On the voltage-current transferring in topological graph theory

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Abstract

It is shown that the voltage-current duality in topological graph theory can be obtained as a consequence of a combinatorial description of the pair (an embedded graph, the embedded dual graph) without any reference to derived graphs and derived embeddings. In the combinatorial description the oriented edges of an embedded graph are labeled by oriented edges of the embedded dual graph.

1 Introduction

In the theory of voltage and current graphs [2, 3] in topological graph theory, both a current graph and an embedded voltage graph have their own derived graphs, and the derived embeddings of the derived graphs are defined. Then, given a pair (an embedded graph K, the embedded dual graph), the voltage-current transferring is defined so that, given a voltage (resp. current) assignment on the oriented edges of the embedded graph K, the transferred current (resp. voltage) assignment on the oriented edges of the embedded dual graph is defined such that both the obtained current graph and the obtained embedded voltage graph determine the same derived embedding of the same derived graph.

In this paper we give a combinatorial description of the pair (an embedded graph, the embedded dual graph) in the case when the graphs are embedded (generally) in a nonorientable surface. In the combinatorial description the oriented edges of an embedded graph K are labeled by oriented edges of the embedded dual graph. If, in addition, the oriented edges of the embedded dual graph are assigned some marks (these marks can be elements of arbitrary set; in particular, they can be elements of a group), then the marks are transferred to become additional labels of the oriented edges of the embedded graph K. Then, as a consequence of the combinatorial description, we obtain the voltage-current transferring, the same as in the theory of voltage and current graphs, without any reference to derived graphs and derived embeddings. This gives a new insight into the voltage-current transferring and suggests that the theory of voltage and current graphs can be outlined in the following (more easier to understand) way: define a voltage graph and the derived graph; given an embedding of a voltage graph, define the derived embedding of the derived graph; given an embedding of a voltage graph, define a current graph as the embedded dual graph such that the voltages are transferred (in accordance with the combinatorial description of the duality) to become additional labels (currents) of the oriented edges of the embedded dual graph.

Note that an embedded voltage graph and a current graph can both be useful to study embeddings, without one entirely supplanting the other. Some aspects are much easier to study with voltage graphs (orientability and connectivity tests, etc.), other aspects are easier to study with current graphs (map-coloring problems, where duality plays the central role, etc.).

The paper is organized as follows. Section 2 gives preliminaries about a combinatorial description of a 2-cell embedding of a graph. In Section 3 we show how, starting with a pair (an embedded graph, the embedded dual graph) we can arrive at the idea of a combinatorial description of the pair such that the oriented edges of the embedded graph are labeled by oriented edges of the embedded dual graph. Section 4 gives a formal exposition of the combinatorial description as a correspondence between some 6-tuples. The content of Section 4 is based on the paper [1]. In Section 5 we show how the voltage-current transferring follows from the combinatorial description.

The reader is assumed to be familiar with the theory of voltage and current graphs, derived graphs and derived embeddings.

2 A combinatorial description of an embedding of a graph

In what follows, by an embedding of a graph we mean a 2-cell embedding of the graph in a surface. Digraphs considered in this paper may have loops and multiple arcs. Permutations are expressed in cyclic form: $(\delta_1, \delta_2, \ldots, \delta_m)$.

In this section we give preliminaries about the combinatorial description of an embedding of a graph proposed by Ringel [4], and introduce some definitions used in further sections to describe embedded dual graphs.

By a set with the involution we mean a finite set with a fixed involutory permutation θ of the set (that is, $\theta(\theta a) = a$ for every element a of the set), the elements a and θa are called (mutually) reverse elements, an element $a = \theta a$ is called a self-reverse element.

Let G be a connected digraph with the vertex set $V(G)$ and the arc set $A(G)$ with the involution θ (called also the involution of G) such that if an arc a is directed from the vertex v to the vertex u, then the arc θa is directed from u to v , the arcs a and θa are called reverse arcs. An arc a such that $a = \theta a$ is called a self-reverse loop.

A *rotation* D of G is a permutation of $A(G)$ whose orbits cyclically permute the arcs directed outwards from each vertex. The rotation D can be represented as $\{D_v : v \in V(G)\}$, where D_v, called a rotation of the vertex v, is a cyclic permutation of the arcs directed outwards from v. We will consider triples $\langle G, W, D \rangle$, where W is a subset of $A(G)$ such that if $a \in W$, then $a \neq \theta a$ and $\theta a \in W$. The subset W is called the set of twisted arcs of G. By the mapping associated with the subset W we mean the mapping $\lambda : A(G) \to \{1, -1\}$ such that $\lambda(a) = -1$ iff $a \in W$.

Given a triple $\langle G, W, D \rangle$, define a permutation $H(G, W, D) = H$ of the set $A(G) \times \{1, -1\}$ as

$$H(a, \tau) = (D^{\sigma} \theta a, \sigma) \tag{1}$$

where $\sigma = \tau \lambda(a)$. If an arc a is directed from v to w, then the arc $D^{\sigma}\theta a$ is directed from w. Hence, if $((a_1, \tau_1), (a_2, \tau_2), \ldots, (a_m, \tau_m))$ is a cycle of H, then $(a_1, a_2, \ldots, a_m)$ is a closed directed walk in G called the *circuit* determined by the cycle. The circuits $(a_1, a_2, \ldots, a_m)$ and $(\theta a_m, \ldots, \theta a_2, \theta a_1)$ are called reverse circuits. Define a permutation ρ of $A(G) \times \{1, -1\}$ as

$$\rho(a, \tau) = (\theta a, -\tau \lambda(a)) \tag{2}$$

It is easy to see that $\rho^2(a, \tau) = (a, \tau)$ for every pair (a, τ). For every cycle $\Omega = ((a_1, \tau_1), (a_2, \tau_2), \ldots, (a_m, \tau_m))$ of H, define the cyclic sequence $M\Omega = (\rho(a_m, \tau_m), \ldots, \rho(a_2, \tau_2), \rho(a_1, \tau_1))$. Clearly, $M(M\Omega) = \Omega$. The notation $(a, \tau) \in \Omega$ will mean that the pair (a, τ) enters into the cycle Ω.

Consider an arbitrary cycle of H. By (1) and (2), one can easily check that if $H(a, \tau) = (b, \sigma)$, then $H\rho(b, \sigma) = \rho(a, \tau)$, hence, $M\Omega$ is a cycle of H also. Now we show that $M\Omega \neq \Omega$ for every cycle Ω of H. Suppose (reductio ad absurdum) that $\Omega = M\Omega$ for some cycle Ω of H. Then for some $k \in \{1, 2, \dots, m\}$, either $(a_k, \tau_k) = \rho(a_k, \tau_k) = (\theta a_k, -\tau_k \lambda(a_k))$ or $\rho(a_k, \tau_k) = (a_{k+1}, \tau_{k+1}) = H(a_k, \tau_k)$. If $(a_k, \tau_k) = (\theta a_k, -\tau_k \lambda(a_k))$, then $a_k = \theta a_k$, $\lambda(a_k) = -1$, a contradiction. If $\rho(a_k, \tau_k) = H(a_k, \tau_k)$, then $(\theta a_k, -\tau_k \lambda(a_k)) = (D^{\sigma} \theta a_k, \sigma)$, where $\sigma = \tau_k \lambda(a_k)$, a contradiction.

Cycles Ω and $M\Omega$ are called reverse cycles, they determine reverse circuits. The set of cycles of H is partitioned into pairs $\{\Omega, M\Omega\}$ of reverse cycles. Choose one cycle from every pair of reverse cycles of H. The obtained collection of cycles is called an *r-set* (representative set) for H. If $(a, \tau) \in \Omega$, then $\rho(a, \tau) \in M\Omega$, hence we have the following claim:

(A1) Let $\mathcal{R}$ be a subset of the set of cycles of H. Then $\mathcal{R}$ is an r-set for H iff the cycles from $\mathcal{R}$ contain exactly one element from every pair $\{(a, \tau), \rho(a, \tau)\}$.

In what follows, given an r-set $\mathcal{R}$, the notation $(a, \tau) \in \mathcal{R}$ will mean that $(a, \tau) \in \Omega$ and $\Omega \in \mathcal{R}$.

Given a connected nonoriented graph K, the associated digraph $\overline{K}$ with the involution is defined as follows. A directed edge of K is an edge endowed with one of the two possible directions. Every edge of K gives rise to two oppositely directed edges called (mutually) reverse arcs of $\overline{K}$, and these arcs for all edges of K form the arc set $A(\overline{K})$. Note that $\overline{K}$ has no self-reverse loops.

The boundary of a face of an embedding of a graph K is a closed walk in K called the boundary cycle of the face. The boundary cycle of a face has two opposite directions, the boundary cycle with a chosen direction is called a directed boundary cycle of the face and is considered to be an oriented closed walk in $\overline{K}$.

Ringel [4] shown that every triple $\langle \overline{K}, W, D \rangle$ generates an embedding of K such that the circuits determined by the cycles of $H = H(\overline{K}, W, D)$ are exactly the directed boundary cycles of the faces of the embedding and that every embedding of K is generated by some triple $\langle \overline{K}, W, D \rangle$. Following Ringel, the circuits determined by cycles of H are constructed as follows. Let a be an arc directed to a vertex v. If a is an arc of a circuit, then the subsequent arc of the circuit is uniquely determined by D_v, $\lambda(a)$ and the behavior with which the circuit passes the first half of the arc a. The circuit has two modes of behavior, normal (depicted as a solid line in figures) and alternative (dashed line). In normal behavior the circuit obeys the rotation given at each vertex. In alternative behavior the circuit acts as if the given rotations are reversed. When the circuit passes an arc a with $\lambda(a) = -1$ its behavior switches modes at the midpoint of the arc. Now we see that to construct a circuit $(a_1, a_2, \ldots, a_m)$ in such a manner, we actually construct a cycle $((a_1, \tau_1), (a_2, \tau_2), \ldots, (a_m, \tau_m))$ of H where $\tau_i = 1$ (resp. $= -1$) means that the circuit passes the first half of the arc a_i with normal (resp. alternative) behavior.

Now we consider some mappings that will be used in a combinatorial description of embedded dual graphs.

Two sets with the involutions are called (mutually) comparable if they have the same number of pairs of reverse elements and the same number of self-reverse elements. The arc sets of an embedded graph and the dual embedded graph are comparable sets.

Given $\langle G, W\rangle$, where G is a digraph with the involution θ, and given a set A^* with the involution θ^* such that $A(G)$ and A^* are comparable, a mapping $\mu : A(G) \to A^*$ is called a *proper mapping* of $A(G)$ into A^* iff (A2) holds:

(A2) For every pair $b \neq \theta^* b$ from A^*, there are exactly two arcs, a_1 and a_2, of G such that $\mu(a_1), \mu(a_2) \in \{b, \theta^* b\}$. The arcs a_1 and a_2 are reverse and the following holds: $\mu(a_1) = \mu(a_2)$ iff $a_1 \in W$. If $b = \theta^* b$, then G has exactly one arc a such that $\mu(a) = b$, this arc a is a self-reverse loop.

The arcs a of an embedded graph G will be labeled by the arcs $\mu(a)$ of the dual embedded graph with the arc set A^* such that μ is a proper mapping of $A(G)$ into A^*.

Given $\langle G, W\rangle$ and a proper mapping μ of $A(G)$ into A^*, define a mapping $T_\mu : A(G) \times \{1, -1\} \to A^*$ as follows:

$$T_\mu(a, \tau) = \begin{cases} \mu(a) & \text{if } \tau = 1, \\ \theta^* \mu(a) & \text{if } \tau = -1. \end{cases} \tag{3}$$

the mapping T_μ will be used to define the arc set of the dual embedded graph. For simplicity, in this section we shall write T instead of T_μ.

Verifying the following claim (A3) is straightforward and is left to the reader.

(A3) $T(a, 1) \neq T(a, -1)$ for $a \neq \theta a$; $T(a, \tau) = T\rho(a, \tau)$ for every pair (a, τ).

Since $T(a, 1), T(a, -1) \in \{\mu(a), \theta^* \mu(a)\}$, by (A2) and (A3), we obtain the following:

(A4) For every element b of A^*, there are exactly two pairs (a_1, τ_1) and (a_2, τ_2) such that $b = T(a_1, \tau_1) = T(a_2, \tau_2)$, and we have $(a_1, \tau_1) = \rho(a_2, \tau_2)$.

By the *log* of a cycle $((a_1, \tau_1), (a_2, \tau_2), \ldots, (a_m, \tau_m))$ of $H = H(G, W, D)$ we mean the cyclic sequence $(T(a_1, \tau_1), T(a_2, \tau_2), \ldots, T(a_m, \tau_m))$. The following claim (A5) follows from (A1) and (A4):

(A5) Let $\mathcal{R}$ be a collection of cycles of H. Then $\mathcal{R}$ is an r-set for H iff every element from A^* appears exactly once in the logs of the cycles from $\mathcal{R}$.

By (A3), we obtain:

(A6) The logs of cycles Ω and $M\Omega$ of H are mutually reverse cyclic sequences.

Given $\langle G, W, D\rangle$, an r-set $\mathcal{R}$ for H, and given a proper mapping μ of $A(G)$ into A^*, define a mapping $\varphi_\mu : A(G) \to A^* \times \{1, -1\}$ as follows:

$$\varphi_\mu a = \begin{cases} (\mu(a), 1) & \text{if } (a,1) \in \mathcal{R}, \\ (\mu(a), -1) & \text{if } (a,1) \notin \mathcal{R}. \end{cases} \tag{4}$$

The mapping φ_μ will be used to define an r-set for the dual embedding.

In what follows, when we write T_μ and φ_μ , it will be always clear from the context what $\langle G, W, D\rangle$, $\mathcal{R}$, A^* and μ are considered.

3 Mutual labellings of arcs in dual embeddings

In this section we show how, starting with a pair (an embedded graph, the embedded dual graph) we can arrive at the idea of a combinatorial description of the pair such that the oriented edges of the embedded graph are labeled by oriented edges of the embedded dual graph.

Given an embedding f of a graph K in a surface, define the embedding f^* of the dual graph K^* in the surface as follows. For every face of f , insert a vertex of K^* inside the face. Then for every edge e of K, draw the dual edge e^* of K^* such that e^* crosses e (and no other edge of K or K^*) and joins the two vertices inside the faces of f which are incident with e. If e is incident with one face only, then e^* is a loop. The taking of duals is involutory, that is, $(K^*)^* = K$. The dual edge e^* gives rise to two reverse arcs called the arcs dual to the arcs corresponding to the edge e of K. Denote the involutions of $\overline{K}$ and $\overline{K^*}$ by θ and θ^*, respectively. To every vertex v of the embedded graph K there corresponds a face of f^* such that v is the unique vertex of K inside the face, denote the face by $F(v)$ (in what follows, when we will speak about a face $F(v)$, it will be always clear from the context what an embedded graph K is considered). There are two possible orientations at v, each of them induces an orientation of the face $F(v)$ and induces a rotation of v. We will say that an orientation of $F(v)$ determines a rotation of v and vice versa, provided that they both are induced by the same orientation at v.

Given the embeddings f and f^*, let $\mathcal{R}^*$ be a collection of orientations of the faces of f^*, one for each face. The collection $\mathcal{R}^*$ determines D and W such that $\langle\overline{K}, W, D\rangle$ generates f. These D and W are determined in the following way. For every vertex v of K, the rotation D_v is determined by the orientation of the face $F(v)$. An orientation of a face can be specified by a direction of the boundary cycle of the face. Hence, $\mathcal{R}^*$ can be considered as a collection of directed boundary cycles, one for each face. Every edge of K^* appears exactly twice as an edge endowed with a direction in the cycles from $\mathcal{R}^*$. An edge e of K gives rise to two reverse arcs from W if and only if the dual edge e^* of K^* has the same direction in the two appearances in the cycles from $\mathcal{R}^*$.

Now we consider quadruples $\langle\overline{K}, W, D\rangle_{\mathcal{R}}$ where $\mathcal{R}$ is a collection of orientations of the faces of the embedding generated by $\langle\overline{K}, W, D\rangle$. Given the embeddings f and f^*, define the correspondence

$$\langle\overline{K}, W, D\rangle_{\mathcal{R}} \to \langle\overline{K^*}, W^*, D^*\rangle_{\mathcal{R}^*}, \tag{5}$$

where $\langle\overline{K}, W, D\rangle$ and $\langle\overline{K^*}, W^*, D^*\rangle$ generate f and f^*, respectively, $\mathcal{R}$ determines D^* and W^*, $\mathcal{R}^*$ determines D and W. Since taking of duals is involutary, and since D and D^* in turns determine $\mathcal{R}^*$ and $\mathcal{R}$ respectively, we get the following duality:

(D1) If $\langle\overline{K}, W, D\rangle_{\mathcal{R}} \to \langle\overline{K^*}, W^*, D^*\rangle_{\mathcal{R}^*}$, then $\langle\overline{K^*}, W^*, D^*\rangle_{\mathcal{R}^*} \to \langle\overline{K}, W, D\rangle_{\mathcal{R}}$.

Given a quadruple $\langle\overline{K}, W, D\rangle_{\mathcal{R}}$, the collection $\mathcal{R}$ can be combinatorially represented by an r-set for $H(\overline{K}, W, D)$, denote this r-set by the same letter $\mathcal{R}$. In what follows we will consider quadruples $\langle\overline{K}, W, D, \mathcal{R}\rangle$, where $\mathcal{R}$ is an r-set for $H(\overline{K}, W, D)$. Then the correspondence (5) takes the form

$$\langle\overline{K}, W, D, \mathcal{R}\rangle \to \langle\overline{K^*}, W^*, D^*, \mathcal{R}^*\rangle, \tag{6}$$

where $\langle\overline{K}, W, D\rangle$ and $\langle\overline{K^*}, W^*, D^*\rangle$ generate f and f^*, respectively, $\mathcal{R}$ (resp. $\mathcal{R}^*$) represents the collection of orientations of the faces of f (resp. f^*) such that this collection determines D^* and W^* (resp. D and W). The duality (D1) takes the form:

(D2) If $\langle\overline{K}, W, D, \mathcal{R}\rangle \to \langle\overline{K^*}, W^*, D^*, \mathcal{R}^*\rangle$, then $\langle\overline{K^*}, W^*, D^*, \mathcal{R}^*\rangle \to \langle\overline{K}, W, D, \mathcal{R}\rangle$.

Given $\langle\overline{K}, W, D, \mathcal{R}\rangle$, the quadruple $\langle\overline{K^*}, W^*, D^*, \mathcal{R}^*\rangle$ in (6) is obtained in the following way. Construct the embedding f of K generated by $\langle\overline{K}, W, D\rangle$. Consider the embedding f^* of the dual graph K^* and construct the triple $\langle\overline{K^*}, W^*, D^*\rangle$ generating f^* such that D^* and W^* are determined by those orientations of the faces of f which are represented by the r-set $\mathcal{R}$. Choose an r-set $\mathcal{R}^*$ for $H(\overline{K^*}, W^*, D^*)$ such that the r-set represents those orientations of the faces of f^* which determine D and W.

Now consider the following problem:

(P1) Given $\langle \overline{K}, W, D, \mathcal{R} \rangle$, obtain $\langle \overline{K^*}, W^*, D^*, \mathcal{R}^* \rangle$ in (6), not considering the embeddings f and f^*.

Here is a problem of describing the arc set of $\overline{K^*}$. We will solve the problem by labeling the oriented edges of an embedded graph by oriented edges of the embedded dual graph.

Considering an embedding of K, since an arc of $\overline{K}$ is a directed edge of K, the embedded edge with the direction can be called the embedded arc and all the embedded arcs are called the arcs of the embedded graph K.

Now we consider how the arcs of an embedded graph can be labeled by arcs of the embedded dual graph. Given a triple $\langle \overline{K}, W, D \rangle$ generating an embedding of K, and given an arc a of $\overline{K}$, by the arc $(a|D)$ of the embedded K^* we mean the arc that is defined in the following way. Let the embedded arc a be directed from a vertex v and cross the edge e^* of K^* incident with vertices g and h (may be $g = h$). We can join v with g by an edge c inside $F(v)$ so that we obtain a closed disc $L \subset F(v)$ such that the boundary of the disc consists of the three components: the edge c; the part B of e^* between g and the intersection point of e^* and a; the part of a between v and the intersection point such that this part is directed from v to the intersection point (see Fig. 1(a), where e^* is depicted in a wavy line). The rotation D_v determines an orientation of $F(v)$, thereby determining an orientation of L. This orientation of L induces a direction of B, thereby determining a direction of e^*.

The edge e^* with this direction is the arc $(a|D)$ (see Fig.1(a), where the orientation of L is indicated by an oriented cycle inside L). It should be noticed that such a definition of the arc $(a|D)$ is given to take into account that graphs may have loops and that an edge can appear twice in the boundary cycle of a face.

Now, given a quadruple $\langle \overline{K}, W, D, \mathcal{R} \rangle$ and the correspondence (6), consider the 6-tuple $\langle \overline{K}, W, D, \mu, \mathcal{R} \rangle (A(\overline{K^*}))$, where $\mu : A(\overline{K}) \to A(\overline{K^*})$ such that $\mu(a) = (a|D)$ for every arc a of $\overline{K}$. We will say that an arc a of $\overline{K}$ is labeled by the arc $\mu(a) = (a|D)$ of $\overline{K^*}$. The sets $A(\overline{K})$ and $A(\overline{K^*})$ are comparable. One can easily see that $\mu(a) = \mu(\theta a)$ if $a \in W$, and $\mu(a) = \theta^* \mu(\theta a)$ if $a \notin W$, that is, μ is a proper mapping of $A(\bar{K})$ into $A(\bar{K}^*)$. Given (6), consider the correspondence

$$\langle \overline{K}, W, D, \mu, \mathcal{R} \rangle (A(\overline{K^*})) \to \langle \overline{K^*}, W^*, D^*, \mu^*, \mathcal{R}^* \rangle (A(\overline{K})), \qquad (7)$$

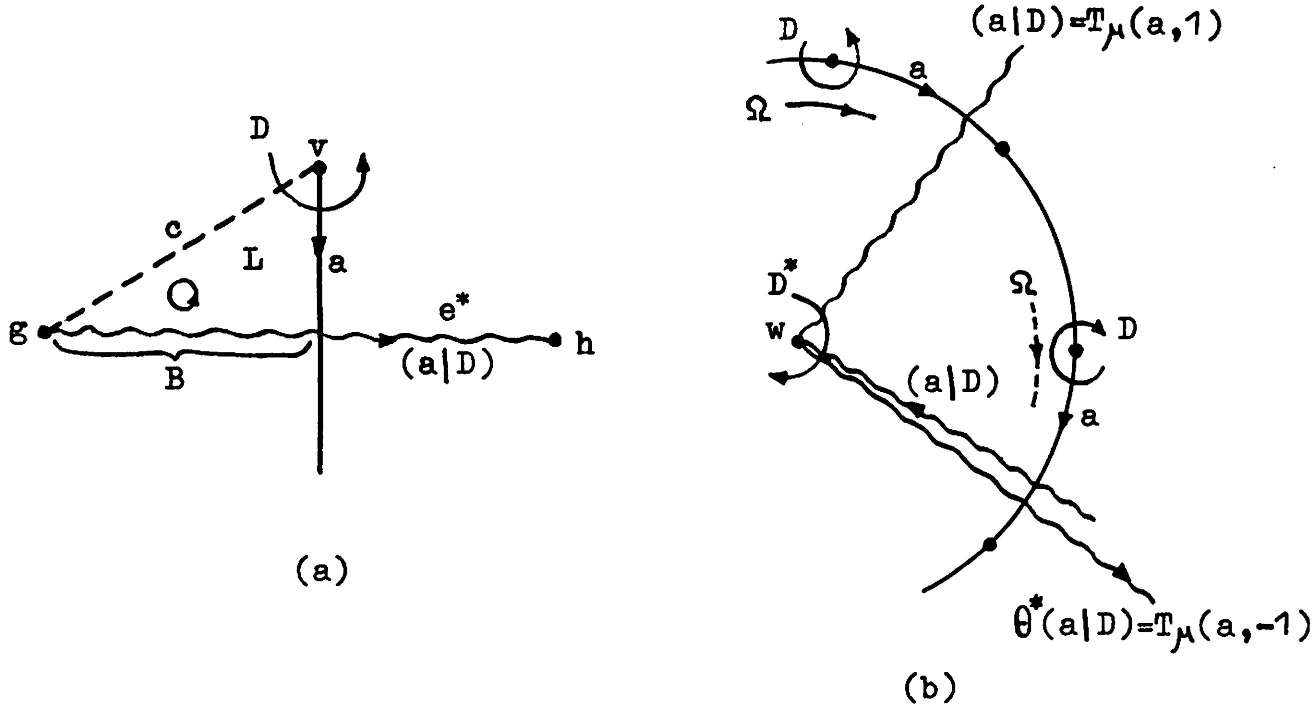

Fig.1

where $\mu^* : A(\overline{K^*}) \to A(\overline{K})$ such that $\mu^*(b) = (b|D^*)$. Analogously, μ^* is a proper mapping of $A(\overline{K^*})$ into $A(\overline{K})$. Since, given f and f^*, the mappings μ and μ^* are uniquely determined by D and D^*, respectively, the duality (D2) implies the following duality:

(D3) If $\langle\overline{K}, W, D, \mu, \mathcal{R}\rangle(A(\overline{K^*})) \to \langle\overline{K^*}, W^*, D^*, \mu^*, \mathcal{R}^*\rangle(A(\overline{K}))$, then $\langle\overline{K^*}, W^*, D^*, \mu^*, \mathcal{R}^*\rangle(A(\overline{K})) \to \langle\overline{K}, W, D, \mu, \mathcal{R}\rangle(A(\overline{K^*}))$.

The motivation to consider the correspondence (7) is that we can solve the following problem:

(P2) Given $\langle\overline{K}, W, D, \mu, \mathcal{R}\rangle(A(\overline{K^*}))$, obtain $\langle\overline{K^*}, W^*, D^*, \mu^*, \mathcal{R}^*\rangle(A(\overline{K}))$ in (7), not considering the embeddings f and f^*.

A solution of (P2) is given below in points (B1)-(B4).

Since $\mu(a) = (a|D)$, taking into account the definition (3) of the mapping T_μ, we have $(T_\mu(a,\tau)|D^*) = a$ (see Fig.1(b) where the arcs of $\overline{K}^*$ are depicted in wavy lines), hence

(B1) If $\Omega = ((a_1,\tau_1),(a_2,\tau_2),\dots,(a_m,\tau_m))$ is a cycle from $\mathcal{R}$ representing the orientation of a face $F(w)$ of f, where $w \in V(K^*)$, then

$$D^*_w = (T_\mu(a_1,\tau_1), T_\mu(a_2,\tau_2), \dots, T_\mu(a_m,\tau_m)),$$

that is, D^*_w is the log of the cycle Ω.

Considering (A5), we see that (B1) also defines $\overline{K^*}$ assigning the initial and terminal vertices to every arc from $A(\overline{K^*})$.

The arc $T_\mu(a,\tau)$ is one of the arcs of $\overline{K^*}$ dual to the arcs a and θa of $\bar{K}$, whence

(B2) $T_\mu(a,\tau) \in W$ iff $(a,1),(a,-1) \in \mathcal{R}$.

We have $(T_\mu(a,\tau)|D^*) = a$ for every $(a,\tau) \in \mathcal{R}$ (see Fig. 1(b)), hence

(B3) $\mu^*(T_\mu(a,\tau)) = a$.

We have $R^* = \{[v] : v \in V(\overline{K})\}$, where $[v]$ is the cycle representing the orientation of the face $F(v)$ of f^*. If $D_v = (a_1, a_2, \dots, a_m)$, then, clearly, the circuit determined by the cycle $[v]$ is $((a_1|D),(a_2|D),\dots,(a_m|D)) = (\mu(a_1),\mu(a_2),\dots,\mu(a_m))$. Taking into account the definition (4) of the mapping φ_μ, and considering Fig.2, we obtain

(B4) $[v] = (\varphi_\mu a_1, \varphi_\mu a_2, \dots, \varphi_\mu a_m)$

(a formal proof of (B4) will be given in Section 4).

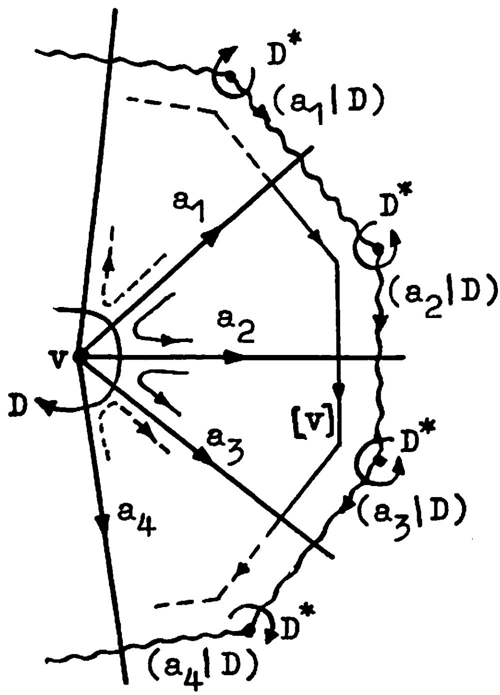

Fig.2

We see that (B1)-(B4) give a purely combinatorial way to solve (P2). The problem of describing the arc set of $\overline{K^*}$ is solved here by extension of a quadruple $\langle \overline{K}, W, D, \mathcal{R}\rangle$ to the 6-tuple $\langle \overline{K}, W, D, \mu, \mathcal{R}\rangle(A(\overline{K^*}))$, that is, by specifying the arc set of $\overline{K^*}$ and by labeling the arcs of $\overline{K}$ by arcs of $\overline{K^*}$.

Now we show how the solution of (P2) can be used to solve (P1). Let A' be an arbitrary set with the involution θ'. Let A' and $A(\overline{K})$ be comparable and μ' be an arbitrary proper mapping of $A(\overline{K})$ into A'. Define the mapping $\omega : A(\overline{K^*}) \to A'$ as follows:

$$\omega(\mu(a)) = \mu'(a),$$

$$\omega(\theta^* \mu(a)) = \theta' \mu'(a),$$

for every arc a of $\overline{K}$. Since μ and μ' are proper mappings, we obtain that ω is a bijection that takes reverse arcs of $A(\overline{K^*})$ to reverse elements of A'. Consider the correspondence

$$\langle\overline{K}, W, D, \mu', \mathcal{R}\rangle(A') \to \langle\overline{K'}, W', D', \mu'', \mathcal{R}'\rangle(A(\overline{K})),$$

the right side being determined by (B1)-(B4), where we replace μ by μ'. The digraphs $\overline{K^*}$ and $\overline{K'}$ have the same vertex set. Since $\omega T_\mu(a,\tau) = T_{\mu'}(a,\tau)$ for every $(a,\tau) \in \mathcal{R}$, we get that that for every $b \in A(\overline{K^*})$, the arcs b and $\omega(b)$ have the same initial vertex. Now it is an easy exercise to check that the following (i)-(iv) hold:

(i) $D^*b_1 = b_2$ iff $D'\omega(b_1) = \omega(b_2)$;

(ii) $b \in W^*$ iff $\omega(b) \in W'$;

(iii) $\mu^*(b) = \mu''(\omega(b))$;

(iv) $((b_1,\tau_1),(b_2,\tau_2),\ldots,(b_m,\tau_m)) \in \mathcal{R}^*$ iff $((\omega(b_1),\tau_1),(\omega(b_2),\tau_2),\ldots,(\omega(b_m),\tau_m)) \in \mathcal{R}'$.

We see that $\langle\overline{K'}, W', D', \mu'', \mathcal{R}'\rangle(A(\overline{K}))$ is obtained from $\langle\overline{K^*}, W^*, D^*, \mu^*, \mathcal{R}^*\rangle(A(\overline{K}))$ by relabeling the arcs of $\overline{K^*}$ so that for every $b \in A(\overline{K^*})$, the arc b of $\overline{K^*}$ becomes the arc $\omega(b) \in A' = A(\overline{K'})$.

Hence, the problem (P1) can be solved in the following way. Given $\langle\overline{K}, W, D, \mathcal{R}\rangle$, choose an arbitrary set A^* comparable with $A(\overline{K})$ to be the arc set of the embedded dual graph K^*. Then choose an arbitrary proper mapping $\mu : A(\overline{K}) \to A^*$. Consider $\langle\overline{K}, W, D, \mu, \mathcal{R}\rangle(A(\overline{K^*})) \to \langle\overline{K^*}, W^*, D^*, \mu^*, \mathcal{R}^*\rangle(A(\overline{K}))$, where the right 6-tuple is determined in accordance with (B1)-(B4). Then $\langle\overline{K^*}, W^*, D^*, \mu^*, \mathcal{R}^*\rangle$ is the required quadruple (up to designation of arcs).

4 The duality between 6-tuples

In this section we consider a generalization of the correspondence (7) where $\overline{K}$ is replaced by a digraph G with the involution, such that G may have self-reverse loops. For the case of such 6-tuples, we prove the duality (D3). Considering digraphs with self-reverse loops is motivated by the fact that a voltage graph with the voltage group of even order may have self-reverse loops assigned voltages of order 2.

Note that if G has self-reverse loops, then a triple $\langle G, W, D\rangle$ can be considered as a triple generating an embedding of a graph K in a surface with holes such that every hole is bounded by an embedded edge of K. Then, to obtain G, we replace every edge bounding (resp. not bounding) a hole by a self-reverse loop (resp. by a pair of reverse arcs). We will not consider the topological interpretation of such triples, and in what follows we will consider the triples and the 6-triples from the combinatorial point of view only.

Let G be a digraph with the involution θ. Let A^* be a set with the involution θ^*, and A^* be comparable with $A(G)$. Now we define the correspondence

$$\langle G, W, D, \mu, \mathcal{R}\rangle(A^*) \to \langle G^*, W^*, D^*, \mu^*, \mathcal{R}^*\rangle(A) \tag{8}$$

where G^* is a digraph with the arc set A^* and the involution θ^*. For simplicity, in this section we will write T,φ,T^*, and φ^* instead of T_μ, φ_μ, T^*_μ, and φ^*_μ, respectively.

The vertex set of G^* is the set of all pairs $\{\Omega, M\Omega\}$ of reverse cycles of $H(G, W, D) = H$.

By (A5), every element of A^* appears exactly once in the logs of the cycles from . For every cycle Ω_i from $\mathcal{R}$, denote by w_i the vertex $\{\Omega_i, M\Omega_i\}$ of G^*. If $b \neq \theta^* b$ and $b, \theta^* b$ appear in the logs of cycles Ω_i and Ω_j ,respectively, then the arc b of G^* is directed from w_i to w_j , and $\theta^* b$ is directed from w_j to w_i . If $b = \theta^* b$ and b appears in the log of a cycle Ω_i , then b is a self-reverse loop incident with the vertex w_i . Note that, by (A6), the constructed digraph G^* does not depend on the choice of the r-set $\mathcal{R}$.

Define the rotation D^* as follows: the rotation of a vertex w_i is the log of the cycle Ω_i from $\mathcal{R}$.

The set W^* is defined to be the set of all arcs $b, \theta^* b$ such that $\{b, \theta^* b\} = \{T(a,1), T(a,-1)\}$, where $(a,1), (a,-1) \in \mathcal{R}$.

Define A to be $A(G)$.

For every arc b of G^*, by (A5), $b = T(a,\tau)$ for some unique pair $(a,\tau) \in \mathcal{R}$, and we define $\mu^* b = \mu^*(T(a,\tau)) = a$. It follows from the definition of W^* that μ^* is a proper mapping of $A(G^*)$ into A.

To define $\mathcal{R}^*$, we need the following lemma.

Lemma 1 *For every $v \in V(G)$, if $D_v = (a_1, a_2, \ldots, a_m)$, then $[v] = (\varphi a_1, \varphi a_2, \ldots, \varphi a_m)$ is a cycle of $H(G^*, W^*, D^*) = H^*$ and the log of the cycle is D_v.*

Proof It suffices to show that

$$H^* \varphi a = \varphi D a, \tag{9}$$

$$T^* \varphi D a = D a \tag{10}$$

for every arc a of G. By (A1), there are two cases: either $(Da, 1)$ or $\rho(Da, 1)$ enters into the cycles from $\mathcal{R}$. Since $\lambda(a) = \lambda(\theta a) = \pm 1$, it follows from (1) that $H(\theta a, \lambda(a)) = (Da, 1)$ and $H(\theta D a, -\lambda(Da)) = (a, -1)$, where, by (2), we have $\rho(\theta a, \lambda(a)) = (a, -1)$ and $\rho(Da, 1) = (\theta Da, -\lambda(Da))$. By (3), we get

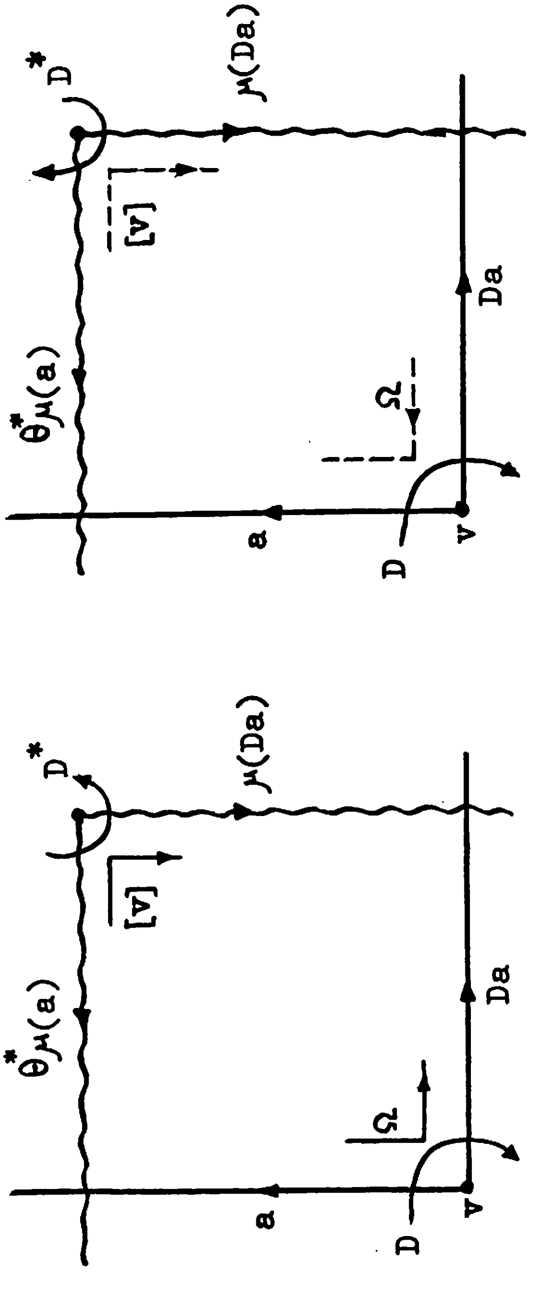

Fig.3

$$T\rho(\theta a, \lambda(a)) = T(a,-1) = \theta^* \mu(a),$$
$$T(\theta Da, -\lambda(Da)) = T(Da, 1) = \mu(Da). \qquad (11)$$

Now we consider the two cases.

Case 1. $\mathcal{R}$ has a cycle $\Omega = (\dots, (\theta a, \lambda(a)), (Da, 1), \dots)$ (see Fig.3(a), where the arcs of G^* are depicted in wavy lines) and $(a,-1) \notin \mathcal{R}$. From (11) and the definition of D^* and μ^* it follows that

$$D^*(\theta^* \mu(a)) = \mu(Da), \mu^*(\theta^* \mu(a)) = \theta a, \mu^*(\mu(Da)) = Da.$$

We have $\varphi Da = (\mu(Da), 1)$ and $T^* \varphi Da = \mu^*(\mu(Da)) = Da$, hence (10) holds. If $(a,1) \in \mathcal{R}$, then $\mu(a) \notin W^*$ (by the definition of W^*) and $\varphi a = (\mu(a), 1)$. Here we get $\lambda(\mu(a)) = 1$, hence $H^* \varphi a = (D^* \theta^* \mu(a), 1) = (\mu(Da), 1) = \varphi Da$. If $(a,1) \notin \mathcal{R}$, then $\mu(a) \in W^*$ and $\varphi a = (\mu(a), -1)$. Here we get $\lambda(\mu(a)) = -1$, hence $H^* \varphi a = (D^* \theta^* \mu(a), 1) = (\mu(Da), 1) = \varphi Da$. Thus (9) holds.

Case 2. $\mathcal{R}$ has a cycle $\Omega = (\dots, (\theta Da, -\lambda(Da)), (a,-1), \dots)$ (see Fig.3(b)) and $(Da, 1) \notin \mathcal{R}$. It follows from (11) that

$$D^*(\mu(Da)) = \theta^* \mu(a), \mu^*(\mu(Da)) = \theta Da, \mu^*(\theta^* \mu(a)) = a.$$

We have $\varphi Da = (\mu(Da), -1)$ and $T^* \varphi Da = \theta \mu^*(\mu(Da)) = Da$, hence (10) holds. If $(a,1) \notin \mathcal{R}$, then $\mu(a) \notin W^*$ and $\varphi a = (\mu(a), -1)$. Here we get $\lambda(\mu(a)) = 1$, hence $H^* \varphi a = ((D^*)^{-1} \theta^* \mu(a), -1) = (\mu(Da), -1) = \varphi Da$. If $(a,1) \in \mathcal{R}$, then $\mu(a) \in W^*$ and $\varphi a = (\mu(a), 1)$. Here we get $\lambda(\mu(a)) = -1$, hence $T^* \varphi a = ((D^*)^{-1} \theta^* \mu(a), -1) = (\mu(Da), -1) = \varphi Da$. Thus (9) holds.
□

Denote $\mathcal{R}^* = \{[v] : v \in V(G)\}$. By Lemma 1, every arc of G appears exactly once in the logs of the cycles from $\mathcal{R}^*$, hence, by (A5), $\mathcal{R}^*$ is an r-set for H^*.

The reader can easily check in a way analogous to that used in Section 3 that taking different $\mathcal{R}^*$ and μ in (8), we will obtain the same right 6-tuple in (8) (up to designation of arcs).

Theorem 1 *If* $\langle G, W, D, \mu, \mathcal{R} \rangle (A^*) \to \langle G^*, W^*, D^*, \mu^*, \mathcal{R}^* \rangle (A)$, *then* $\langle G^*, W^*, D^*, \mu^*, \mathcal{R}^* \rangle (A) \to \langle G, W, D, \mu, \mathcal{R} \rangle (A^*)$.

Proof. Let $\langle G^*, W^*, D^*, \mu^*, \mathcal{R}^* \rangle (A) \to \langle G', W', D', \mu', \mathcal{R}' \rangle (A^*)$. The vertex set of G' is the set of all pairs $\{[v], M[v]\}$, $v \in V(G)$. For every vertex v of G, denote by v the vertex $\{[v], M[v]\}$ of G'. Then, by Lemma 1, we get $G' = G$ and $D' = D$.

For every arc a of G, $\varphi a = (\mu(a), \tau) \in \mathcal{R}^*$ and, by Lemma 1, $T^*\varphi a = a$, hence $\mu'(a) = \mu'(T^*\varphi a) = \mu'(T^*(\mu(a), \tau) = \mu(a)$, that is, $\mu' = \mu$. Since μ' is a proper mapping of $A(G') = A(G)$ into $A^* = A(G^*)$, we get $W' = W$.

It remains to show that $\mathcal{R}' = \mathcal{R}$. We have $G' = G$, $W' = W$ and $D' = D$, thus $\mathcal{R}'$ and $\mathcal{R}$ are r-sets for the same permutation $H(G, W, D)$. Hence, it suffices to prove the following:

(i) for every $(a, \tau) \in \mathcal{R}$, we have $(a, \tau) \in \mathcal{R}'$.

For every arc a of G, either $(a, 1) \in \mathcal{R}$, or $\rho(a, 1) \in \mathcal{R}$. Now, to prove (i), we show that for every arc a of G, if $(a, 1) \in \mathcal{R}$ (resp. $\rho(a, 1) \in \mathcal{R}$), then $\varphi^*\mu(Da) = (a, 1) \in \mathcal{R}'$ (resp. $= \rho(a, 1) \in \mathcal{R}'$).

If $(a, 1) \in \mathcal{R}$, then $T(a, 1) = \mu(a)$, $\mu^*(\mu(a)) = a$ and $\varphi a = (\mu(a), 1) \in \mathcal{R}^*$, hence $\varphi^*\mu(a) = (\mu^*(\mu(a)), 1) = (a, 1) \in \mathcal{R}'$.

Let $(\mathrm{a}, 1) = (\theta \mathrm{a}, -\lambda(\mathrm{a})) \in \mathcal{R}$. Then $\varphi a = (\mu(a), -1) \in \mathcal{R}^*$ and, by (11) (where replace Da by a), we have $T\rho(a, 1) = \mu(a)$, hence $\mu^*(\mu(a)) = \theta a$. If $a = \theta a$, then $(\mathrm{a}, 1) = (\mathrm{a}, -1) \in \mathcal{R}$, $\mu^*(\mu(a)) = a$, $\mu(a) = \theta^*\mu(a)$ and $(\mu(\mathrm{a}), -1) = (\mu(\mathrm{a}), 1)$, hence $(\mu(a), 1) \notin \mathcal{R}^*$ and $\varphi^*\mu(a) = (\mu^*(\mu(a)), -1) = (a, -1) \in \mathcal{R}'$. Let $a \neq \theta a$. If $(\mu(a), 1) \in \mathcal{R}^*$, then $a \in W$ and $\varphi^*\mu(a) = (\mu^*(\mu(a)), 1) = \rho(a, 1) \in \mathcal{R}'$. If $(\mu(a), 1) \notin \mathcal{R}^*$, then $a \notin W$ and $\varphi^*\mu(a) = (\mu^*(\mu(a)), -1) = (\theta a, -\lambda(a)) = \rho(a, 1) \in \mathcal{R}'$. □

Taking the theorem into account, we will say that the 6-tuples in (8) are mutually dual. The theorem reduces the problem of construction of a 6-tuple to the problem of construction of the dual 6-tuple. As a consequence of the theorem, it will be shown in Section 5 that the problem of construction of an embedded voltage graph is reduced to the problem of construction of the dual current graph.

5 Transferring voltages and currents

In this section we first briefly review the voltage-current transferring. Then we show how the transferring follows from the combinatorial description of embedded dual graphs given in Sections 3 and 4.

Let Φ be a group and G be a digraph with the involution θ. By a *voltage assignment* on G we mean a mapping $\varphi : A(G) \to \Phi$ such that $\varphi(a) = (\varphi(\theta a))^{-1}$ for every arc a. By an embedded voltage graph we mean a quadruple $\langle G, W, \varphi, D\rangle$, where φ is a voltage assignment on G. Here $\varphi(a)$ is called the voltage of the arc a.

By a current graph we mean a quadruple $\langle G, W, \eta, D\rangle$, where $\eta : A(G) \to \Phi$ such that $\eta(a) = (\eta(\theta a))^{-1}$ if $a \notin W$, and $\eta(a) = \eta(\theta a)$ if $a \in W$. Here η is called a *current assignment*, $\eta(a)$ the current of the arc a.

In the theory of voltage and current graphs, both a current graph and an embedded voltage graph have their own derived graphs, and the derived embeddings of the derived graphs are defined (the derived graphs and the derived embeddings are defined in different ways in both of the cases; see [2] as a good expository paper on this subject). Then, in the theory, the voltage-current transferring is defined in the following way. Consider the correspondence

$$\langle \overline{K}, W, D, \mathcal{R} \rangle \leftrightarrow \langle \overline{K^*}, W^*, D^*, \mathcal{R}^* \rangle,$$

where $\langle \overline{K}, W, D \rangle$ and $\langle \overline{K^*}, W^*, D^* \rangle$ generate embeddings of graphs K and K^*, respectively. Then, given $\langle \overline{K}, W, \varphi, D \rangle$, where φ is a voltage assignment, the transferred current assignment η is defined such that $\langle \overline{K}, W, \varphi, D \rangle$ and $\langle \overline{K^*}, W^*, \eta, D^* \rangle$ determine the same derived embedding of the same derived graph, the embedded voltage graph and the current graph are said to be mutually dual. The (voltage $\to$ current) transferring is defined as follows: for every $b \in A(\overline{K^*})$, $\eta(b) = \varphi((b|D^*))$ (see Fig.4). And vice versa, given $\langle \overline{K^*}, W^*, \eta, D^* \rangle$, where η is a current assignment,

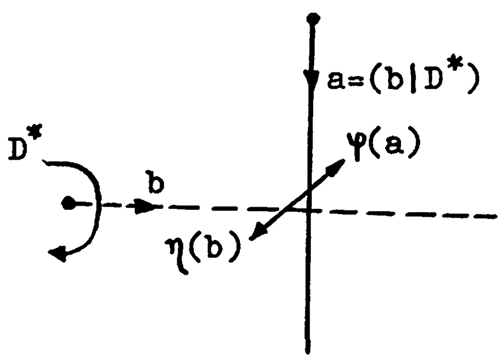

Fig.4

the transferred voltage assignment φ is defined such that $\langle \overline{K}, W, \varphi, D \rangle$ and $\langle \overline{K^*}, W^*, \eta, D^* \rangle$ determine the same derived embedding of the same derived graph. The (current $\to$ voltage) transferring is defined as follows: for every $b \in A(\overline{K^*})$, $\varphi((b|D*)) = \eta(b)$ and $\varphi(\theta(b|D*)) = (\eta(b))^{-1}$, where θ is the involution of $\overline{K}$. If we first transfer a current assignment to a voltage assignment and then transfer back again, we regain the original current assignment. Similarly, the two processes are inverses of each other in the other order.

Now we show how the voltage-current duality follows from the combinatorial description of embedded dual graphs given in Sections 3 and 4.

Let G be a digraph with the involution θ. Let $\varphi : A(G) \to \Phi$ be a voltage assignment on G. By the voltage digraph G_φ with the involution θ' we mean the digraph with the vertex set $V(G)$ and the arc set $\{\langle a, \varphi(a)\rangle : a \in A(G)\}$, where arcs a of G and $\langle a, \varphi(a)\rangle$ of G_φ have the same initial vertex and the same terminal vertex, and $\theta'\langle a, \varphi(a)\rangle = \langle \theta a, (\varphi(a))^{-1}\rangle$. Consider

$$\langle G_\varphi, W, D, \mu, \mathcal{R}\rangle(A(G^*)) \leftrightarrow \langle G^*, W^*, D^*, \mu^*, \mathcal{R}^*\rangle(A(G_\varphi)). \qquad (12)$$

Suppose we speak about an arc $\langle a, \varphi(a)\rangle$ of G_φ as an arc with the voltage $\varphi(a)$, and about an arc b of G^* such that $\mu^*(b) = \langle c, \varphi(c)\rangle$ as an arc with the current $\eta(b) = \varphi(c)$. Then, since $\mu^*(b) = (b|D^*)$, we get

$$\eta(b) = \varphi((b|D^*))$$

and, clearly,

$$\varphi(\theta'(b|D^*)) = (\eta(b))^{-1}.$$

Hence, if given the voltage assignment φ on G_φ , we consider η as the transferred current assignment, and if, given the current assignment η, we consider φ as the transferred voltage assignment, then we obtain the same voltage-current transferring as in the theory of voltage and current graphs.

Considering the combinatorial description of the correspondence (12) given in Section 4, we obtain the following combinatorial description of the voltage-current duality:

(Voltage $\to$ current) transferring: For every $(a, \tau) \in \mathcal{R}$, we have $\mu^*(T_\mu(a, \tau)) = a$, hence

$$\eta(T_\mu(a, \tau)) = \varphi(a).$$

(Current $\to$ voltage) transferring: For every $(b, \tau) \in \mathcal{R}^*$, we have (see (3))

$$T_{\mu^*}(b, \tau) = \begin{cases} \mu^*(b) & \text{if } \tau = 1, \\ \theta'\mu^*(b) & \text{if } \tau = -1, \end{cases} \qquad (13)$$

hence

$$\varphi(T_{\mu^*}(b, \tau)) = \begin{cases} \eta(b) & \text{if } \tau = 1, \\ (\eta(b))^{-1} & \text{if } \tau = -1. \end{cases} \qquad (14)$$

Taking (A5) into account, we see that (13) and (14) determine the transferred currents and voltages of all arcs of G^* and G_φ , respectively.

The duality (12) reduces the problem of construction of an embedded voltage graph to the problem of construction of the dual current graph.

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