Average degrees of critical graphs

Xuechao Li
Division of Academic Enhancement
The University of Georgia
Athens, GA 30602
email:xcli@uga.edu

Abstract

Let G be a simple graph with the average degree d_{ave} and the maximum degree Δ. It is proved, in this paper, that G is not critical if $d_{ave} \leq \frac{103}{12}$ and $\Delta \geq 12$. It also improves current result by L.Y. Miao and J.L. Wu [7] on the number of edges of critical graphs for $\Delta \geq 12$.

Key words: edge chromatic number, critical graph.

1 Introduction

A graph is $k-$*edge colorable* if its edges can be colored with k colors in such a way that adjacent edges receive different colors. The *edge chromatic number* of a graph G, denoted by $\chi_e(G)$, is the smallest integer k such that G is $k-$edge colorable. A simple graph G is *class one* if it is $\Delta-$edge colorable, where Δ is the maximum degree of G. Otherwise, Vizing's Theorem [10] guarantees that it is $(\Delta+1)-$edge colorable, in which case, it is said to be *class two*. An edge is called $i-$colored edge if it is colored by color i.

A *critical graph* G is a connected graph such that G is class two and $G-e$ is class one for any edge e of G. Surfaces, in this paper, are compact, connected two manifolds without boundary. Embeddings considered in this paper are 2-cell embeddings. Let S be a surface, denote c_S the Euler characteristic of the surface S. Let V, E and F be the vertex set, edge set and face set of a given embedded graph G, respectively. Let $|V|, |E|$ and $|F|$ be the cardinality of V, E and F of G, respectively. A k-vertex is a vertex of degree of k. We call a vertex z is a i^+-vertex if $i \leq d(z) \leq 8$ ($i \leq 8$). Let $\phi : E(G) \rightarrow C$ be an edge-coloring. Denote by $\phi^{-1}(i)$ the set of edges colored with the color i where $i \in C$. Denote d_{ave} the average degree of G. Let $x \in V(G)$. Denote $N(x)$ the set of vertices adjacent to x. We call a vertex y is a neighbor of x if $y \in N(x)$. For $V' \subseteq V(G)$, let $N(V') = \cup_{x \in V'} N(x)$.

1.1 Main Results

In 1965, Vizing [11] proved that any planar graph of maximum degree at least 8 is class one. In 1968, he [13] also made the following conjecture:

Conjecture 1.1 *(Vizing planar graph conjecture) Every planar graph with maximum degree 6 or 7 is of class one.*

For maximum degree $\Delta \leq 5$, there are graphs of class two.

Vizing planar graph conjecture seems to be very difficult. The case $\Delta = 7$ was recently confirmed independently by Zhang [16] and D. Sanders and Y.Zhao [9]. The case $\Delta = 6$ remains open.

H. Hind and Y.Zhao([3]) and Z.Yan and Y.Zhao([14]) proved that if a graph G can be embedded in a surface S of characteristic $c_S = -5, -4, \ldots, -1, 0$ and $\Delta(G) \geq 11, 11, 10, 9, 9$ and 8, respectively, then G is class one.

The following conjecture was proposed by Vizing [13] concerning the sizes and average degrees of critical graphs.

Conjecture 1.2 *If $G = (V, E)$ is a critical simple graph, then*

$$|E| \geq \frac{1}{2}(|V|(\Delta - 1) + 3).$$

That is, if

$$d_{ave} < (\Delta - 1) + \frac{3}{|V|},$$

then G is not critical.

In 1981, Yap [15] gave some lower bounds on the number of edges of critical graphs with $\Delta \leq 7$.

Theorem 1.3 *(Yap [15] 1981) Let $G = (V, E)$ be a critical graph with the maximum degree Δ.*

(1) If $\Delta = 6$, then $|E| \geq \frac{(9|V|+1)}{4}$;

(2) If $\Delta = 7$, then $|E| \geq \frac{5|V|}{2}$.

Lian-Ying Miao and Jian-liang Wu [7] gave another lower bound:

Theorem 1.4 *(Miao and Wu [7] 2002) Let $G = (V, E)$ be a critical graph.*

If $\Delta \geq 8$, then $|E| \geq 3(|V| + \Delta - 8)$.

Currently, R. Luo and C.Q. Zhang [6] improved the Yap's result:

Theorem 1.5 *(Luo and Zhang [6] 2002) Let $G = (V, E)$ be a critical graph.*

(1) If $\Delta = 8$, then $|E| \geq 3|V| + 1$;

(2) If $\Delta \geq 9$, then $|E| > \frac{10}{3}|V|$.

The following main theorem of this paper is motivated by Conjecture 1.2 and partial results mentioned above.

Theorem 1.6 *Let G be a graph with maximum degree $\Delta \geq 12$ and average degree $d_{ave} \leq \frac{103}{12}$. Then G is not critical.*

The following corollary of the main Theorem 1.6 gives a lower bound on the number of edges of critical graphs with $\Delta \geq 12$.

Corollary 1.7 *Let $G = (V, E)$ be a critical graph with $\Delta \geq 12$, then $|E| > \frac{103}{24}|V|$.*

2 Application

We exclude a special case as following in order to present a theorem which extends the results by Z. Yan and Y. Zhao ([14]) for $c_S \geq -5$.

Lemma 2.1 *Let G be a graph with $\Delta = 12$ and $|V(G)| = 13$ and G can be embedded in a surface S with $c_S \geq -6$. Then G is not critical.*

Proof. Suppose that G is critical. Denote F the set of faces of G embedded in surface S with $c_S \geq -6$. Since G is simple, $|F| \leq \frac{2|E|}{3}$, thus, by Euler formula that $|V|-|E|+|F| \geq -6$, we have $\frac{|E|}{3} \leq |V|+6$. And

$$d_{ave} = \frac{2|E|}{|V|} \leq 6(1 + \frac{6}{|V|}) = 6(1 + \frac{6}{13}) \leq 8.8 \qquad (1)$$

Let $x \in V(G)$ be a 12-vertex and $y \in N(x)$ with $d(y) = d = min\{d(z); z \in N(x)\}$.

If $d = 12$, then $d_{ave} = 12$ which contradicts to equation (1). Hence, we assume that $d < 12$ and x is adjacent to at least $(12-d+1)$ 12-vertices by Lemma 3.1. So, G has at least $(12-d+2)$ 12-vertices. Obviously, the number of 12-vertices is no larger than d (degree of $d(y)$). Thus, we have $12 - d + 2 \leq d$, so, d is at least 7 which leads to $d(z) \geq 7$ for each $z \in N(x)$ by the definition of $d(y)$.

(i) Claim that $d \neq 7$.

If $d = 7$, then, G must have at least seven $12-$vertices.

Thus,

$d_{ave} = \frac{\sum_{v\in V(G)} d(v)}{13} \geq \frac{7\times 12+(13-7)\times 7}{13} > 9.69$ which contradicts to the equation (1).

(ii) Claim that $d \neq 8$.

If $d = 8$, then, G must have at least six 12−vertices.

$d_{ave} = \frac{\sum_{v\in V(G)} d(v)}{13} \geq \frac{6\times 12+(13-6)\times 8}{13} > 9.8$ which contradicts to the equation (1).

By using similar arguments as of in (i), we know that $d \neq 9, 10, 11$.

This contradiction completes the proof. ■

Theorem 2.2 *A simple graph embedded in a surface S with $c_S = -6$ and $\Delta \geq 12$ is class one.*

Proof. Let G be the smallest counterexample to the theorem with respect to the number of edges. Then, G is critical. By the Lemma 2.1, $|V(G)| \geq 14$.

Since G is simple, $|F| \leq \frac{2|E|}{3}$, thus, by Euler formula that $|V| - |E| + |F| = -6$, we have $\frac{|E|}{3} \leq |V| + 6$.

And $d_{ave} = \frac{2|E|}{|V|} \leq 6(1 + \frac{6}{|V|}) \leq 6(1 + \frac{6}{14}) \leq 8.58 \leq \frac{103}{12}$. Then, by Theorem 1.6, G is not critical, a contratiction.

■

3 Some Adjacent Lemmas

The main tools used in the proof of Theorem 1.6 are Vizing's Adjacent Lemma, Zhang's Adjacent Lemma and charge-discharge method. We will first introduce these known results.

Lemma 3.1 *(Vizing's Adjacent Lemma [10]) If H is a critical graph with maximum degree Δ, that is, $\chi_e(H) = \Delta + 1$ and $\chi_e(H - e) = \Delta$ for every edge $e \in E(H)$, and if u and v are adjacent vertices of H, where the degree of v is d, then,* *(i) if $d < \Delta$, then u is adjacent to at least $\Delta - d + 1$ vertices of degree Δ,*

and,

(ii) if $d = \Delta$, then u is adjacent to at least two vertices of degree Δ.

From the Vizing's Adjacent Lemma, one can get the following corollary:

Corollary 3.2 *Let H is a critical graph with maximum degree Δ. Then*

(1) every vertex is adjacent to at most one 2-vertex and at least two Δ-vertices.

(2) the sum of the degree of any two adjacent vertices is at least $\Delta + 2$.

(3) every vertex is adjacent to at least two Δ-vertices.

(4) if a vertex is adjacent to a 2-vertex, then the rest of its neighbors are Δ-vertices.

We define $N(x, y)$ to be the set of all vertices in $N(x) \cup N(y)$ for $x, y \in V(G)$.

Lemma 3.3 *(Zhang's Adjacent Lemma [16]) Let G be critical, $xy \in E(G)$ and $d(x) + d(y) = \Delta + 2$. The following hold:*

(1) every vertex of $N(x, y) \setminus \{x, y\}$ is a Δ-vertex;

(2) every vertex of $N(N(x, y)) \setminus \{x, y\}$ is of degree at least $\Delta - 1$;

(3) if $d(x), d(y) < \Delta$, then every vertex of $N(N(x, y)) \setminus \{x, y\}$ is a Δ-vertex.

Lemma 3.4 *(R.Luo and C.Q. Zhang [6])*

Let G be a critical graph with the maximum degree $\Delta \geq 5$. Assume that there is a 3-vertex x with each of $N(x)$ of degree Δ. Then, there must be a vertex $y \in N(x)$ such that $d(y') \geq \Delta - 1$ for each $y' \in N(y) \setminus \{x\}$.

The following lemma generalizes the Lemma 3.4 and its corollaries will be used in the proof of main theorem.

Lemma 3.5 *Let G be a critical graph with the maximum degree Δ. Let $d(x) = d$ and each of $N(x)$ is of degree Δ. Then there is a vertex $y \in N(x)$ such that there are at most $d-3$ vertices of $N(y) \setminus \{x\}$ with degree $\leq \Delta - d + 1$.*

Proof.

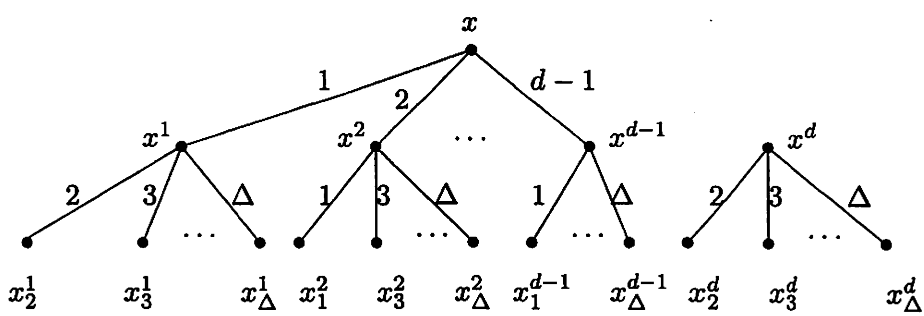
By contradiction, assume that for each vertex $y \in N(x)$, there is at least $d-2$ vertices of $N(y) \setminus \{x\}$ with degree $\leq \Delta - d + 1$.

Let $N(x) = \{x^1, x^2, \dots, x^d\}$ and $d(x^i) = \Delta$ $(i = 1, 2, \dots, d)$.

Let $G' = G - xx^d$. Since G is critical graph, G' has a $\Delta-$edge coloring $\phi : E(G) \setminus \{xx^d\} \to C = \{1, 2, 3, ..., \Delta\}$.

For a vertex $v \in V(G')$, denote by $\phi(v)$ the set of colors appearing at the edges incident with the vertex v and denote $\overline{\phi(v)} = C \setminus \phi(v)$.

The coloring ϕ of G' can be regarded as a partial edge coloring of G with the edge xx^d uncolored. Without loss of generality, let $\phi(xx^i) = i$ $(i = \{1, 2, \dots, d-1\})$ (see the following figure).



(1) We claim that $|\phi(x) \cap \phi(x^d)| = d-2$.

Since $d(x^d) = \Delta - 1$ in G', we have $d-1 \geq |\phi(x^d) \cap \{1,2,\ldots,d-1\}| \geq d-2$. Suppose that $|\phi(x^d) \cap \{1,2,\ldots,d-1\}| = d-1$. That is, $\phi(x) \subseteq \phi(x^d)$ and therefore, $C \setminus [\phi(x) \cup \phi(x^d)] \neq \emptyset$ as $|\phi(x^d)| = \Delta - 1$ and $|C| = \Delta$. Let $a \in C \setminus \phi(x^d)$. Now the partial coloring ϕ can be extended to a $\Delta-$edge coloring of G by coloring the edge xx^d with the color a, a contradiction. Thus, $|\phi(x) \cap \phi(x^d)| = d-2$.

Without loss of generality, in G', let $\phi(x^d) = \{2,3,\ldots,\Delta\}$, $N(x^i) = \{x, x_2^i, x_3^i, \ldots, x_{i-1}^i, x_{i+1}^i, \ldots, x_\Delta^i\}$ $(i = \{1,2,\ldots,d-1\})$ and $N(x^d) = \{x, x_2^d, x_3^d, \ldots, x_\Delta^d\}$.

By (1), we can assume that $\phi(x^i x_j^i) = j$ $(i = 1,2,\ldots,d,\ j = 1,2,\ldots,\Delta)$.

(2) For each pair $\{i,j\} \in C \times C$, every (i,j)-bi-colored component of $\phi^{-1}(i) \cup \phi^{-1}(j)$ is a path or an even cycle (a cycle of even order). Furthermore, for each pair $(i,j) \in [\phi(x) \setminus \phi(x^d)] \times [\phi(x^d) \setminus \phi(x)]$, the (i,j)-bi-colored path P containing one of $\{x, x^d\}$ must containing both of $\{x, x^d\}$ and x, x^d must be the endvertices of the path P. Otherwise, we can extend the coloring ϕ of G' to G by alternating colors i and j along path P and then color the edge xx^d with the color j if $x \notin V(P)$, or with the color i if $x^d \notin V(P)$.

We denote $P_{i,j}(y)_\phi$ the (i,j)-bi-colored path starting at y with $i \in \phi(y)$ and $j \in \overline{\phi(y)}$.

(3) Clearly, for each $i \in \phi(x)$ and for each $j \in \{d, d+1, \ldots, \Delta\}$, $P_{i,j}(x)_\phi$ ends at x^d as $j \in \phi(x^d) \setminus \phi(x)$.

(4) We claim that $d(x_a^1) \geq \Delta - d + 2, a \in \{d, d+1, \ldots, \Delta\}$.

We will show that $\{1, d, d+1, \ldots, \Delta\} \subseteq \phi(x_a^1)$ with $a \in \{d, d+1, \ldots, \Delta\}$. If $x_a^1 = x^d$, then $d(x_a^1) = \Delta$. So we may assume that $x_a^1 \neq x^d$.

(4-1) $\{1, a\} \subseteq \phi(x_a^1)$.

Let $a \in \{d, d+1, \ldots, \Delta\}$. Since $(1,a)$-bi-colored path $P_{1,a}(x)_\phi$ starting at x must ends at x^d by (3) and $x_a^1 \neq x^d$, the vertex x_a^1, which is on the path, must be incident with a 1-colored edge and an a-colored edge and therefore, $\{1,a\} \in \phi(x_a^1)$.

(4-2) Assume that $k \notin \phi(x_a^1)$ for some $k \in \{d, d+1, \ldots, \Delta\}$ with $k \neq a$. The $P_{1,k}(x)_\phi$ with two endvertices x and x^d does not pass through the vertex x_a^1, since $1 \in \phi(x_a^1)$ by (4-1) and $k \notin \phi(x_a^1)$. Let ϕ' be the new coloring obtained from ϕ by alternating the colors 1 and k along the $(1,k)$-bi-colored path $P_{1,k}(x_a^1)_\phi$ starting at x_a^1. ($P_{1,k}(x_a^1)_\phi$ could be a single edge with color 1.) Then, in the coloring ϕ', $P_{1,a}(x)_{\phi'} = xx^1x_a^1$, $\phi'(xx^1) = \phi(xx^1) = 1$ and $\phi'(x^1x_a^1) = \phi(x^1x_a^1) = a$. A $\Delta-$edge coloring of G can be obtained from ϕ' by alternating the colors 1 and a along the path $xx^1x_a^1$ and then color the edge xx^d with color 1, a contradiction. So, $\{1, d, d+1, \ldots, \Delta\} \subseteq \phi(x_a^1)$ for $a \in \{d, d+1, \ldots, \Delta\}$ if $x_a^1 \neq x^d$, and of course, $d(x_a^1) \geq \Delta - d + 2, a \in \{d, d+1, \ldots, \Delta\}$ for $x_a^1 \neq x^d$.

(5) By the contradiction assumption and by (4), we have $d(x_j^1) \geq \Delta - d + 2$ if $j \notin \phi(x) = \{1, 2, ..., d-1\}$ and $d(x_j^1) \leq \Delta - d + 1$ if $j \in \phi(x)$ where $j \neq 1$.

(6) Claim that for each $i \in \{d, d+1, \ldots, \Delta\}$, $P_{2,i}(x)_\phi$ passes through x^1.

Otherwise, let ϕ' be the new coloring obtained from ϕ by alternating the colors 2 and i for all edges of $[\phi^{-1}(2) \cup \phi^{-1}(i)] - P_{2,i}(x)_\phi$. Notice that $\phi'(y) = \phi(y)$ for each vertex $y \in \{x, x^1, x^2, \ldots, x^d\}$ and the edge $x^1x_i^1$ is now colored with color 2 and edge $x^1x_2^1$ is colored with color i. By the same argument as of (4), we have $d(x_2^1) \geq \Delta - d + 2$, which contradicts to (5).

(7) Claim that for $i \in \{d, d+1, \ldots, \Delta\}$, $d(x_i^d) \geq \Delta - d + 2$ and each path $P_{2,i}(x)_\phi$ passes through x^d.

A new coloring ϕ' can be obtained from ϕ by uncolor the edge xx^2 and color the edge xx^d with color 2. Notice that $\phi(e) = \phi'(e)$ for each edge $e \in E \setminus \{xx^2, xx^d\}$ and therefore, $P_{2,i}(x)_\phi = P_{2,i}(x)_{\phi'}$ for each $i \in C \setminus \{2\}$. By applying the same argument as of (4), we know that (7) holds.

(8) By contradiction assumption and (7), we have $d(x_j^d) \geq \Delta - d + 2$ if $j \notin \phi(x)$ and $d(x_j^1) \leq \Delta - d + 1$ if $j \in \phi(x)$ where $j \neq 1$

(9) Claim that $d(x_i^b) \geq \Delta - d + 2$ for $i = d, d+1, \ldots, \Delta$.

If $x_i^2 \in N(x)$, then $d(x_i^2) = \Delta$. There is nothing to prove. So, assume that $x_i^b \neq x^1, x^d$ and $b = 2, 3, \ldots, d-1$.

Without loss of generality, we show that $d(x_i^2) \geq \Delta - d + 2$ for $i = d, d+1, \ldots, \Delta$.

We will show, sufficiently, that $\{1, d, d+1, \ldots, \Delta\} \subseteq \phi(x_i^2)$ for $i = d, d+1, \ldots, \Delta$.

Let $i \in \{1, d, d+1, \ldots, \Delta\}$, by (6) and (7), $P_{2,i}(x)_\phi$ passes through both x^1 and x^d, and since the vertex x_i^2 is not an endvertex of the path, therefore, $\{2, i\} \subseteq \phi(x_i^2)$.

Assume that there exists a $k \in \{d, d+1, \ldots, \Delta\} \setminus \phi(x_i^2)$ where $k \neq 2, i$. Let ϕ' be the new coloring obtained from ϕ by alternating the colors 2 and k along the path $P_{2,k}(x_i^2)_\phi$. Notice that $P_{2,i}(x)_{\phi'} = xx^2x_i^2$ since $2 \notin \phi'(x_i^2)$. Let ϕ'' be a coloring obtained from ϕ' by alternating the colors 2 and i along the path $xx^2x_i^2$. Since $P_{2,k}(x)_\phi$ and $P_{2,k}(x_i^2)_\phi$ are vertex disjoint and by (6) and (7), the $(2,k)$-bi-colored path $P_{2,k}(x)_\phi$ passes through the vertex x^1, so, the path $P_{2,k}(x_i^2)_\phi$ doesn't pass through the vertex x^1. Therefore, in the coloring ϕ'', $\phi''(e) = \phi(e)$ for each edge e incident with x^1, $\phi(x^1x_i^1) = \phi''(x^1x_i^1) = \phi''(xx^2) = i$. By applying the same argument as of (4), we have $d(x_2^1) \geq \Delta - d + 2$ which contradicts to (5).

(10) By contradiction assumption and (9), we have $d(x_j^b) \geq \Delta - d + 2$ if $j \in \{d, d+1, \ldots, \Delta\}$ and $d(x_j^b) \leq \Delta - d + 1$ if $j \in \phi(x) = \{1, 2, \ldots, d-1\}$ where $b = 2, 3, \ldots, d-1$.

(11) Claim that each $P_{1,i}(x)_\phi$ passes through the vertex x_j^b and x_j^b is not an endvertex for each $i \in \{d, d+1, \ldots, \Delta\}$ and $j \in \phi(x) \setminus \{b\}$ where $b = 2, 3, \ldots, d-1$.

Without loss of generality, we are to show that each $P_{1,i}(x)_\phi$ passes through the vertex x_j^2 and x_j^2 is not an endvertex for each $i \in \{d, d+1, \ldots, \Delta\}$ and $j \in \{1, 3, \ldots, d-1\}$.

Suppose that $P_{1,i}(x)_\phi$ doesn't pass through the vertex x_j^2 for some $j \in \{1, 3, \ldots, d-1\}$. Let ϕ' be the new coloring obtained from ϕ by alternating the colors 1 and i along the path $P_{1,i}(x)_\phi$. Then, in the coloring ϕ', $\phi'(e) = \phi(e)$ for each edge e incident with the vertex x_j^2 and $\phi'(xx^1) = i$. Since $\phi'(xx^1) = i$ and $\phi'(x) = \{2, 3, \ldots, d-1, i\}$, under coloring ϕ', by applying the same argument as of in (9), we have $d(x_i^2) \leq \Delta - d + 1$ which contradicts to (10). Therefore, $P_{1,i}(x)_\phi$ contains the vertex x_j^2. Notice that $x^d \neq x_j^2$ since $d(x^d) = \Delta > \Delta - d + 1 \geq d(x_j^2)$ where $j \in \{1, 3, 4, \ldots, d-1\}$. By (2), x and x^d are the endvertices of the path $P_{1,i}(x)_\phi$. Therefore, x_j^2 cannot be an endvertex of $P_{1,i}(x)_\phi$.

Hence, $\{1, d, d+1, \ldots, \Delta\} \subseteq \phi(x_j^2)$, and so, $d(x_j^2) \geq \Delta - d + 2$ where $j \in \{1, 3, 4, \ldots, d-1\}$, which contradicts to (10).

Now,we have completed the proof.

■

If $d = 4$ or 5, then, we have the following corollaries directly induced from Lemma 3.5 and Lemma 3.1.

Corollary 3.6 *Let G be a critical graph with the maximum degree Δ. Let $d(x) = 4$ and each of $N(x)$ is of degree Δ, then there is a vertex $y \in N(x)$ such that there are at most one vertex of $N(y) \setminus \{x\}$*

with degree $\leq \Delta - 3$ and at least $\Delta - 3$ vertices of $N(y) \setminus \{x\}$ are of degree Δ-vertices.

Corollary 3.7 *Let G be a critical graph with the maximum degree Δ. Let $d(x) = 5$ and each of $N(x)$ is of degree Δ, then there is a vertex $y \in N(x)$ such that there are at most two vertices in $N(y) \setminus \{x\}$ with degree $\leq \Delta - 4$ and at least $\Delta - 4$ vertices of $N(y) \setminus \{x\}$ are of degree Δ-vertices.*

4 Proof of the Main Theorem

For the sake of convenience, we re-state the main theorem:

Let G be a graph with maximum degree $\Delta \geq 12$ and average degree $d_{ave} \leq \frac{103}{12}$. Then G is not critical.

Proof. By contradiction, suppose that G is critical.

For each vertex v, denote $m_v = min\{d(w) : w \in N(v)\}$.

Let $c(x) = d(x) - \frac{103}{12}$ be the initial charge of x for each $x \in V(G)$. We are going to reassign a new charge denoted by $c'(x)$ to each $x \in V$ according to the following dischargeing rules starting from R2:

R2. Every 2-vertex v receives $\frac{10}{3}$ from each of its neighbors;

R3. For each 3-vertex v,

(i) v receives $\frac{5}{3}$ from each of the adjacent Δ-vertex and receives $\frac{7}{3}$ from each of adjacent $(\Delta - 1)$-vertex if $m_v = \Delta - 1$,

(ii) v receives $\frac{10}{3}$ from the adjacent Δ-vertex whose neighbors are all of degree at least $(\Delta - 1)$ except for v and receives $\frac{7}{6}$ from the other two adjacent Δ-vertices if $m_v = \Delta$.

R4. For each 4-vertex v,

(i) v receives $\frac{10}{9}$ from each of the adjacent Δ-vertex and receives $\frac{4}{3}$ from each adjacent $(\Delta - 2)$-vertex if $m_v = \Delta - 2$,

(ii) v receives $\frac{41}{36}$ from each adjacent Δ-vertex and receives $\frac{7}{6}$ from each adjacent $(\Delta - 1)$-vertex if $m_v = \Delta - 1$,

(iii) v receives $\frac{5}{3}$ from the adjacent Δ-vertex which has at most two adjacent vertex with degree$\leq \Delta - 3$ and receives $\frac{10}{9}$ from each of the rest adjacent Δ-vertex if $m_v = \Delta$.

R5. For each 5-vertex v,

(i) v receives $\frac{5}{6}$ from each of the adjacent Δ-vertex and receives $\frac{1}{3}$ from the adjacent $(\Delta - 3)$-vertex if $m_v = \Delta - 3$,

(ii) v receives $\frac{19}{24}$ from each of its adjacent Δ-vertex, receives $\frac{7}{9}$ from each of its $(\Delta - 1)$-vertex and receives $\frac{2}{3}$ from each of its adjacent $(\Delta - 2)$-vertex if $m_v = \Delta - 2$,

(iii) v receives $\frac{19}{24}$ from each of its adjacent Δ-vertex and receives $\frac{7}{9}$ from each of its adjacent $(\Delta - 1)$-vertex if $m_v = \Delta - 1$,

(iv) v receives $\frac{19}{18}$ from the adjacent Δ-vertex which has at most three adjacent vertices with degree $\leq \Delta - 4$ and receives $\frac{19}{24}$ from each of the rest adjacent Δ-vertex if $m_v = \Delta$.

R6. For each 6-vertex v,

(i) v receives $\frac{2}{3}$ from each of the adjacent Δ-vertex if $m_v = \Delta - 4$, or $m_v = \Delta - 3$,

(ii) v receives $\frac{4}{9}$ from each of its adjacent vertex if $m_v \geq \Delta - 2$.

R7. For each 7-vertex v,

(i) v receives $\frac{5}{9}$ from each of the adjacent Δ-vertex if $(\Delta - 5) \leq m_v \leq (\Delta - 2)$,

(ii) v receives $\frac{5}{18}$ from each of its adjacent vertices otherwise.

R8. For each 8-vertex v, v receives $\frac{1}{3}$ from each of the adjacent Δ-ertex.

We are going to show that after discharging, with new charge c', each vertex has non-negative charge, and the total original charge is non-positive which leads a contradiction.

(1) Claim that $c'(x) > 0$ for $d(x) = 2$ or $9 \leq d(x) \leq \Delta - 4$.

If $d(x) = 2$, then by Lemma 3.1, x is adjacent to two Δ-vertices, and $c'(x) = -\frac{79}{12} + 2 \times \frac{10}{3} = \frac{1}{12} > 0$ by R2.

If $9 \leq d(x) \leq \Delta - 4$, $c(x) \geq 9 - \frac{103}{12} > 0$.

(2) Let $y \in N(x)$ such that $d(y) = min\{d(z) : z \in N(x)\}$. Note that if $d(y) = i$, x is adjacent to at least $(\Delta - i + 1)$ Δ-verices by Lemma 3.1. We are going to use the value of $d(y)$ for discharging.

(3) Claim that $c'(x) > 0$ if $d(x) = 3$ where $c(x) = 3 - \frac{103}{12} = -\frac{67}{12}$ and $d(y) \geq \Delta - 1$ by Corollary 3.2(2).

If $d(y) = \Delta - 1$, then, by (2), x is adjacent to at least two Δ-vertices and therefore, by R3(i), $c'(x) = -\frac{67}{12} + 2 \times \frac{5}{3} + \frac{7}{3} = \frac{1}{12} > 0$

If $d(y) = \Delta$, then, all three neighbors of x are Δ-vertices, say x_1, x_2, x_3. Let $\mu_i = min\{d(z) : z \in N(x_i) \setminus \{x\}\}(i = 1, 2, 3)$. And therefore, by Lemma 3.4, at least one of $\{\mu_1, \mu_2, \mu_3\}$ is of at least $\Delta - 1$. By R3(ii), $c'(x) = -\frac{67}{12} + \frac{10}{3} + 2 \times \frac{7}{6} = \frac{1}{12} > 0$.

(4) Claim that $c'(x) \geq 0$ if $d(x) = 4$ where $c(x) = -\frac{55}{12}$ and $d(y) \geq \Delta - 2$ by Corollary 3.2.

(4-1) If $d(y) = \Delta - 2$, By Lemma 3.1(i), the rest 3 neighbors of x are all of degree Δ. Therefore, by R4(i), $c'(x) = -\frac{55}{12} + 3 \times \frac{10}{9} + \frac{4}{3} = \frac{1}{12} > 0$.

(4-2) $d(y) = \Delta - 1$.

(4-2-1) By R4(ii), $c'(x) \geq -\frac{55}{12} + 2 \times \frac{7}{6} + 2 \times \frac{41}{36} > 0$ if x is adjacent to two $(\Delta - 1)$-vertices and two Δ-vertices.

(4-2-2) Again by R4(ii), $c'(x) \geq -\frac{55}{12} + \frac{7}{6} + 3 \times \frac{41}{36} = 0$ if x is adjacent to one $(\Delta - 1)$-vertex and three Δ-vertices.

(4-3) If $d(y) = \Delta$, then, all four neighbors of x are Δ-vertices. By Corollary 3.6 and R4(iii), $c'(x) = -\frac{55}{12} + \frac{5}{3} + 3 \times \frac{10}{9} = \frac{5}{12} > 0$.

(5) Claim that $c'(x) > 0$ if $d(x) = 5$. Note that $c(x) = -\frac{43}{12}$ and $d(y) \geq \Delta - 3$ by (2). Note that x is adjacent to at least Δ-vertices by Corollary 3.2.

(5-1) If $d(y) = \Delta - 3$, then, by Lemma 3.1(i), the rest four neighbors of x are all of degree Δ. Therefore, by R5(i), $c'(x) = -\frac{43}{12} + \frac{1}{3} + 4 \times \frac{5}{6} = \frac{1}{12} > 0$.

(5-2) If $d(y) = \Delta - 2$, by Lemma 3.1(i), x is adjacent to at least three Δ-vertices. Therefore, by R5(ii), $c'(x) = -\frac{43}{12} + \frac{2}{3} + 4 \times \frac{38}{4 \cdot 12} = \frac{1}{4} > 0$ if x is adjacent to one $(\Delta - 2)$-vertex and four Δ-vertices.

Or $c'(x) = -\frac{43}{12} + 2 \times \frac{2}{3} + 3 \times \frac{19}{24} = \frac{-43+16+28.5}{12} > 0$, if x is adjacent to two $(\Delta - 2)$-vertices and three Δ-vertices.

Or $c'(x) = -\frac{43}{12} + \frac{2}{3} + \frac{7}{9} + 3 \times \frac{19}{24} > -\frac{43+8+7+28.5}{12} > 0$ if x adjacent one $(\Delta - 2)$-vertex, one $(\Delta - 1)$-vertex and three Δ-vertices.

(5-3) If $d(y) = \Delta - 1$, then, by R5(iii), $c'(x) = -\frac{43}{12} + \frac{7}{9} + 4 \times \frac{19}{24} > 0$ if x is adjacent to one $(\Delta - 1)$-vertex and four Δ-vertices.

Or $c'(x) = -\frac{43}{12} + 2 \times \frac{7}{9} + 3 \times \frac{19}{24} > 0$ if x is adjacent to two $(\Delta - 1)$-vertices and three Δ-vertices.

Or $c'(x) = -\frac{43}{12} + 3 \times \frac{7}{9} + 2 \times \frac{19}{24} > 0$ if x is adjacent to three $(\Delta - 1)$-vertices and two Δ-vertices.

(5-4) If $d(y) = \Delta$, by Corollary 3.7 and R5(iv), $c'(x) = -\frac{43}{12} + \frac{19}{18} + 4 \times \frac{19}{24} > 0$.

(6) Claim that $c'(x) > 0$ if $d(x) = 6$ where $c(x) = -\frac{31}{12}$ and $d(y) \geq \Delta - 4$.

If $d(y) = \Delta - 4$, by Lemma 3.1, the rest neighbors of x are all of degree Δ. Therefore, by R6(i), $c'(x) = -\frac{31}{12} + 5 \times \frac{8}{12} > 0$.

If $d(y) = \Delta - 3$, by Lemma 3.1, x is adjacent to at least four Δ-vertices. Therefore, by R6(i), $c'(x) = -\frac{31}{12} + 4 \times \frac{2}{3} = \frac{1}{12} > 0$.

If $d(y) \geq \Delta - 2$, by R6(ii), x receives $\frac{4}{9}$ from each of its neighbors, then, $c'(x) = -\frac{31}{12} + 6 \times \frac{4}{9} = \frac{1}{12} > 0$.

(7) Claim that $c'(x) > 0$ if $d(x) = 7$ where $c(x) = -\frac{19}{12}$ and $d(y) \geq \Delta - 5$.

If $\Delta - 5 \leq d(y) \leq \Delta - 2$, by Lemma 3.1(i), x is adjacent to at least three Δ-vertices. By R7(i), $c'(x) \geq -\frac{19}{12} + 3 \times \frac{5}{9} = \frac{1}{12} > 0$.

If $d(y) \geq \Delta - 1$, by R7(ii), $c'(x) = -\frac{19}{12} + 7 \times \frac{5}{18} > \frac{1}{12} > 0$.

(8) Claim that $c'(x) > 0$ if $d(x) = 8$ where $c(x) = -\frac{7}{12}$ and $d(y) \geq \Delta - 6$.

By R8, $c'(x) \geq -\frac{7}{12} + 2 \times \frac{1}{3} = \frac{1}{12} > 0$.

(9) Claim that $c'(x) > 0$ if $d(x) = \Delta - 3$ where $c(x) \geq \frac{5}{12}$.

Be aware that only rule R5(i) affects the charge of x. x sends at most $\frac{1}{3}$ out, $c'(x) \geq \frac{5}{12} - \frac{1}{3} = \frac{1}{12} > 0$.

(10) Claim that $c'(x) > 0$ if $d(x) = \Delta - 2$ where $c(x) \geq \frac{17}{12}$ and $d(y) \geq 4$. Note that only rules R4(i), R5(ii) and R6(ii) affect the charge of x.

If $d(y) = 4$, by Lemma 3.1 and definiton of $d(y)$ (see (2)), the rest neighbors of x are all of degree Δ. By R4(i), x only sends $\frac{4}{3}$ out, $c'(x) = \frac{17}{12} - \frac{4}{3} = \frac{1}{12} > 0$.

If $d(y) = 5$, then x is adjacent to at most two 5^+-vertices. By R5(ii) and R6(ii), x sends at most $2 \times \frac{2}{3}$ out, $c'(x) = \frac{17}{12} - \frac{4}{3} = \frac{1}{12} > 0$.

If $d(y) \geq 6$, x is adjacent to at most three 6^+-vertices. By R6(ii), x sends at most $3 \times \frac{4}{9}$ out. $c'(x) = \frac{17}{12} - 3 \times \frac{4}{9} > 0$.

For the sake of convenience of discussion from now on, we let $\{x_1, x_2, x_3, \ldots, x_q\}^{\oplus n}$ be the set with each element of the form as $x_{j_1} + x_{j_2} + \ldots + x_{j_n}$ where $\{j_1, j_2, \ldots, j_n\} \in \{1, 2, \ldots, q\}$.

(11) Claim that $c'(x) \geq 0$ if $d(x) = \Delta - 1$ where $c(x) \geq \frac{29}{12}$ and $d(y) \geq 3$.

If $d(y) \geq 9$, all neighbors of x are of degree of at least 9, then, original charge $c(x)$ is not affected. So we only discuss it when $d(y) = 3, 4, 5, 6, 7, 8$.

If $d(y) = 3$, then, $d(x) + d(y) = \Delta + 2$, by Lemma 3.3(i), the rest neighbors of $N(x) \setminus \{y\}$ are all Δ-vertices. Therefore, x sends $\frac{7}{3}$ to y only. $c'(x) \geq \frac{29}{12} - \frac{7}{3} > 0$.

If $d(y) = 4$, by Lemma 3.1(i), x is adjacent to at most two 4^+-

vertices and all rest of its neighbors are Δ-vertices. Therefore, by R4(ii), R5(iii), R6(ii), R7(ii), x sends at most $\frac{7}{3}$ out, where $\frac{7}{3} = max\{\frac{7}{6}, \frac{7}{9}, \frac{4}{9}, \frac{5}{18}\}^{\oplus 2} = \frac{7}{6} + \frac{7}{6} = \frac{7}{3}$, so, $c'(x) \geq \frac{29}{12} - \frac{7}{3} > 0$.

If $d(y) = 5$, then, x is adjacent to at most three 5^+-vertices and all rest of its neighbors are Δ-vertices. Therefore, by R5(ii), R5(iii), R6(ii), R7(ii), x sends at most $\frac{14}{9}$ out to its neighbors, where $\frac{7}{3} = max\{\frac{7}{9}, \frac{4}{9}, \frac{5}{18}\}^{\oplus 3} = 3 \times \frac{7}{9}$. So, $c'(x) \geq \frac{29}{12} - \frac{7}{3} > 0$.

If $d(y) = 6$, then, x is adjacent to at most four 6^+-vertices and all rest of its neighbors are Δ-vertices. Therefore, by R6(ii), R7(ii), x sends at most $\frac{16}{9}$ out to its neighbors, where $\frac{16}{9} = max\{\frac{4}{9}, \frac{5}{18}\}^{\oplus 4} = 4 \times \frac{4}{9}$. thus, $c'(x) \geq \frac{29}{12} - \frac{16}{9} = \frac{23}{36} > 0$.

If $d(y) = 7$, then, x is adjacent to at most five 7^+-vertices and all rest of its neighbors are Δ-vertices. Therefore, by R7(ii), x sends at most $\{\frac{5}{18}\}^{\oplus 5} = \frac{25}{18}$ out to its neighbors. Hence, $c'(x) \geq \frac{29}{12} - \frac{25}{18} > 0$.

(12) Claim that $c'(x) \geq 0$ if $d(x) = \Delta$ where $c(x) \geq \frac{41}{12}$ and $d(y) \geq 2$.

If $d(y) \geq 9$, then by R2, x sends nothing out to its neighbors, original charge $c(x)$ is not affected. So, we assume that $d(y) = 2, 3, \ldots, 8$.

If $d(y) = 2$, by Lemma 3.3(1), the rest neighbors of x are all of degree Δ. Then, x only sends $\frac{10}{3}$ out to y. $c'(x) \geq \frac{41}{12} - \frac{10}{3} = \frac{1}{12} > 0$.

If $d(y) = 3$, then by Lemma 3.3(1), x is adjacent to at most two 3^+-vertices and all rest of its neighbors are Δ-vertices.

By Lemma 3.4, and R3-R8, x sends at most $\frac{10}{3}$ out, where $\frac{10}{3} = max\{\frac{5}{3}, \frac{7}{6}, \frac{10}{9}, \frac{41}{36}, \frac{5}{3}, \frac{5}{6}, \frac{19}{24}, \frac{19}{24}, \frac{19}{18}, \frac{2}{3}, \frac{4}{9}, \frac{5}{9}, \frac{5}{18}, \frac{1}{3}\}^{\oplus 2}, \frac{10}{3}\}$. We have $c'(x) \geq \frac{41}{12} - \frac{10}{3} > 0$.

If $d(y) = 4$, then, x is adjacent to at most three 4^+-vertices and all rest of its neighbors are Δ-vertices. Therefore, by Corollary 3.6 and rules R4-R8, x sends at most $\frac{41}{12}$ out to its neighbors where $\frac{41}{12} = max\{\frac{10}{9}, \frac{41}{36}, \frac{10}{9}, \frac{5}{6}, \frac{19}{24}, \frac{19}{18}, \frac{2}{3}, \frac{5}{9}, \frac{4}{9}, \frac{5}{18}, \frac{1}{3}\}^{\oplus 3}, 2 \times \frac{5}{3}\}$.

If $d(y) = 5$, then, x is adjacent to at most four 5^+-vertices and all rest of its neighbors are Δ-vertices. Therefore, by R5-R8, x sends at most $\frac{4\times 19}{24} = \frac{19}{6}$ out to its neighbors where $\frac{4\times 19}{24} = max\{\frac{5}{6}, \frac{19}{24}, \frac{2}{3}, \frac{4}{9}, \frac{5}{9}, \frac{5}{18}, \frac{1}{3}\}^{\oplus 4}, 3\times\frac{19}{18}\}$, and $c'(x) \geq \frac{41}{12} - \frac{19}{6} = \frac{3}{12} > 0$.

If $d(y) = 6$, then, x is adjacent to at most five 6^+-vertices and all rest of its neighbors are Δ-vertices. Therefore, by R6-R8, x sends at most $\frac{10}{3}$ out to its neighbors where $\frac{10}{3} = max\{\frac{2}{3}, \frac{4}{9}, \frac{5}{9}, \frac{5}{18}, \frac{1}{3}\}^{\oplus 5}$.

So, $c'(x) \geq \frac{41}{12} - \frac{10}{3} = \frac{1}{12} > 0$.

If $d(y) = 7$, then, x is adjacent to at most six 7^+-vertices and all rest of its neighbors are Δ-vertices. Therefore, by R7 and R8, x sends at most $\frac{30}{9} = max\{\frac{5}{9}, \frac{5}{18}, \frac{1}{3}\}^{\oplus 6}$ out to its neighbors. Thus, $c'(x) = \frac{41}{12} - \frac{30}{9} = \frac{1}{12} > 0$.

If $d(y) = 8$, then, x is adjacent to at most seven 8-vertices and all rest of its neighbors are Δ-vertices. Therefore, by R8, x sends at most $7\times\frac{1}{3}$ out to its neighbors. Again, we have $c'(x) \geq \frac{41}{12} - \frac{7}{3} > 0$.

From all above arguments, we conclude that $c'(x) \geq 0$ for each vertex x in G.

And we have that

$$\sum_{x\in V(G)} c'(x) = 0 \qquad (*)$$

since $0 \leq \sum_{x\in V(G)} c'(x) = \sum_{x\in V(G)} c(x) \leq 0$.

Note that each vertex carries strict positive charge under new charge c' except 4-vertices and Δ-vertices. So, graph G only contains 4-vertices and Δ-vertices.

(13) Claim that there is no 4-vertex.

Suppose that G has a 4-vertex x. Note that $c'(x) = 0$ occurs only in the case (4-2-2). Thus, x must be adjacent to one $(\Delta - 1)$-vertex, say, z, and other three Δ-vertices. But $c'(z) > 0$ by (11) which contradicts to equation (*), so, there is no 4-vertex in G.

(14) From (13), we know that graph G only contains Δ-vertices which means that average degree of G is $\Delta \geq 12$, which contradicts

to $d_{ave} \leq \frac{103}{12}$.

This contradiction completes the proof of the theorem.

■

ACKNOWLEDGEMENTS

The author would like to thank Dr. Rong Luo for his helpful suggestions.

References

[1] L.W. Beineke, S. Fiorini, On small graphs critical with respect to edge-colourings, *Discrete Math.*, 16(1976), 109-121.

[2] S. Fiorini, Some remarks on a paper by Vizing on critical graphs, *Math. Proc. Camb. Phil. Soc.*, 77(1975), 475-483.

[3] Hugh Hind and Yue Zhao, Edge colorings of graphs embeddable in a surface of low genus, *Discrete Math.*, 190(1998), 107-114.

[4] I. T. Jacobsen, On critical graphs with chromatic index 4, *Discrete Math.*, 9(1974), 265-276.

[5] K. Kayathri, On the size of edge-chromatic critical graphs, *Graphs and Combinatorics*, 10(1994), 139-144.

[6] Rong Luo and C.Q. Zhang, Edge coloring of graphs with small average degrees, preprint.

[7] Lian-Ying Miao and Jian-liang Wu, Edge-coloring critical graphs with high degree, *Discrete Math, 2002, In press.*

[8] L. S. Mel'nikov, The chromatic class and location of a graph on a closed surface, *Mat. Zametki*, 7(1970), 671-681 (Math. Notes 7 (1970) 405 - 411).

[9] Daniel Sanders and Yue Zhao, Planar graphs of maximum degree seven are class I, *J. Combin. Theory Ser. B.*, 83(2)(2001), 201-212.

[10] V. G. Vizing, On an estimate of the chromatic class of a p-graph, *Metody Diskret. Analiz*, 3(1964), 25-30.

[11] V. G. Vizing, Critical graphs with given chromatic class (in Russian), *Metody Diskret. Analiz.*, 5(1965), 9-17.

[12] V. G. Vizing, The chromatic class of a multigraph (in Russian), *Kiebernetika (Kiev)*, 3(1965), 29-39.

[13] V. G. Vizing, Some unsolved problems in graph theory (in Russian), *Uspekhi Mat. Nauk,* 23(1968), 117-134; English translation in Russian Math. Surveys, 23 (1968) 125-141.

[14] Zhongde Yan and Yue Zhao, Edge Colorings of Embedded Graphs,*Graphs and Combinatorics*, 16(2)(2000), 245-256.

[15] H. P. Yap, On graphs critical with respect to edge-colorings, *Discrete Math.*, 37(1981), 289-296.

[16] Limin Zhang, Every planar graph with maximum degree 7 is of class 1, *Graphs and Combinatorics*, 16(4)(2000), 467-495.