_{v\in P_\Delta}(d(v)+1) + \sum_{v\in P_\delta}(d(v)+1) + \sum_{v\in P_\Theta}(d(v)+1) - \sum_{v\in M_\Delta}(d(v)+1) - \sum_{v\in M_\delta}(d(v)+1) - \sum_{v\in M_\Theta}(d(v)+1) \geq q(\Delta+2)n-(\Delta+1)n$.

We know that $d(v) = \Delta$ for all v in P_Δ or M_Δ, and $d(v) = \delta$ for all v in P_δ or M_δ. For any vertex v in either P_Θ or M_Θ, $\delta+1 \leq d(v) \leq \Delta-1$. Therefore, we have $(\Delta+1)|P_\Delta| + (\delta+1)|P_\delta| + \Delta|P_\Theta| - (\Delta+1)|M_\Delta| - (\delta+1)|M_\delta| - (\delta+2)|M_\Theta| \geq q(\Delta+2)n - (\Delta+1)n$. For $i \in \{\Delta, \delta, \Theta\}$, we replace $|P_i|$ with $|V_i| - |M_i|$ in the above inequality. Therefore, we have $(\Delta+1)|V_\Delta| + (\delta+1)|V_\delta| + \Delta|V_\Theta| \geq q(\Delta+2)n - (\Delta+1)n + 2(\Delta+1)|M_\Delta| + 2(\delta+1)|M_\delta| + (\Delta+\delta+2)|M_\Theta|$. It follows that

$$\begin{aligned}
&[2(\Delta+1) - (\Delta+2)q]n\\
&\geq 2(\Delta+1)|M_\Delta| + 2(\delta+1)|M_\delta| + (\Delta+\delta+2)|M_\Theta| + (\Delta-\delta)|V_\delta| + |V_\Theta|\\
&= 2(\Delta+1)|M_\Delta| + 2(\delta+1)|M_\delta| + (\Delta+\delta+2)|M_\Theta| + (\Delta-\delta)(|P_\delta| + |M_\delta|)\\
&+(|P_\Theta| + |M_\Theta|)\\
&= 2(\Delta+1)|M_\Delta| + (\Delta+\delta+2)|M_\delta| + (\Delta+\delta+3)|M_\Theta| + (\Delta-\delta)|P_\delta| + |P_\Theta|\\
&\geq (\Delta+\delta+2)|M_\Delta| + (\Delta+\delta+2)|M_\delta| + (\Delta+\delta+2)|M_\Theta|\\
&= (\Delta+\delta+2)|M|.
\end{aligned}$$

Therefore,

$$|M| \leq \frac{2(\Delta+1) - (\Delta+2)q}{\Delta+\delta+2}n.$$

So

$$\begin{aligned}
\gamma_q(G) &= |P| - |M| = n - 2|M|\\
&\geq n - \frac{4(\Delta+1) - 2(\Delta+2)q}{\Delta+\delta+2}n = \frac{\delta - 3\Delta - 2 + 2(\Delta+2)q}{\Delta+\delta+2}n.
\end{aligned}$$

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Corollary 1.1[4] For any graph G of order n and maximum Δ, minimum degree δ,

$$\gamma_s(G) \geq \frac{\delta - \Delta + 2}{\Delta+\delta+2}n.$$

Corollary 1.2 For any graph G of order n and maximum Δ, minimum degree δ,

$$\gamma_{maj}(G) \geq \frac{\delta - 2\Delta}{\Delta + \delta + 2} n.$$

Theorem 2 *Let c, d be positive integers such that $gcd(c, d) = 1$ and $0 < q = \frac{c}{d} \leq 1$. If all vertices of graph G are odd vertices ,and $|V(G)| = n$, then*

$$\gamma_q(G) \geq \frac{\delta - 3\Delta - 2 + 2(\Delta + 3)q}{\Delta + \delta + 2} n.$$

Proof. Since all vertices of G are odd vertices, note that for at least q of the vertices $v \in V$, $f(N[v]) \geq 2$, i.e., $\sum_{v \in V} f(N[v]) \geq 2qn - (\Delta + 1)(1 - q)n = q(\Delta + 3)n - (\Delta + 1)n$. Similar to the proof of Theorem 1, we can complete the proof of Theorem 2. ∎

By Theorem 1 and Theorem 2, we can obtain Theorem 3(See [1]).

Theorem 3 *Let c, d be positive integers such that $gcd(c, d) = 1$ and $0 < q = \frac{c}{d} \leq 1$. For every r-regular graph $G = (V, E)$ of order n,*

$$\gamma_q(G) \geq \begin{cases} (q\frac{r+3}{r+1} - 1)n & for\ r\ odd \\ (q\frac{r+2}{r+1} - 1)n & for\ r\ even. \end{cases}$$

Corollary 3.1[5] For every r-regular graph $G = (V, E)$ of order n,

$$\gamma_s(G) \geq \begin{cases} \frac{2n}{r+1} & for\ r\ odd \\ \frac{n}{r+1} & for\ r\ even. \end{cases}$$

Corollary 3.2[6] For every r-regular graph $G = (V, E)$ of order n,

$$\gamma_{maj}(G) \geq \begin{cases} \frac{1-r}{2(r+1)} n & for\ r\ odd \\ \frac{-r}{2(r+1)} n & for\ r\ even. \end{cases}$$

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