Constructions of some classes of neighbor balanced designs

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Abstract: Some classes of neighbor balanced designs in two-dimensional blocks are constructed. Some of these designs are statistically optimal and others are highly efficient when errors arising from units within each block are correlated.

Key words: Block design; efficiency; finite field; method of differences; neighbor balance.

1. Introduction

Let there be bpq experimental units which are grouped into b blocks each comprising a $p \times q$ arrangement of plots (units) where both p and q are greater than 1. A design for v treatments for these units is said to be connected if it permits all elementary treatment contrasts to be estimated under a standard linear model for the design. For given v, b, p, q, we denote the class of all such connected designs by $D(v, b, p, q)$.

The problems of optimality and construction of designs in $D(v, b, p, q)$ when observations within blocks are correlated have been addressed by several researchers, see [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 17, 18, 19, 20, 21], for example. In these papers, universally optimal and highly efficient two dimensional designs have been determined and constructed for some design parameters under certain error processes in conjunction with simplifying model assumptions. The present paper addresses the optimality and construction problems under the following block effects model with the autonormal error process

$$Y_d = X_d\tau + Z\beta + \epsilon, \quad \text{cov}(\epsilon) = \Sigma, \tag{1.1}$$

where Y_d is the response vector, τ is the vector of treatment effects, X_d is an $bpq \times v$ plot-treatment design matrix that defines the allocation of treatments to units, β is the $b \times 1$ vector of fixed block effects, and $Z = I_b \otimes J_{pq\times 1}$ is the plot-block incidence matrix. Note that the model has no parameters for rows and/or columns within blocks. The error covariance matrix Σ considered here is given by

$$\sigma^2\Sigma^{-1} = I_b \otimes (I_{pq} - \alpha_1(I_p \otimes H_q) - \alpha_2(H_p \otimes I_q) - \alpha_3(H_p \otimes H_q)).$$

Here the error covariance matrix parameters α_1, α_2, and α_3 are assumed positive. Note that the error correlations arising from units within each block are functions of these parameters. Furthermore, for Σ to be positive definite, α_1, α_2, and α_3 must satisfy

$$\alpha_2 \cos\frac{\pi}{p+1} + \alpha_1 \cos\frac{\pi}{q+1} + 2\alpha_3 \cos\frac{\pi}{p+1}\cos\frac{\pi}{q+1} < \frac{1}{2}.$$

The square matrices H_p and H_q of orders p and q, respectively, are

$$(H_p)_{ll'} = \begin{cases} 1, & \text{if } (l-l') = \pm 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad (H_q)_{ll'} = \begin{cases} 1, & \text{if } (l-l') = \pm 1 \\ 0, & \text{otherwise.} \end{cases}$$

The generalized least squares information matrix C_d for estimation of treatment contrasts under (1.1) is

$$C_d = X_d'\Sigma^{-1}X - X'\Sigma^{-1}Z(Z'\Sigma^{-1}Z)^{-}Z'\Sigma^{-1}X, \qquad (1.2)$$

where $(Z'\Sigma^{-1}Z)^{-}$ denotes a generalized inverse of $Z'\Sigma^{-1}Z$. To simplify C_d for an arbitrary design d, we introduce the following notations:

c_{di}^j = replication of treatment i in the four corner plots of block j, the plots $(1,1), (1,q), (p,1), (p,q)$ being the corner plots in a $p \times q$ block,

e_{dri}^j = replication of treatment i in the $2(q-2)$ edge plots of the first and last rows of block j, these edge plots being the plots $(1,u)$ and (p,u) for $u = 2,3,\ldots,q-1$ in a $p \times q$ block,

e_{dci}^j = replication of treatment i in the $2(p-2)$ edge plots of the first and last columns of block j, these edge plots being the plots $(u',1)$ and (u',q) for $u' = 2,3,\ldots,p-1$ in a $p \times q$ block,

m_{di}^j = replication of treatment i in the $(p-2)(q-2)$ interior plots of block j, the plots (u',u) for $u' = 2,3,\ldots,p-1$, $u = 2,3,\ldots,q-1$ being the interior plots,

r_{di}^j = replication of treatment i in block j,

$n_{dii'}^r$ = the number of times that treatments i and i' occur as row neighbors in the b blocks,

$n_{dii'}^c$ = the number of times that treatments i and i' occur as column neighbors in the b blocks,

$n^d_{dii'}$ = the number of times that treatments i and i' occur as diagonal neighbors in the b blocks,

$w = pq - 2p(q-1)\alpha_1 - 2q(p-1)\alpha_2 - 4(p-1)(q-1)\alpha_3$,

$y^j_{di} = (1-\alpha_1-\alpha_2-\alpha_3)c^j_{di}+(1-2\alpha_1-\alpha_2-2\alpha_3)e^j_{dri}+(1-\alpha_1-2\alpha_2-2\alpha_3)e^j_{dci} +(1-2\alpha_1-2\alpha_2-4\alpha_3)m^j_{di}$.

With these notations, the (i,i')-th element $c_{dii'}$ of the matrix C_d given by (1.2) simplifies as follows:

$$c_{dii} = \sum_{j=1}^{b} r^j_{di} - 2\alpha_1 n^r_{dii} - 2\alpha_2 n^c_{dii} - 2\alpha_3 n^d_{dii} - \frac{1}{w}\sum_{j=1}^{b}(y^j_{di})^2, \qquad (1.3)$$

$$c_{dii'} = -\alpha_1 n^r_{dii'} - \alpha_2 n^c_{dii'} - \alpha_3 n^d_{dii'} - \frac{1}{w}\sum_{j=1}^{b}(y^j_{di})(y^j_{di'}), \quad i \neq i'. \qquad (1.4)$$

It follows from Proposition 1 of Kiefer's [3] that, under model (1.1) with Σ as specified, a design d^* is universally optimal in $D(v,b,p,q)$ if

(i) trace$(C_{d^*}) \geq$ trace(C_d) for all $d \in D$ and

(ii) C_{d^*} is completely symmetric with the diagonal elements all equal and the off-diagonal elements all equal.

Using equations (1.3) and (1.4) in conjunction with Kiefer's [3] sufficient conditions (i) and (ii), we state the following theorem.

Theorem 1.1. *A design d^* is universally optimal in $D(v,b,p,q)$ for generalized least squares estimation of treatment contrasts under* (1.1) *if*

(a) $\sum_{i=1}^{v}(\alpha_1 n^r_{d^*ii} + \alpha_2 n^c_{d^*ii} + \alpha_3 n^d_{d^*ii}) \leq \sum_{i=1}^{v}(\alpha_1 n^r_{dii} + \alpha_2 n^c_{dii} + \alpha_3 n^d_{dii})$ *for all* $d \in D(v,b,p,q)$,

(b) $\sum_{j=1}^{b}\sum_{i=1}^{v}(y^j_{d^*i})^2 \leq \sum_{j=1}^{b}\sum_{i=1}^{v}(y^j_{di})^2$ *for all* $d \in D(v,b,p,q)$,

(c) *for all* $i \neq i'$, $n^r_{d^*ii'} = \lambda_r$, $n^c_{d^*ii'} = \lambda_c$, $n^d_{d^*ii'} = \lambda$,

(d) $\sum_{j=1}^{b}(y^j_{d^*i})(y^j_{d^*i'}) = y$, *a constant, for all* $i \neq i'$. ◇

The conditions (a) and (b) are needed for the maximal trace condition (i) above, and (c) and (d) give the complete symmetry condition (ii). We refer to conditions (a) and (c) as *neighbor conditions* and conditions (b) and (d) as *replication* conditions. For the purpose of constructions, the optimality conditions (a) – (d) need to be expressed more simply. For $v \geq 4$, condition (a) is satisfied if no treatment is neighbored by itself in rows, in columns, and in diagonals. For all $v \geq 2$, condition (b) is satisfied if, for all $i \neq i'$,

$$\begin{aligned} y^j_{d^*i} &= \bar{y} \\ &= \sum_{i=1}^{v}\sum_{j=1}^{b} y^j_{di}/(bv) \\ &= (1-2\alpha_1-2\alpha_2-4\alpha_3)pq/v(\alpha_1+2\alpha_3)2p/v \\ &\quad + (\alpha_2+2\alpha_3)2q/v-(4/v)\alpha_3 \\ &= w/v. \end{aligned}$$

For all i and j, the value $y^j_{d^*i}$ can be made exactly equal to $\bar{y}$ if $v = 2, 4$, and $p \equiv 0 \pmod{v/2}$, $q \equiv 0 \pmod{v/2}$. These two special cases are considered in Section 2.

For all other p, q, and v, the value y^j_{di} cannot be made exactly equal to $\bar{y}$ for all i and j. Then, expressing the replication conditions even more simply and constructing universally optimal designs are hardly possible without some restrictions on the design parameters and/or on the class of competing designs. We address this problem in section 3.

2. Optimal designs for $v = 2$ and $v = 4$

Consider first the class $D(v = 4, b, p = 2m, q = 2n)$ where m and n are any positive integers. Then condition (a) of Theorem 1.1 is satisfied by a design d if

(e) $n^r_{dii} = n^c_{dii} = n^d_{dii} = 0$ for all i,

and condition (b) with $y^j_{di} = \bar{y}$ (hence (d) too) is satisfied if

(f) $(c^j_i, e^j_{dri}, e^j_{dci}, m^j_{di}) = (1, n-1, m-1, (m-1)(n-1))$ for all i and j.

Hence we state

Theorem 2.1. *Let $m \geq 1$ and $n \geq 1$ be integers. A design d^* is universally optimal in $D(v = 4, b, p = 2m, q = 2n)$ for generalized least squares estimation of treatment contrasts under (1.1), with Σ as specified, if d^* satisfies (c), (e) and (f).* ◇

Universally optimal designs of Theorem 2.1 are given below with $b = 3$ blocks. Define

$$A = J_{m\times 1} \otimes (A_1, A_2, \ldots, A_n) \quad \text{where} \quad A_i = \begin{cases} \begin{pmatrix} 0 & \infty \\ 1 & 2 \end{pmatrix}, & \text{if } i \text{ is odd} \\ \begin{pmatrix} 1 & 2 \\ 0 & \infty \end{pmatrix}, & \text{if } i \text{ is even.} \end{cases}$$

Then by inspection one may verify that the following three blocks (with ∞ as an invariant under addition)

$$A, \quad A + J_{2m\times 2n}(\text{mod } 3), \quad A + 2J_{2m\times 2n}(\text{mod } 3),$$

give a universally optimal design in $D(4,3,2m,2n)$ for every pair of integers $m \geq 1$, $n \geq 1$. If necessary, one may use multiple copies of these blocks to increase the number of error degrees of freedom.

Example 1. Theorem 2.1 designs for some m and n.

For $m = n = 1$, the three blocks are

$$\begin{pmatrix} 0 & \infty \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & \infty \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & \infty \\ 0 & 1 \end{pmatrix}.$$

For $m = 1$, $n = 2$, the three blocks are

$$\begin{pmatrix} 0 & \infty & 1 & 2 \\ 1 & 2 & 0 & \infty \end{pmatrix}, \begin{pmatrix} 1 & \infty & 2 & 0 \\ 2 & 0 & 1 & \infty \end{pmatrix}, \begin{pmatrix} 2 & \infty & 0 & 1 \\ 0 & 1 & 2 & \infty \end{pmatrix}.$$

For $m = 1$, $n = 3$, the three blocks are

$$\begin{pmatrix} 0 & \infty & 1 & 2 & 0 & \infty \\ 1 & 2 & 0 & \infty & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & \infty & 2 & 0 & 1 & \infty \\ 2 & 0 & 1 & \infty & 2 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 2 & \infty & 0 & 1 & 2 & \infty \\ 0 & 1 & 2 & \infty & 0 & 1 \end{pmatrix}.$$

For $m = 2$, $n = 2$, the three blocks are

$$\begin{pmatrix} 0 & \infty & 1 & 2 \\ 1 & 2 & 0 & \infty \\ 0 & \infty & 1 & 2 \\ 1 & 2 & 0 & \infty \end{pmatrix}, \begin{pmatrix} 1 & \infty & 2 & 0 \\ 2 & 0 & 1 & \infty \\ 1 & \infty & 2 & 0 \\ 2 & 0 & 1 & \infty \end{pmatrix}, \begin{pmatrix} 2 & \infty & 0 & 1 \\ 0 & 1 & 2 & \infty \\ 2 & \infty & 0 & 1' \\ 0 & 1 & 2 & \infty \end{pmatrix}.$$

For $v = 2$, no connected design can have $n^r_{dii} = n^c_{dii} = n^d_{dii} = 0$ for all i. Hence to satisfy the condition (a) of Theorem 1.1, it is necessary to know which of the three α's is equal to $\min\{\alpha_1, \alpha_2, \alpha_3\}$. For the second order conditional auto-normal error covariance process considered in this paper, the conditional expectation of any observation given all other observations is a linear combination of the neighboring observations where the weights of row, column, and diagonal neighbor(s) are α_1, α_2, and α_3 respectively (see [2]). Hence it is not unreasonable, especially in agricultural field trials, to assume that both α_1 and α_2 are greater than α_3. Under this assumption, condition (a) of Theorem 1.1 is satisfied if

$$n^r_{dii} = n^c_{dii} = 0, \quad \text{for all } i, \tag{2.1}$$

and condition (b) is satisfied if

$$(c^j_i, e^j_{dri}, e^j_{dci}, m^j_{di}) = (2, q-2, p-2, (p-2)(q-2)) \text{ for all } i \text{ and } j. \quad (2.2).$$

This last condition (2.2) implies that $y^j_{di} = \bar{y}$ for all i and j and hence that condition (d) is satisfied. However, (2.1) and (2.2) cannot be achieved simultaneously if both p and q are odd. Hence

Theorem 2.2. *Let $p \geq 2$ and $q \geq 2$ be integers at least one of which is even. A design d^* is universally optimal in $D(v = 2, b, p, q)$ for generalized least squares estimation of treatment contrasts under* (1.1) *with $\alpha_u > \alpha_3$, $u = 1, 2$ if* (2.1) *and* (2.2) *are satisfied.* ◇

Each block of a universally optimal design from Theorem 2.2 has the following property:

(iii) no treatment is neighbored by itself in rows and in columns.

The condition (2.1) follows immediately, and condition (2.2) from the restriction that at least one of p and q is even.

As noted above, the conditions (2.1) and (2.2) cannot be met simultaneously if both p and q are odd. In this case, we have failed to show that the design d^* having the property (iii) is optimal for all α_1, α_2 and α_3. However, we offer the following result.

Theorem 2.3. *Let $p \geq 3$ and $q \geq 3$ be odd integers. The design d^* having the property (iii) is universally optimal in $D(v = 2, b, p, q)$ under* (1.1) *with $\alpha_i > \alpha_3$, $i = 1, 2$ if $\alpha_1 + \alpha_2 - \alpha_3 \geq \frac{1}{4w}$.* ◇

Proof: It is sufficient to prove the theorem just for $b = 1$ and so ignore the superscript j. For the design d^*, we have

$$\begin{aligned} n^r_{d^*ii} &= n^c_{d^*ii} = 0, \\ n^d_{d^*ii} &= 2(p-1)(q-1), \; i = 1, 2, \\ y_{d^*1} &= \frac{w+1}{2}, \\ y_{d^*2} &= \frac{w-1}{2}. \end{aligned} \tag{2.3}$$

Using (2.3), we obtain $\text{trace}(C_{d^*}) = pq - 4(p-1)(q-1)\alpha_3 - \frac{w^2+1}{2w}$.

Now let $d \in D$ be any design with neighbor properties different from those of d^* described in (2.3). Then d must have $n_{dii}^d \leq 2(p-1)(q-1)-1$, $n_{dii}^r \geq 1$ for least one i, and $n_{dii}^c \geq 1$ for at least one i. Since $v = 2$, a decrease in n_{dii}^d by 1 results in an increase of at least 1 for each of n_{dii}^r and n_{dii}^c. This implies that $\alpha_1 n_{dii}^r + \alpha_2 n_{dii}^c + \alpha_3 n_{dii}^d \geq 2\alpha_1 + 2\alpha_2 + 2(2(p-1)(q-1)-1)\alpha_3$. Thus

$$\begin{aligned}\text{trace}(C_d) &< pq - 2\alpha_1 - 2\alpha_2 - 2(2(p-1)(q-1)-1)\alpha_3 - w/2\\ &\leq \text{trace}(C_{d^*}) \ \text{ if } \ \alpha_1 + \alpha_2 - \alpha_3 \geq \frac{1}{4w}.\end{aligned}$$

This completes the proof. ◇

Combining Theorems 2.2 and 2.3, we see that if $\max(\alpha_1, \alpha_2) > \frac{1}{4w}$, then d^* is universally optimal for all p and q. As we have assumed $\alpha_1 > \alpha_3$ and $\alpha_2 > \alpha_3$, our Theorem 2.3 does not cover values of α_1, α_2, and α_3, if any, for which $\alpha_1+\alpha_2-\alpha_3 \in (\max(\alpha_1, \alpha_2), \frac{1}{4w})$. It is this set of α's for which we have failed to determine a universally optimal design in $D(v, b, p, q)$ when both p and q are odd.

3. Neighbor balanced designs for $v = 3$ and $v \geq 5$

For $v = 3$ and $v \geq 5$, it is hardly possible to construct designs that satisfy all four optimality conditions of Theorem 1. By relaxing the replication condition, various classes of designs that are highly efficient with respect to the universally optimal designs have been introduced in the literature, see [17, 18, 19]. Here we introduce the following class of neighbor balanced designs.

Definition 1. Neighbor Balanced Design. A design $d^* \in D(v, b, p, q)$ is said to be a neighbor balanced design, henceforth referred to as an $NBD(v, b, p, q)$, if it satisfies the neighbor conditions (a) and (c) of Theorem (1.1) and the following condition:

(e) d^* is a balanced block design for v treatments in b blocks each of size $k = pq$.

To construct an NBD for $v = 3$, denote the three treatments by 0, 1, 2. Construct a $p \times q$ block satisfying the condition in (2.1) such that the three treatments appear equally or as nearly equally as possible. Call this block A. Then the three blocks A, $A + 1$ (mod 3), $A + 2$ (mod 3) give an

$NBD(v=3, b=3, p, q)$. As an example, the following three blocks give an $NBD(3,3,4,4)$:

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}.$$

For $v \geq 5$, we concentrate only on the construction of designs with binary blocks for which condition (a) of Definition 1 is satisfied with no treatment neighboring itself in rows, in columns, and in diagonals. For the purpose of our constructions, we let $F_s(x) = \{0, x^0, x^1, \ldots, x^{s-2}\}$ denote the finite field of order s with primitive root x, where s is an odd prime power. Also, we use the notation $BBD(v,b,k)$ for a balanced block design for v treatments in b blocks of size k each. A $BBD(v,b,k)$ is a $BIBD(v,b,k)$ whenever $k < v$. Furthermore, we use the convention that $\infty + x = \infty$ for all finite x.

We utilize the method of differences (see [14], Chapter 2), to construct an NBD with binary blocks. Using this technique, one needs to construct a set of initial blocks such that the differences arising from the elements of these initial blocks satisfy some conditions. If $v = 2m+1$ is a prime or a prime power, then m initial binary blocks of an $NBD(v, b = mv, p, q)$ can always be constructed for all $p \geq 2$ and $q \geq 2$ such that $pq \leq v$. We state this result in the following lemma.

Lemma 1. *Let $v = 2m+1$ ($m \geq 2$) be a prime or prime power. Let A be a $p \times q$ block of pq distinct elements of $F_v(x)$. Then the blocks $A_i = x^{i-1}A$, $i = 1, 2, \ldots, m$, are the m initial blocks of an $NBD(v, b = v(v-1)/2, p, q)$ with $\lambda_r = p(q-1)$, $\lambda_c = q(p-1)$, and $\lambda_d = 2(p-1)(q-1)$.* ◇

Neighbor balanced designs for some non-prime power numbers of treatments in $2 \times v/2$ complete blocks can be found in [13]. Neighbor balanced designs in 2×2 blocks are balanced incomplete block designs with nested rows and columns and are constructed in [15]. Here we offer a method for constructing NBDs for $v = s+1$ treatments in $p \times q$ blocks for some other values of p and q where s is an odd prime power. We describe our method in the following theorem.

Theorem 3.1. *Let s be an odd prime or prime power. Let m_1, u, p, and q be positive integers such that $m_1(s+1) = upq$. Suppose that $p \times q$ blocks $A_1, A_2, \ldots, A_u$ of the elements of $\{F_s(x) \cup \infty\}$ can be constructed satisfying the following conditions:*

(*iv*) each block is an arrangement of pq distinct elements of $\{F_s(x) \cup \infty\}$, (*v*) The symbol ∞ appears in exactly m_1 blocks where $1 \le m_1 \le u$,

(*vi*) the symmetric differences arising from the finite elements of all blocks are each nonzero elements of $F_s(x)$ exactly $\lambda = m_1(pq-1)$ times.

(*vii*) the symmetric finite row, column, and diagonal neighbor differences arising from all u blocks are, respectively, $\lambda_r = 2up(q-1)/(s+1)$, $\lambda_c = 2u(p-1)q/(s+1)$, and $\lambda_d = 4u(p-1)(q-1)/(s+1)$ copies of the nonzero elements of $F_s(x)$.

(*viii*) The symbol ∞ is neighbored by λ_r, λ_c, and λ_d finite elements in rows, columns, and diagonals, respectively.

Then the following us blocks A_{ij}, $i = 1, 2, \ldots, u$, $j = 1, 2, \ldots, s$, form an NBD for $v = s+1$ treatments in $b = us$ binary blocks each of size $p \times q$.

$$A_{ij} = \begin{cases} A_i, & i = 1, 2, \ldots, u;\ j = 1 \\ A_i + x^{j-1} J_{p\times q}, & i = 1, 2, \ldots, u;\ j = 2, \ldots, s. \end{cases}$$

Here we use the convention that $\infty + x^{j-1} = \infty$ for all j.

Proof: By construction. The conditions (*iv*) – (*vi*) imply that blocks $A_1, A_2, \ldots, A_u$ are initial blocks of the BIB design with $\lambda = m_1(pq-1)$. The conditions (*vii*) and (*viii*) guarantee that the design is balanced for nearest row, column, and diagonal neighbors. ◇

As an application of the above theorem, we have the following corollary.

Corollary 1. *Let $s = 4m+1$ $(m \ge 2)$ be a prime or prime power. Define $2m+1$ blocks each of size 2×3 as follows.*

$$A_1 = \begin{pmatrix} \infty & 1 & 1+x^m \\ 2 & x^{2m} & 1-x^m \end{pmatrix}, \quad A_{m+1} = \begin{pmatrix} \infty & x^m & x^m(1+x^m) \\ 2x^m & x^{3m} & x^m(1-x^m \end{pmatrix},$$

$$A_{2m+1} = \begin{pmatrix} x^{2m} & \infty & x^{3m} \\ 1 & 0 & x^m \end{pmatrix},$$

$$A_{i+1} = \begin{pmatrix} 0 & 1 & 1+x^m \\ 2 & x^{2m} & 1-x^m \end{pmatrix} x^i,\ i = 1, \ldots, 2m-1,\ i \ne m.$$

Then the blocks $A_1, A_2, \ldots, 2m+1$ are initial blocks of an $NBD(s+1, s(s+1)/2, 2, 3)$ with $\lambda_r = 4$, $\lambda_c = 3$, and $\lambda_d = 4$. ◇

Proof: Denote the $s+1$ treatments by the $s+1$ elements of $\{F_s(x) \cup \infty\}$. We first show that the blocks $A_1, A_2, \ldots, A_{2m+1}$ are initial blocks of a BIB design. More specifically, we must show that the symmetric differences arising from the finite elements of the $2m+1$ initial blocks are 15 copies of the nonzero elements of $F_s(x)$. To see this, first consider the initial blocks

A_1 and A_{m+1}. Replace the symbol ∞ by zero in A_1 and A_{m+1} and denote the resulting blocks by B_1 and B_{m+1}, respectively. Then, utilizing $x^{2m} = -1 = s - 1$, the symmetric differences arising from within-block elements of $B_1, A_2, \ldots, A_m, B_{m+1}, A_{m+2}, \ldots, A_{2m}$ are

$$\pm \{1, 2, s-1, 1+x^m, 1-x^m, s-1, 2, x^m, x^m, 3, 1-x^m, 1+x^m, x^m+2, 2-x^m, 2x^m\} \otimes \{x^0, x^1, x^2, \ldots, x^{2m-1}\}$$

which are each nonzero element of $F_s(x)$ exactly 15 times. Hence the symmetric differences arising from within-block finite elements of initial blocks $A_1, A_2, \ldots, A_{2m}$ are

(15 copies of the nonzero elements of $F_s(x)$) except

$\pm\{1, 2, s-1, (1+x^m), (1-x^m), x^m, 2x^m, x^m, (1-x^m), 1+x^m)\}$.

The symmetric differences arising from the finite elements of the last initial block A_{2m+1} are

$$\pm\{1, 2, s-1, 1+x^m, 1-x^m, x^m, 2x^m, x^m, 1-x^m, 1+x^m\}.$$

So the differences arising from within-block finite elements of initial blocks $A_1, A_2, \ldots, A_{2m+1}$ are 15 copies of the nonzero elements of $F_s(x)$, and hence $\lambda = 15$ for pairs of finite elements. As ∞ appears in 3 blocks, which have 15 finite elements between them, every pair of the form (∞, f), $f \in F_s(x)$ also appears 15 times. Thus the *BIB* design condition is satisfied.

Similarly, we see that the row, column, and diagonal neighbor differences arising from finite elements of $A_1, A_2, \ldots, A_{2m+1}$ are $\lambda_r = 4$, $\lambda_c = 3$ and $\lambda_d = 4$ copies, respectively, of the nonzero elements of $F_s(x)$. Thus the neighbor balanced condition is satisfied, as the symbol ∞ is neighbored by $\lambda_r = 4$ finite elements in rows, $\lambda_c = 3$ finite elements in columns, and $\lambda_d = 4$ finite elements in diagonals. Hence the design generated from the initial blocks $A_1, A_2, \ldots, A_{2m+1}$ is an *NBD* design. ◇

Example 2. Corollary 1 design for $v = 14$ in 2×3 blocks.

Here $s = 13$, $m = 3$. The 7 initial blocks each of size 2×3 are as follows.

$$\begin{pmatrix} \infty & 1 & 9 \\ 2 & 12 & 6 \end{pmatrix}, \begin{pmatrix} \infty & 8 & 7 \\ 3 & 5 & 9 \end{pmatrix}, \begin{pmatrix} 12 & \infty & 5 \\ 1 & 0 & 8 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 5 \\ 4 & 11 & 12 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 4 & 10 \\ 8 & 9 & 11 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 1 \\ 6 & 10 & 5 \end{pmatrix}, \begin{pmatrix} 0 & 6 & 2 \\ 12 & 7 & 10 \end{pmatrix}.$$

The design constructed in corollary 1 is in fact universally optimal under the block effects model (1.1) for at least two error covariance processes; doubly geometric and conditional autonormal error processes [19]. One may verify that the corner design (whose blocks consist of four corner plots) and the edge design (whose blocks consist of two edge plots) of this series of designs are BIB designs. Hence all the universal optimality conditions of Theorems 2.1 and 2.2 given in [19] are satisfied. Corollary 1 designs are therefore universally optimal under the conditions stated in [19].

Theorem 2.1 reduces the construction problem of our NBDs to construction of u initial blocks. However, this latter construction is tedious and challenging. We offer designs in 2×3 blocks in our corollary 1. Even blocks of size $2 \times q$, $4 \leq q < v/2$, are not known for many design parameters. However, the initial blocks of Lemma 1 can be utilized, in some cases, to construct initial blocks of Theorem 2.1. This is illustrated in the following example.

Example 3. $s = 11, v = 12, p = 2, q = 4.$

For $s = 11$ treatments, take the five 2×4 initial blocks of Lemma 1 as follows.

$$\begin{pmatrix} 0 & 1 & 3 & 2 \\ 5 & 6 & 9 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 6 & 4 \\ 10 & 1 & 7 & 8 \end{pmatrix}, \begin{pmatrix} 0 & 4 & 1 & 8 \\ 9 & 2 & 3 & 5 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 8 & 2 & 5 \\ 7 & 4 & 6 & 10 \end{pmatrix}, \begin{pmatrix} 0 & 5 & 4 & 10 \\ 3 & 8 & 1 & 9 \end{pmatrix}.$$

These are the initial blocks of an $NBD(11, 55, 2, 4)$. To illustrate how these initial blocks for 11 treatments can be used to construct initial blocks of an NBD for 12 treatments in 2×4 blocks, denote the 12th treatment by ∞ and choose $m_1 = 4$ and $u = 6$. Replace four elements in four of the above initial blocks by ∞ in such a way that the symbol ∞ is neighbored by λ_r, λ_c, and λ finite elements in rows, columns, and diagonals, respectively. The resulting initial blocks are

$$\begin{pmatrix} \infty & 1 & 3 & 2 \\ 5 & 6 & 9 & 4 \end{pmatrix}, \begin{pmatrix} \infty & 2 & 6 & 4 \\ 10 & 1 & 7 & 8 \end{pmatrix}, \begin{pmatrix} 0 & \infty & 1 & 8 \\ 9 & 2 & 3 & 5 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 8 & \infty & 5 \\ 7 & 4 & 6 & 10 \end{pmatrix}, \begin{pmatrix} 0 & 5 & 4 & 10 \\ 3 & 8 & 1 & 9 \end{pmatrix}.$$

Insertion of ∞ in some units that had finite elements upsets the row, column, and diagonal neighbor difference counts of finite elements. Also the BIB design difference counts are now unbalanced. All these lost differences can be recovered using an extra 2×4 block with eight finite elements. This extra block can be constructed using a computer program. We used

SAS PROC IML (SAS Interactive Matrix Language Procedure) program to obtain the block

$$\begin{pmatrix} 0 & 1 & 6 & 9 \\ 2 & 5 & 7 & 3 \end{pmatrix}.$$

The six blocks together satisfy all conditions of our Theorem 3.1. This theorem does not give initial blocks of an NBD; it gives only a general description of our initial blocks. The above example illustrates how these initial blocks can be constructed utilizing initial blocks from Lemma 1. The procedure works well for designs in $2 \times q$ binary blocks for small v. Some designs in 2×4, 2×5 blocks constructed this way can be found in [16].

4. Efficiency of Neighbor balanced Designs.

Here we consider the A-efficiency under (1.1), with Σ as specified, of the neighbor balanced designs considered in this paper. Let $\mu_{d1}, \ldots, \mu_{d(v-1)}$ be the $v-1$ nonzero eigenroots of the information matrix C_d of a design $d \in D(v,b,p,q)$. A hypothetical universally optimal design d^* would have $\mu_{d^*1} = \mu_{d^*2} = \ldots = \mu_{d^*(v-1)} = \text{trace } (C_{d^*})/(v-1)$. Then a lower bound of the A-efficiency under (1.1), with Σ as specified, of a design d with respect to the hypothetical universally optimal design d^* is given by

$$A\text{-efficiency} \geq \frac{(v-1)/\sum_{i=1}^{v-1}(\frac{1}{\mu_{di}})}{\text{trace}(C_{d^*})/(v-1)}.$$

The lower bound of this A-efficiency of the design from example 3 is displayed in Table 1 for some combinations of α_1, α_2 and α_3. Table 2 shows this lower bound for the $NBD(3,3,4,4)$ presented in the beginning of section 3. Similar numerical efficiency results can be found in [13,16] for some other NBDs. Note that most of these efficiencies in Tables 1 and 2 are more than 95%.

Table 1. A-efficiency Lower bound of the design of example 3.

α_1	α_2	α_3	A-Efficiency	α_1	α_2	α_3	A-Efficiency
0.1	0.1	0.05	0.969	0.2	0.4	0.05	0.981
0.1	0.2	0.05	0.973	0.2	0.4	0.10	0.974
0.1	0.3	0.05	0.977	0.2	0.4	0.15	0.945
0.1	0.4	0.05	0.981	0.2	0.5	0.05	0.978
0.1	0.5	0.05	0.985	0.2	0.5	0.10	0.941
0.1	0.6	0.05	0.988	0.3	0.1	0.05	0.973
0.1	0.7	0.05	0.987	0.3	0.2	0.05	0.974
0.2	0.1	0.05	0.973	0.3	0.2	0.10	0.966
0.2	0.2	0.05	0.976	0.3	0.2	0.15	0.943
0.2	0.2	0.10	0.976	0.3	0.3	0.05	0.970
0.2	0.2	0.15	0.974	0.3	0.3	0.10	0.945
0.2	0.3	0.05	0.979	0.3	0.4	0.05	0.945
0.2	0.3	0.10	0.977	0.4	0.1	0.05	0.961
0.2	0.3	0.15	0.970	0.4	0.2	0.05	0.943

Table 2. A-efficiency Lower bound of $NBD(3,3,4,4)$.

α_1	α_2	α_3	A-Efficiency
0.1	0.1	0.05	0.949
0.1	0.2	0.05	0.952
0.1	0.3	0.05	0.954
0.1	0.4	0.05	0.952
0.2	0.1	0.05	0.952
0.2	0.2	0.05	0.954
0.2	0.2	0.1	0.909
0.2	0.3	0.05	0.952
0.3	0.1	0.05	0.954
0.3	0.2	0.05	0.952
0.4	0.1	0.05	0.952

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