3-RESTRICTED EDGE CONNECTIVITY OF VERTEX TRANSITIVE GRAPHS[1]

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Abstract A 3-restricted edge cut is an edge cut that disconnects a graph into at least two components each having order at least 3. The cardinality λ_3 of minimum 3-restricted edge cuts is 3-restricted edge connectivity. Let G be a connected k-regular graph of girth $g(G)$ at least 4 and order at least 6. Then $\lambda_3 \leq 3k-4$. It is proved in this paper that if G is a vertex transitive graph then either $\lambda_3 = 3k-4$ or λ_3 is a divisor of $|G|$ such that $2k-2 \leq \lambda_3 \leq 3k-5$ unless $k=3$ and $g(G)=4$. If $k=3$ and $g(G)=4$, then $\lambda_3 = 4$. The extreme cases where $\lambda_3 = 2k-2$ and $\lambda_3 = 3k-5$ are also discussed.

Key words Graph, Vertex transitive, Restricted edge connectivity, Fragment

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1 Introduction

All graphs considered in this paper are undirected, connected, finite, simple, k-regular ($k \geq 3$) of order $\nu(G) \geq 6$. An m-restricted edge cut is an edge cut that disconnects a connected graph into components each having order at least m. The m-restricted edge connectivity λ_m is the cardinality of a minimum m-restricted edge cut. Simplify m-restricted edge cut and m-restricted edge connectivity as R_m-edge cut and R_m-edge connectivity respectively. These concepts generalize the restricted edge cut and restricted edge connectivity, proposed first by Harary [1] in 1983 and then studied by Esfahanian and Hakimi in [2]. As was pointed out in [2], m-restricted edge

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connectivity is a more precise measure for the fault-tolerance and reliability of networks than the traditional measures such as edge connectivity, and has been studied in [3-5].

Let X be a subset of $V(G)$ or a subgraph of graph G. Then $G \setminus X$ denotes the graph obtained by removing the vertices in X from G, $G \setminus \{w\}$ is simplified as $G \setminus w$. Let $\partial(X)$ denote the degree of X, namely the number of edges with one end in X and the other in $G \setminus X$. Let $\xi_3(G) = \min\{\partial(X) : X$ is a connected vertex-induced subgraph of order 3 $\}$. Then $\lambda_3(G) \leq \xi_3(G)$. A graph G is called maximal 3-restricted edge connected if the the last inequality becomes equality. Let $g(G)$ indicate the girth of graph G, namely the length of the shortest cycle. We present the following result in this paper.

Theorem 3.1 Let G be a connected k-regular vertex transitive graph of order at least 6. Then

(1) If $g(G) \geq 5$, then G is maximal 3-restricted edge connected.

(2) If $g(G) = 4$ and $k \geq 4$, then either graph G is maximal 3-restricted edge connected or $\lambda_3(G)$ is a divisor of $|G|$ such that $2k-2 \leq \lambda_3(G) \leq 3k-5$.

(3) If $g(G) = 4$ and $k = 3$, then $\lambda_3(G) = 4$.

Theorem 3.2 gives a necessary and sufficient condition for $\lambda_3(G) = 2k-2$. In section 3 we also describe a class of graphs with $\lambda_3(G) = 3k-5$. Theorem 3.1 generalize Mader's result on the edge connectivity of vertex transitive graphs [6], which states that k-regular connected vertex transitive graphs are k-edge connected; on the other hand, it also strengthens Xu Junming's [8] recent result which states that connected K_3-free k-regular vertex transitive graphs are maximal R_2-edge connected if $k \geq 3$.

Before proving these results, we introduce some more notation and terminology. The two components resulting from the removal of a minimum R_3-edge cut S from G are called R_3-fragments or simply fragments of G corresponding to S, the smaller one (with less vertices) is called normal fragment. If we denote by X one of the two fragments of G corresponding to S, then the other one is denoted by X^c. We often describe a fragment by its vertex set. Fragments of smallest order are atoms. It is worth noting that fragments defined here are connected vertex-induced subgraphs that correspond to minimum R_m-edge cut and appear in pairs.

Denote by $\varepsilon(G)$ the size of graph G. Let A and B be two disjoint subsets of $V(G)$, then $[A, B]$ represents the set of edges with one end in A and the other in B, we often simplify $[\{v\}, B]$ as $[v, B]$ and $[A, V(G) - A]$ as $I(A)$. The cardinality of set A is denoted by $|A|$. A k-regular graph G is vertex-transitive if for any two vertices u and v of G, there is an automorphism τ in $Aut(G)$ such that $\tau(u) = v$, where $Aut(G)$ is the automorphism group of graph G. Let S be a subset of $E(G)$, then $G - S$ represents the graph obtained by removing the edges of S from G but preserving their end points. For other terminology we follow [7].

2 Auxiliaries

A flower F is a connected graph of order at least 6 that contains a cut vertex w such that no component of $F \setminus w$ has order more than 2. We refer to the vertex w as its stamen and the components of $F \setminus w$ as its petals. It is easy to see that every flower has only one stamen and at least three petals.

Lemma 2.1 Let G be a connected graph with $\nu(G) \geq 7$. If G is not a flower, then G contains R_3-edge cuts.

Proof Since G is connected, $\varepsilon \geq \nu - 1$. We prove at first that the lemma is true when $\varepsilon = \nu - 1$. In this case G is a tree. Every edge is thus a bridge. Since G is not a flower, it contains either a path of length at least 5 or at least two vertices of degree 3 each. In the first case, let $e = uv$ be an edge of the path such that the vertex u and v are at distance at least 2 from the ends of the path. In the second case, let e be an edge on the path that connects the two vertices of degree at least 3. In either case, every components of $G - e$ has order at least 3. Therefore edge e forms an R_3-edge cut of G.

Inductively accept the truth of the lemma when $\varepsilon(G)$ is smaller. When $\varepsilon(G)$ is larger than $\nu - 1$, we are going to verify that there is an edge e in G such that $G - e$ is not any flower. By induction, there is an R_3-edge cut T of $G - e$ such that $T \cup \{e\}$ is an R_3-edge cut of G.

If there is no such edge in G, then for any given edge e that is not cut edge, $G - e$ is a flower with stamen v and petals $G_i, i = 1, 2, 3, ..., p, p \geq 3$. Edge e must join two petals of $G - e$, say G_1 and G_2, since G is not a flower and the union of G_1, G_2 and edge e is the only possible component of $G \setminus v$ different from that of $\{G - e\} \setminus v$. Since $\nu(G_1) + \nu(G_2) \geq 3$, $\nu(G_1) \leq 2$

and $\nu(G_2) \leq 2$, at least one equality in the last two inequalities holds, say $\nu(G_1) = 2$. It is not difficult to see that e and v are contained in some cycle C.

Claim firstly that v has only one neighbor in G_1 and G_2 respectively. If otherwise G_1 contains two neighbors a and b of v, then $G - av$ is not any flower since no vertex of G is the stamens x (To see this, we need only understand that either the component of $Q = \{G - av\} \setminus x$ that contains G_3 or the component that contains G_1 and G_2 has order more than 3), which is a contraction. Let f be the edge of C joining v to G_1, put $G - f = H$. Then H is a flower with stamen w by the hypothesis.

Claim secondly that w must be the unique vertex u in G_2 that is incident with v. Since if w is not in G_2, then either the component of $H \setminus w$ containing G_1 and G_2 has order at least 3 or the component of $H \setminus w$ containing $G_i, i \neq 1$, has order at least 3. If w is vertex of G_2 not incident with v, then the component of $H \setminus w$ containing vertices u, v and subgraph G_3 has order at least 3. In either case, H is not any flower, this contradiction confirm our claim.

Since H is a flower with stamen $w = u$, $(G - e) \setminus v$ contains exactly three components with G_3 being an isolated vertex (since $G_i, i \geq 3$, are contained in the same component of $H \setminus w$ as vertex v). Therefore

$$\nu(G) = \nu(G_1) + \nu(G_2) + \nu(G_3) + 1 \leq 6 < 7 \leq \nu(G)$$

This contraction completes the proof. □

In the rest of this paper we restrict ourselves to k-regular connected vertex transitive graphs with $k \geq 3$ and $g(G) \geq 4$. By Lemma 2.1, this kind of graphs contains R_3-edge cuts. Since every connected subgraph of order 3 of such graphs is a path of length 2, it follows that $\xi_3(G) = 3k - 4$.

Lemma 2.2[6] k-regular connected vertex transitive graph is k-edge connected.

Lemma 2.3[8] For every K_3-free, k-regular, connected, vertex transitive graph with $k \geq 3$, $\lambda_2(G) = 2k - 2$.

Lemma 2.4 $\lambda_2(G) \leq \lambda_3(G) \leq \lambda_3(G)$.

Proof Since every R_3-edge cut is an R_2-edge cut, the first inequality of lemma 2.4 is true. Let P be a path of G with length 2, and G_1 be a component of $G \setminus P$. If G_1 is an isolated vertex or an isolated edge, then $g(G) = 3$. This contradiction implies that $I(G_1)$ is an R_3-edge cut with size at most $\partial(P) = \xi_3(G)$, and the second inequality thus follows. □

Lemma 2.5 Let X and Y be two fragments of G. $X \cap Y$ is a fragment if the following two conditions hold:

(1) $\nu(X \cap Y) \geq 3$ (2) $\partial(X \cap Y) \leq \lambda_3(G)$

Proof Let X and X^c be the two fragments corresponding to a minimum R_3-edge cut S, Y and Y^c be the two fragments corresponding to another minimum R_3-edge cut T. Define

$$A = X \cap Y, B = X \cap Y^c, C = X^c \cap Y, D = X^c \cap Y^c$$

Since X^c is contained in $G \setminus A$, the order of $G \setminus A$ is not less than 3. In order to complete the proof, we now need only to prove that both A and A^c are connected.

(a) A^c is connected.

Since X^c and Y^c are two connected fragments of G, A^c is connected if D is not empty. If D is empty, we claim that $[B, C]$ is not empty, and A^c is thus also connected. Otherwise $S = [A, C], T = [A, B]$ and $I(A) = S \cup T$. Therefore

$$\partial(A) = |S| + |T| = 2\lambda_3(G) > \lambda_3(G),$$

which is a contradiction to the fact that $X \cap Y$ is a fragment.

(b) A is connected.

Suppose, to the contrary, that A is not connected with components $A_i, i = 1, 2, ..., m$. Then

$$\sum_{i=1}^{m} \partial(A_i) = \partial(A) \leq \lambda_3(G) \leq 3k - 4$$

If $m \geq 3$, then there is some A_j such that $\partial(A_j) \leq k - 1$, and $I(A_j)$ is an edge cut with size less than k, which contradicts Lemma 2.2. Hence, $m = 2$. Since $\nu(A) \geq 3$, A_1 or A_2, say A_1, has order at least 2. Therefore, $I(A_1)$ is a restricted edge cut (R_2-edge cut), from Lemma 2.3 we have $\partial(A_1) \geq 2k - 2$. Hence

$$\partial(A_2) = \partial(A) - \partial(A_1) \leq 3k - 4 - (2k - 2) = k - 2$$

This is a contradiction to Lemma 2.2 and completes our proof. □

Lemma 2.6 Let X and Y be two distinct normal fragments of G. If $\nu(X \cap Y) \geq 3$, then $X \cap Y$ is a fragment.

Proof According to Lemma 2.5, it suffices to prove that $\partial(X \cap Y) \leq \lambda_3(G)$. Let X and Y be the two fragments corresponding to R_3-edge cuts

S and T respectively. Define A, B, C and D as in the proof of Lemma 2.5. Since

$$\nu(A) + \nu(B) = \nu(X) \leq \nu(G)/2 \leq \nu(Y^c) = \nu(B) + \nu(D)$$

Therefore

$$\nu(D) \geq \nu(A) \tag{1}$$

We claim that

$$\partial(D) \geq \lambda_3(G) \tag{2}$$

Formula (2) is obviously true if D is connected since, in this case, $I(D)$ is an R_3-edge cut. If D is not connected with components $D_i, i = 1, 2, ..., p$, then according to lemma 2.2 we have $\partial(D_i) \geq k$. Formula (2) is thus true when $p \geq 3$. When $p = 2$, formula (1) implies that one of the two components of D, say D_1, contains at least two vertices. From Lemma 2.2 and 2.3, we find that $\partial(D_1) \geq 2k - 2$ and $\partial(D_2) \geq k$. Hence

$$\partial(D) = \partial(D_1) + \partial(D_2) \geq 3k - 2 > \lambda_3(G)$$

Our claim is thus true in either case.

Combining (2) and formula

$$\begin{aligned}
\partial(A) + \partial(D) &= |[A, B]| + |[A, C]| + 2|[A, D]| + |[B, D]| \\
&\quad + |[D, C]| + 2|[B, C]| - 2|[B, C]| \\
&= |S| + |T| - 2|[B, C]| \leq |S| + |T| = 2\lambda_3(G)
\end{aligned}$$

we have $\partial(X \cap Y) = \partial(A) \leq \lambda_3(G)$ as desired. □

Since atoms are normal fragments that contain no fragments as their proper subgraphs (i.e. fragments with less vertices), according to Lemma 2.6, we can easily prove the following

Corollary 2.7 If X and Y are two distinct atoms of G, then $\partial(X \cap Y) \leq 2$. □

Lemma 2.8 If G is not maximal R_3-edge connected, i.e. $\lambda_3(G) \neq \xi_3(G)$, then the atoms are disjoint unless $k = 3$ and $\lambda_3(G) = 3k - 5$ and $g(G) = 4$.

Proof Suppose, to the contrary, that X and Y are two atoms of G such that $X \cap Y \neq \emptyset$ and at last one of the three equalities fails. Define A, B, C and D as in the proof of Lemma 2.5. We are going to obtain a contradiction by proving that B or C is a fragment contained in an atom. Since G is not maximal R_3-edge connected, it follows that $\lambda_3(G) \leq 3k-5$. Let H be an arbitrary connected subgraph of G with order 3 or 4. Computing $\partial(H)$, we find that H cannot be a fragment of G unless $k=3$, $g(G)=4$, $\lambda_3(G)=3k-5$ and H is a cycle of length 4. This implies that $\nu(X) \geq 5$ and $\nu(Y) \geq 5$. According to Corollary 2.7, $\nu(X \cap Y) \leq 2$, therefore $\nu(B) \geq 3$ and $\nu(C) \geq 3$. On the other hand, since $\partial(B)+\partial(C) \leq \partial(X)+\partial(Y) = 2\lambda_3(G)$, $\partial(B) \leq \lambda_3(G)$ or $\partial(C) \leq \lambda_3(G)$. Hence, B or C is a fragment by Lemma 2.5. □

Lemma 2.9 Let X be an atom of G. If $\lambda_3(G) \leq 3k-5$, then X is vertex-transitive unless $k=3$ and $g(G)=4$.

Proof Let u and v be two vertices of X. Since G is vertex transitive, there is an automorphism $\tau \in Aut(G)$ such that $\tau(u)=v$. Since $\tau(X)$ is also an atom of G and the intersection of $\tau(X)$ and X is not empty, it follows from Lemma 2.8 that $\tau(X)=X$. Since $\tau|_X$, the restriction of τ on X, is an automorphism of X, X is vertex transitive. □

Lemma 2.10 Let X be a fragment of G. If G is not maximal R_3-edge connected, then $\nu(X) \geq 2k-2$.

Proof Since $g(G) \geq 4$, X contains no triangles. Therefore, $\varepsilon(X) \leq \nu(X)^2/4$. Let $d_X(u)$ denote the degree of vertex u in X. Combining the previous result with $\partial(X)=\lambda_3(G) \leq 3k-5$, we have

$$\begin{aligned}
k\nu(X)-(3k-5) &\leq k\nu(X)-\lambda_3(G)=k\nu(X)-\partial(X) \\
&= \sum_{u\in X} d_X(u) = 2\varepsilon(X) \leq \nu(X)^2/2 \\
&\Rightarrow (\nu(X)-3)(\nu(X)-(2k-3))-1 \geq 0 \\
&\Rightarrow (\nu(X)-3)(\nu(X)-(2k-3)) > 0
\end{aligned}$$

Since G is not maximal R_3-edge connected, it follows that $\nu(X)-3>0$. Hence $\nu(X) \geq 2k-2$. □

3 Main results

Proof Theorem 3.1 A 2-regular connected graph is obviously maximal R_3-edge connected, we thus need only consider the case where $k \geq 3$.

Suppose, to the contrary, that graph G is not maximal 3-restricted edge connected. Let X be an atom of G. According to Lemma 2.10, we have

$$\nu(X) \geq 2k-2 \tag{3}$$

By Lemma 2.9, X is an r-regular vertex transitive graph with $1 \leq r \leq k-1$. This implies

$$\lambda_3(G) = \partial(X) = (k-r)\nu(X) \tag{4}$$

Combining formula (3) with (4), we have

$$(k-r)(2k-2) \leq (k-r)\nu(X) = \partial(X) = \lambda_3(G) \leq 3k-5$$

Therefore $k - r = 1, r = k-1$ and

$$3k-5 \geq \lambda_3(G) = \partial(X) = \nu(X) \tag{5}$$

Let u be an arbitrary vertex of X and $N_X(u) = \{w_1, ..., w_{k-1}\}$ be the neighbors of vertex u in X, let $N_i = N_X(w_i) - \{u\}$. Since $g(G) \geq 5$, the intersection of N_i and N_j is empty whenever $i \neq j$. Hence

$$\nu(X) \geq (k-1)^2 + 1 \tag{6}$$

Combining (5) and (6), we have

$$3k - 5 \geq (k-1)^2 + 1 \Rightarrow$$

$$0 \geq (k-2)(k-3) + 2$$

This contradicts the condition that $k \geq 3$, the first result of the theorem is thus true.

For the second result, we suppose that G is not maximal R_3-edge connected, namely $\lambda_3(G) \leq 3k-5$. Let X be an arbitrary atom of G. Then by Lemma 2.9, X is an r-regular transitive graph. With similar reasoning employed in the proof of the first part, we can show without difficulty that $r = k-1$ and $\partial(X) = \nu(X)$. Since, by Lemma 2.8, the atoms of graph G are disjoint, the union of $\tau(X), \tau \in Aut(G)$, is a spanning subgraph of G. Set $m = |\{\tau(X) : \tau \in Aut(G)\}|$. Then, $\nu(G) = m\nu(X) = m\partial(X) = m\lambda_3(G)$. Thus, $\lambda_3(G)$ is a divisor of $|G| = \nu(G)$. By Lemma 2.3 and 2.4, $2k-2 \leq \lambda_2(G) \leq \lambda_3(G)$. The second result follows.

On the one hand, by Lemma 2.3 and 2.4, we have $4 = 2k-2 = \lambda_2(G) \leq \lambda_3(G)$; on the other hand, for any 4-cycle C of graph G, $G \setminus C$ contains no isolated vertex and isolated edge. Thus $I(C)$ is an R_3-edge cut with $\partial(C) = 4k-8 = 4$, which implies that $\lambda_3(G) \leq 4$. The last result is thus also true and our proof finishes. □

Theorem 3.2 Let G be a connected, k-regular ($k \geq 4$), transitive graph with order at least 6 and girth 4. Then $\lambda_3(G) = 2k - 2$ if and only if G contains $K_{k-1,k-1}$ as its vertex-induced subgraph.

Proof If $X = K_{k-1,k-1}$ is a vertex-induced subgraph of G, then $\lambda_3(G) \leq \partial(X) = 2k - 2$. The sufficiency follows from the combination of this observation with Lemma 2.3 and 2.4.

If $\lambda_3(G) = 2k - 2$, then graph G is not maximal 3-restricted edge connected since $k \geq 4$. Let X be a 3-restricted atom of G. Then X is a $(2k-2)$-regular vertex transitive graph with girth 4 by Lemma 2.9 and 2.10. By the same reasoning employed in the proof of the first result of Theorem 3.1, we can show that $|X| = \partial(X) = 2k - 2$. This implies that

$$2\varepsilon(X) = \sum_{u \in X} d_X(u) = (k-1)(2k-2) \Rightarrow \varepsilon(X) = |X|^2/4$$

Thus, X is the complete bipartite graph of order $2k - 2$. □

To discuss the extreme case where $\lambda_3(G) = 3k-5$, we introduce a class of vertex transitive graphs known as circulants. Denote by $G(n; a_1, a_2, ..., a_h)$ the graph that has vertex set $\{1, 2, ..., n\}$, where vertex j has neighborhood $N(j) = \{j \pm a_1, j \pm a_2, ..., j \pm a_h (\mathrm{mod} n)\}$. Then $G(n; a_1, a_2, ..., a_h)$ is a circulant of order n with jump sequence $(a_1, ..., a_h)$.

Lemma 3.3[9, Proposition 1] The circulant $G(n; a_1, a_2, ..., a_h)$ is connected if and only if $\gcd(a_1, a_2, ..., a_h) = 1$.

Lemma 3.4[10, Theorem 1] Let $G(n; a_1, a_2, ..., a_h)$ be a connected circulant with $h \geq 2$ and $a_h < n/2$. Then $\lambda_2(G) = 4h - 2$.

Let k be a sufficiently large odd integer, H be a $(3k-5)$-cube with vertex set $\{u_i : i = 1, 2, ..., 2^{3k-5}\}$, $G = G(3k-5; a_1, a_2, ..., a_{(k-1)/2})$ be a circulant with jump sequence $(a_1, a_2, ..., a_{(k-1)/2}) = (1, 3, 5, ..., k-2)$. By Lemma 2.2 and 2.3, $\lambda_2(H) = 3k - 5$ and $\lambda(G) = k - 1$. By Lemma 3.3 and 3.4, $\lambda_2(G) = 2k - 4$.

Let G_i be one of the 2^{3k-5} copies of the circulant $G(3k-5; 1, 3, ..., k-2)$ with vertex set $V(G_i) = \{u_{i,1}, ..., u_{i,3k-5}\}$. Since H is a $(3k-5)$-regular bipartite graph, H has a $(3k-5)$ edge coloring $C = (1, 2, ..., 3k-5)$. Substitute G_i for vertex u_i in the graph H. If edge $u_i u_j$ is colored with color m in the coloring C, then join the vertex $u_{i,m}$ to the vertex $u_{j,m}$. Denote the resulting graph by Q, we are going to prove that Q is a k-regular vertex transitive graph with $\lambda_3(G) = 3k - 5$.

Let $u_{i,s}$ and $u_{j,t}$ be arbitrary two vertices of Q. Let τ_i be the permutation on $V(Q)$ such that its restriction on the graph G_i is an automorphism that maps $u_{i,s}$ to $u_{i,t}$, and that $\tau_i(v) = v$ for other vertex v of Q not in G_i. Let σ_{ij} be the automorphism of H that maps vertex i to vertex j. Then $\sigma_{ij}\tau_1\tau_2 \cdots \tau_{3k-5}$ is the automorphism of Q that maps vertex $u_{i,s}$ to $u_{j,t}$. Consequently, Q is a k-regular vertex transitive graph.

Now, one can show that $I(G_i)$ is a 3-restricted edge cut of size $3k-5$, which implies that $\lambda_3(Q) \leq 3k-5$. Let X be an arbitrary 3-restricted fragment of Q. If X is a subgraph of some G_i, $|X| \geq 2k-2$ by Lemma 2.10, we have

$$\begin{aligned}\lambda_3(Q) &= \partial(X) \geq \lambda_2(G_i) + |[X, Q \setminus G_i]| = 2k - 4 + |X| \\ &\geq 4k - 6 > 3k - 5.\end{aligned}$$

If there are at least three G_is with $X \cap G_i \neq \emptyset$, then

$$\lambda_3(Q) = \partial(X) = \sum_i |X \cap G_i| \geq 3\lambda(G_i) = 3k.$$

If X intersects exactly two G_is, then

$$\lambda_3(Q) = \partial(X) \geq \lambda_2(G_i) + \lambda(G_i) = 3k - 2.$$

These observations show that $\lambda_3(Q) \geq 3k-5$. Therefore $\lambda_3(Q) = 3k-5$ as we claim.

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