Resolving Edge Colorings in Graphs

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To the Memory of W. T. Tutte

ABSTRACT

For edges e and f in a connected graph G, the distance $d(e, f)$ between e and f is the minimum nonnegative integer ℓ for which there exists a sequence $e = e_0, e_1, \ldots, e_\ell = f$ of edges of G such that e_i and e_{i+1} are adjacent for $i = 0, 1, \ldots, \ell - 1$. Let c be a proper edge coloring of G using k distinct colors and let $\mathcal{D} = \{C_1, C_2, \ldots, C_k\}$ be an ordered partition of $E(G)$ into the resulting edge color classes of c. For an edge e of G, the color code $c_\mathcal{D}(e)$ of e is the k-tuple $(d(e, C_1), d(e, C_2), \ldots, d(e, C_k))$, where $d(e, C_i) = \min\{d(e, f) : f \in C_i\}$ for $1 \leq i \leq k$. If distinct edges have distinct color codes, then c is called a resolving edge coloring of G. The resolving edge chromatic number $\chi_{re}(G)$ is the minimum number of colors in a resolving edge coloring of G. Bounds for the resolving edge chromatic number of a connected graph are established in terms of its size and diameter and in terms of its size and girth. All nontrivial connected graphs of size m with resolving edge chromatic number 3 or m are characterized. It is shown that for each pair k, m of integers with $3 \leq k \leq m$, there exists a connected graph G of size m with $\chi_{re}(G) = k$. Resolving edge chromatic numbers of complete graphs are studied.

Key Words: distance, resolving decomposition, resolving edge coloring.

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1 Introduction

For edges e and f in a connected graph G, the *distance* $d(e, f)$ between e and f is the minimum nonnegative integer ℓ for which there exists a sequence $e = e_0, e_1, \ldots, e_\ell = f$ of edges of G such that e_i and e_{i+1} are adjacent for $i = 0, 1, \ldots, \ell - 1$. Thus $d(e, f) = 0$ if and only if $e = f$, $d(e, f) = 1$ if and only if e and f are adjacent, and $d(e, f) = 2$ if and only if e and f are nonadjacent edges that are adjacent to a common edge of G. Also, this distance is the standard distance between the *vertices* e and f in the line graph $L(G)$ of G. For an edge e of G and a subgraph F of G, we define the *distance* between e and F as

$$d(e, F) = d(e, E(F)) = \min\{d(e, f) : f \in E(F)\}.$$

A *decomposition* of a graph G is a collection of subgraphs of G, none of which have isolated vertices, whose edge sets provide a partition of $E(G)$. A decomposition into k subgraphs is a *k-decomposition*. A decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ is *ordered* if the ordering $(G_1, G_2, \ldots, G_k)$ has been imposed on $\mathcal{D}$. We write $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ and $\mathcal{D} = \{E(G_1), E(G_2), \ldots, E(G_k)\}$ interchangeably. For an ordered k-decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ of a connected graph G and $e \in E(G)$, the *$\mathcal{D}$-code* (or simply the *code*) of e is the k-vector

$$c_{\mathcal{D}}(e) = (d(e, G_1), d(e, G_2), \ldots, d(e, G_k)).$$

Hence exactly one coordinate of $c_{\mathcal{D}}(e)$ is 0, namely the ith coordinate if $e \in E(G_i)$. The decomposition $\mathcal{D}$ is said to be a *resolving decomposition* for G if every two distinct edges of G have distinct $\mathcal{D}$-codes. The minimum k for which G has a resolving k-decomposition is its *decomposition dimension* $\dim_d(G)$. A resolving decomposition of G with $\dim_d(G)$ elements is a *minimum resolving decomposition* for G. Thus if G is a connected graph of size at least 2, then $\dim_d(G) \geq 2$.

The topic of resolvability in graphs has previously appeared in the literature [5, 6, 7]. Slater described its usefulness when working with U.S. sonar and coast guard Loran (Long range aids to navigation) stations [6, 7]. Harary and Melter [5] discovered these concepts independently as well. Resolving decompositions in graphs were introduced and studied in [3]. Resolving concepts were studied from the point of view of graph colorings in [1, 2]. We refer to the book [4] for graph theory notation and terminology not described here.

A decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ of a connected graph G is *independent* if $E(G_i)$ is independent for each i $(1 \leq i \leq k)$ in G. This concept can be considered from an edge coloring point of view. Recall that a *proper edge coloring* (or simply, an *edge coloring*) of a nonempty graph G is an assignment c of colors (positive integers) to the edges of G so that adjacent edges are colored differently; that is, $c : E(G) \to \mathbb{N}$ is a mapping such that $c(e) \neq c(f)$ if e and f are adjacent edges of G. The minimum k for which there is an edge coloring of G using k distinct colors is called the *edge chromatic number* $\chi_e(G)$ of G. If $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ is an independent decomposition of a graph G, then by assigning color i to all edges in G_i for each i with $1 \leq i \leq k$, we obtain an edge coloring of G using k distinct colors. On the other hand, if c is an edge coloring of a connected graph G, using the colors $1, 2, \ldots, k$ for some positive integer k, then c produces a decomposition $\mathcal{D}$ of $E(G)$ into color classes (independent sets) $C_1, C_2, \ldots, C_k$, where the edges of C_i are colored i for $1 \leq i \leq k$. For an edge e in a graph G, the k-vector

$$c_{\mathcal{D}}(e) = (d(e, C_1), d(e, C_2), \ldots, d(e, C_k))$$

is called the *color code* (or simply the *code*) $c_{\mathcal{D}}(e)$ of e. If distinct edges of G have distinct color codes, then c is called a *resolving edge coloring* (or *independent resolving decomposition*) of G. Thus a resolving edge coloring of G is an edge coloring that distinguishes the edges of G in terms of their distances from the resulting color classes. A *minimum resolving edge coloring* uses a minimum number of colors and this number is the *resolving edge chromatic number* $\chi_{re}(G)$ of G. Suppose that G is a connected graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$, where $m \geq 2$. By assigning the color i to e_i for $1 \leq i \leq m$, we obtain a resolving edge coloring of G. Thus $\chi_{re}(G)$ is defined for every connected graph G. Since every resolving edge coloring is both an edge coloring and a resolving decomposition, it follows that

$$2 \leq \max\{\dim_d(G), \chi_e(G)\} \leq \chi_{re}(G) \leq m \tag{1}$$

for each connected graph G of size $m \geq 2$.

To illustrate these concepts, consider the graph G of Figure 1. Let $\mathcal{D}_1 = \{G_1, G_2, G_3\}$ be the decomposition of G, where $E(G_1) = \{v_1v_2, v_2v_5\}$, $E(G_2) = \{v_2v_3, v_2v_6, v_3v_6\}$, and $E(G_3) = \{v_3v_4, v_3v_5\}$. Since $\mathcal{D}_1$ is a minimum resolving decomposition of G, it follows that $\dim_d(G) = 3$. Define an edge coloring $c : E(G) \to \mathbb{N}$ by assigning the color 1 to v_1v_2 and v_3v_5, the color 2 to v_2v_5 and v_3v_6, the color 3 to v_2v_3, and the color 4 to v_2v_6 and v_3v_4. The coloring c is shown in Figure 1(a). Since c is a minimum edge coloring of G, it follows that $\chi_e(G) = 4$. However, c is not a resolving edge coloring. To see this, let $\mathcal{D}_2 = \{C_1, C_2, C_3, C_4\}$ be the decomposition of G into color classes resulting from c, where the edges in C_i are colored i by c. Then $c_{\mathcal{D}_2}(v_2v_5) = (1, 0, 1, 1) = c_{\mathcal{D}_2}(v_3v_6)$. On the other hand, define an edge coloring $c^* : E(G) \to \mathbb{N}$ by assigning the color 1 to v_1v_2 and v_3v_5, the color 2 to v_2v_3, the color 3 to v_2v_5 and v_3v_4, the color 4 to v_2v_6, and the color 5 to v_3v_6. The coloring c^* is shown in Figure 1(b).

Let $D^* = \{C_1^*, C_2^*, \ldots, C_5^*\}$ be the decomposition of G into color classes of c^*. Then

$$\begin{aligned} c_{\mathcal{D}^*}(v_1v_2) &= (0,1,1,1,2), & c_{\mathcal{D}^*}(v_2v_3) &= (1,0,1,1,1), \\ c_{\mathcal{D}^*}(v_2v_5) &= (1,1,0,1,2), & c_{\mathcal{D}^*}(v_2v_6) &= (1,1,1,0,1), \\ c_{\mathcal{D}^*}(v_3v_4) &= (1,1,0,2,1), & c_{\mathcal{D}^*}(v_3v_5) &= (0,1,1,2,1), \\ c_{\mathcal{D}^*}(v_3v_6) &= (1,1,1,1,0). \end{aligned}$$

Since the D^*-codes of the edges of G are distinct, it follows that c^* is a resolving edge coloring. Moreover, G has no resolving edge coloring with 4 colors and so $\chi_{re}(G) = 5$.

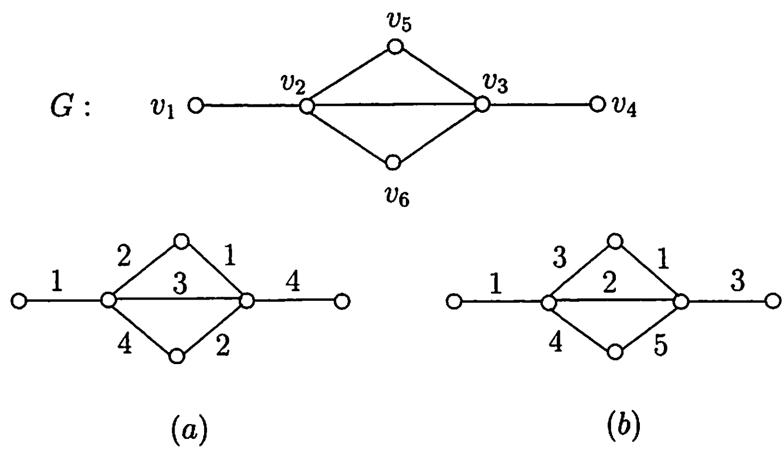

Figure 1: A graph G with $\dim_d(G) = 3$, $\chi_e(G) = 4$, and $\chi_{re}(G) = 5$

The example just presented illustrates an important point. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ be a resolving decomposition of G. If $e \in E(G_i)$ and $f \in E(G_j)$, where $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$, then $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$ since $d(e, G_i) = 0$ and $d(e, G_j) \neq 0$. Thus, when determining whether a given decomposition $\mathcal{D}$ of a graph G is a resolving decomposition for G, we need only verify that the edges of G belonging to the same subgraph in $\mathcal{D}$ have distinct $\mathcal{D}$-codes. The following two observations are useful to us, the first of which is a consequence of (1) and the fact that the edge chromatic number $\chi_e(G)$ of a graph G is bounded below by the maximum degree $\Delta(G)$ of G.

Observation 1.1 *For every graph G, $\chi_{re}(G) \geq \Delta(G)$.*

Observation 1.2 *Let G be a connected graph. Then $\dim_d(G) = \chi_{re}(G)$ if and only if G contains a minimum resolving decomposition, each of whose elements is independent in $E(G)$.*

2 Bounds for Resolving Edge Chromatic Numbers

First, we review some common terminology from graph theory. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G; the *diameter* of G is the greatest distance between any two vertices in G and is denoted by $\operatorname{diam} G$. A $u - v$ path of length $d(u, v)$ is also referred to as a $u - v$ *geodesic*. A vertex w is said to *lie between* two vertices u and v if w is an interior vertex of some $u - v$ geodesic in G. Two vertices u and v are called *antipodal vertices* of G if $d(u, v) = \operatorname{diam} G$. The *girth* of G is the length of a smallest cycle in G. In this section, we establish upper bounds for the resolving edge chromatic number of a connected graph in terms of (1) its size and diameter and (2) its size and girth. In order to do this, we need to establish some preliminary results. We first state, without proof, the resolving edge chromatic numbers of paths, stars, and cycles.

Proposition 2.1 *Let $n \geq 3$. Then*

(*a*) $\chi_{re}(P_3) = 2$ *and* $\chi_{re}(P_n) = 3$ *for* $n \geq 4$;

(*b*) $\chi_{re}(K_{1,n-1}) = n - 1$;

(*c*) $\chi_{re}(C_n) = 3$ *if n is odd and* $\chi_{re}(C_n) = 4$ *if n is even.*

Corollary 2.2 *Let G be a connected graph of size $m \geq 2$.*

(*a*) *Then* $\chi_{re}(G) = 2$ *if and only if* $G = P_3$.

(*b*) *If* $m \geq 3$, *then* $3 \leq \chi_{re}(G) \leq m$.

Lemma 2.3 *Let G be a connected graph of order n such that $G \neq K_n, P_n, C_n$. Then G contains three distinct vertices u, v, and w such that u and v are antipodal vertices of G and w lies between u and v and* $\deg w \geq 3$.

Proof. Assume, to the contrary, that there exists a connected graph G of order n such that $G \neq K_n, P_n, C_n$ and no vertex of degree 3 or more lies between any two antipodal vertices of G. Since $G \neq K_n$, it follows that $\operatorname{diam} G = d \geq 2$.

Let u and v be two antipodal vertices of G and let P be a $u - v$ geodesic in G. Then every interior vertex of P has degree 2 in G. Since $G \neq P_n$, there exists $w \in V(G) - V(P)$ such that w is adjacent to u or v, say w is adjacent to u. Then $d - 1 \leq d(w, v) \leq d$. Furthermore, every $w - v$ geodesic in G and P are internally disjoint. Let Q be a $w - v$ geodesic in G. Then the edge uw and the paths P and Q produce a cycle C of length k in G, say

$$C : u_0 = u, u_1, u_2, \ldots, u_d = v, u_{d+1}, \ldots, u_{k-1} = w, u_k = u,$$

where $k = 2d + 1$ or $k = 2d$. Next, we show that $d(w, u_{d-1}) = d$. Assume, to the contrary, that $d(w, u_{d-1}) \leq d - 1$. Then there exists a $w - u_{d-1}$ geodesic L in G. We consider two cases.

Case 1. $k = 2d + 1$. Then $d(w, v) = d$ and C is an odd cycle and every vertex of $V(C) - \{u, v, w\}$ has degree 2 in G. Since $G \neq C_n$, there exists $x \in V(G) - V(C)$ such that x is adjacent to u, v or w, say x is adjacent to u. Thus $\deg u \geq 3$. Then the graph G contains the graph of Figure 2 as a subgraph. If $d(w, u_{d-1}) \leq d - 2$, then $d = d(w, v) \leq d(w, u_{d-1}) + d(u_{d-1}, v) \leq (d - 2) + 1 = d - 1$, which is impossible. Thus $d(w, u_{d-1}) = d - 1$. Note that $V(L) - \{w, u_{d-1}\}$ contains no interior vertex P and Q. Thus, the vertex immediately preceding u_{d-1} on L must be v. However, then, $d(w, v) \leq d - 2$, which is a contradiction.

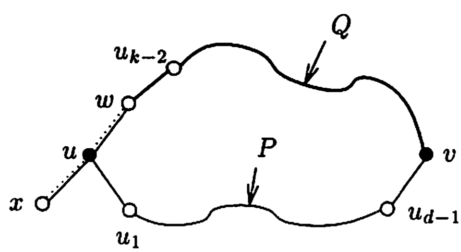

Figure 2: A subgraph of G

Case 2. $k = 2d$. Then $d(w, v) = d - 1$ and C is an even cycle. Then the $u - v$ path P' obtained from Q by joining u to w is also a $u - v$ geodesic. Thus every interior vertex in P' has degree 2 in G. In particular, $\deg w = 2$ in G. Since $G \neq C_n$, there exists $x \in V(G) - V(C)$ such that x is adjacent to u or v, say x is adjacent to u. Again, G contains the graph of Figure 2 as a subgraph. If $d(w, u_{d-1}) \leq d - 3$, then $d - 1 = d(w, v) \leq d(w, u_{d-1}) + d(u_{d-1}, v) \leq (d - 3) + 1 = d - 2$, which is impossible. Thus $d(w, u_{d-1}) = d - 2$ or $d(w, u_{d-1}) = d - 1$. Let $L : w = w_0, w_1, \ldots, w_\ell = u_{d-1}$, where $\ell \in \{d - 1, d - 2\}$. Since $\deg w = 2$, it follows that $w_1 = u$ or $w_1 = u_{k-2}$. Assume first that $w_1 = u$. Since $d(u, u_{d-1}) = d - 1$, it follows that $d(w, u_{d-1}) = d$, as desired. Thus we may assume that $w_1 = u_{k-2} \in V(Q)$. Let w_i $(1 \leq i \leq \ell - 1)$ be the last vertex on L that belongs to Q. Since w_i is adjacent to some vertex not on Q, it follows that $w_i = v$. However, then, $d - 1 = d(w, v) = i \leq \ell - 1 \leq d - 2$, which is a contradiction.

Therefore, $d(w, u_{d-1}) = d$ and so w and u_{d-1} are antipodal vertices of G. Since u lies between w and u_{d-1} and $\deg u \geq 3$, a contradiction is produced. ■

We are now prepared to present an upper bound for the resolving edge chromatic number of a connected graph in terms of its size and diameter.

Theorem 2.4 *If G is a connected graph of size $m \geq 3$ and diameter d, then*

$$\chi_{re}(G) \leq m - d + 3.$$

Moreover, $\chi_{re}(G) = m - d + 3$ if and only if G is a path of size $m \geq 3$.

Proof. If $d \leq 3$, then $\chi_{re}(G) \leq m \leq m - d + 3$ by (1). Thus, we may assume that $d \geq 4$. Let u and v be vertices of G for which $d(u, v) = d$, and let $P_{d+1} : u = v_1, v_2, \ldots, v_{d+1} = v$ be a $u - v$ path of length d in G. Let $e_i = v_i v_{i+1}$ for $1 \leq i \leq d$. Let $E(G) - E(P_{d+1}) = \{f_1, f_2, \ldots, f_{m-d}\}$. Assign the color 1 to e_1, the color 2 to e_i if i is even, the color 3 to e_i if i is odd, and the color $j + 3$ to f_j for $1 \leq j \leq m - d$. Since this is a resolving edge coloring of G with $m - d + 3$ colors, it follows that $\chi_{re}(G) \leq m - d + 3$.

If $G = P_n$, where $n \geq 4$, then $m = d = n - 1$ and $\chi_{re}(P_n) = 3$ by Proposition 2.1. Therefore, $\chi_{re}(G) = m - d + 3$ if G is a path of size $m \geq 3$. It remains to verify the converse. Assume, to the contrary, that there is a connected graph G of size $m \geq 3$ and diameter d such that $\chi_{re}(G) = m - d + 3$, but G is not a path. By Proposition 2.1, we may assume that G is not a cycle. If $d \leq 2$, then $m - d + 3 \geq m + 1$. Since $\chi_{re}(G) \leq m$ by (1), a contradiction is produced. So assume that $d \geq 3$. By Lemma 2.3, G contains three distinct vertices u, v, and w such that $d(u, v) = d$ and w lies between u and v and $\deg w \geq 3$. Let $P : u = v_1, v_2, \ldots, v_{d+1} = v$ be a $u - v$ geodesic in G, let $w = v_j$, where $j \in \{2, 3, \ldots, d\}$, and let $e = v_j x$, where $x \notin V(P)$. Furthermore, let

$$Q_1 : x, v_j, v_{j-1}, \ldots, v_1 \quad \text{and} \quad Q_2 : x, v_j, v_{j+1}, \ldots, v_{d+1}$$

be the paths obtained, respectively, from the $v_1 - v_j$ subpath of P by joining x to v_j and from the $v_j - v_{d+1}$ subpath of P by joining x to v_j. We consider two cases.

Case 1. *There exists no induced cycle C in G such that C contains the edge e and an $x - y$ subpath of Q_i for any $y \in V(Q_i) - \{x, v_j\}$ and for any $i \in \{1, 2\}$.* Define an edge coloring c of G by assigning the color 1 to $v_i v_{i+1}$ if i is odd and $1 \leq i \leq d$, the color 2 to $v_i v_{i+1}$ if i is even and $1 \leq i \leq d$, the color 3 to e, and distinct colors from $\{4, 5, \ldots, m - d + 2\}$ to the remaining $m - d - 1$ edges in $E(G) - (E(P) \cup \{e\})$. Let $\mathcal{D} = \{C_1, C_2, \ldots, C_{m-d+2}\}$ be the decomposition of G resulting from c. Observe that

$$d(v_{j+i}v_{j+i+1}, C_3) = i + 1 \quad \text{for } 0 \le i \le d - j,$$
$$d(v_{j-i}v_{j-i+1}, C_3) = i \quad \text{for } 1 \le i \le j - 1.$$

Thus the codes $c_{\mathcal{D}}(v_iv_{i+1})$, $1 \le i \le d$, are distinct. Therefore, c is a resolving edge coloring of G using $m - d + 2$ colors. Therefore, $\chi_{re}(G) \le m - d + 2$, which is a contradiction.

Case 2. There exists an induced cycle C in G such that C contains the edge e and an $x-y$ subpath of Q_i for some $y \in V(Q_i)-\{x, v_j\}$ and for some $i \in \{1, 2\}$. Without loss of generality, assume that C is an induced cycle in G such that C contains an $x-y$ subpath of Q_1 for some $y \in V(Q_1)-\{x, v_j\}$. Let f be the edge of C that is adjacent to e but not on P. We consider two subcases.

Subcase 2.1. The edge f is adjacent to an edge of P. Since P is a shortest $u - v$ path, it follows that either (1) $f = v_{j-1}x$ or $f = v_{j-2}x$ or (2) $f = v_{j+1}x$ or $f = v_{j+2}x$. We may assume that (1) occurs. Then C is either a 3-cycle or a 4-cycle. In this subcase, the edge coloring c of G described in Case 1 is also a resolving edge coloring of G with $m - d + 2$ colors. Therefore, $\chi_{re}(G) \le m - d + 2$, which is a contradiction.

Subcase 2.2. The edge f is not adjacent to any edge of P. Define an edge coloring c of G by assigning the color 1 to v_1v_2, the color 2 to f and v_iv_{i+1} if i is even and $1 \le i \le d$, the color 3 to v_iv_{i+1} if i is odd and $1 \le i \le d$, and distinct colors from $\{4, 5, \ldots, m - d + 2\}$ to the remaining $m - d - 1$ edges in $E(G) - (E(P) \cup \{f\})$. Since f is not adjacent to any edge of P, it follows that c is an edge coloring of G. Let $\mathcal{D} = \{C_1, C_2, \ldots, C_{m-d+2}\}$ be the decomposition of G resulting from c. Observe that

$$d(v_iv_{i+1},\ C_1) = i - 1 \quad \text{for } 2 \le i \le d,$$
$$d(v_iv_{i+1},\ C_3) = 1 \quad \text{for } 2 \le i \le d,$$
$$d(f,\ C_3) = 2.$$

It follows that c is a resolving edge coloring of G using $m - d + 2$ colors. Thus $\chi_{re}(G) \le m - d + 2$, producing a contradiction.

Therefore, if G is a connected graph of size $m \ge 3$ and diameter d with resolving edge chromatic number $m - d + 3$, then G is a path. ■

Next, we establish an upper bound for the resolving edge chromatic number of a connected graph in terms of its size and girth.

Theorem 2.5 *If G is a connected graph of size m and girth ℓ, where $m \ge \ell \ge 3$, then*

$$\chi_{re}(G) \le \begin{cases} m - \ell + 3 & \text{if } \ell \text{ is odd,} \\ m - \ell + 4 & \text{if } \ell \text{ is even.} \end{cases}$$

Moreover, if ℓ is even and G is not an even cycle, then $\chi_{re}(G) \leq m - \ell + 2$.

Proof. If G is a cycle, then $m = \ell$. By Proposition 2.1, if G is an odd cycle, then $\chi_{re}(G) = 3 = m - \ell + 3$; while if G is an even cycle, then $\chi_{re}(G) = 4 = m - \ell + 4$. Thus we may assume that G is not a cycle and so $m > \ell$. Let $C_\ell : v_1, v_2, \ldots, v_\ell, v_1$ be a cycle of order $\ell \geq 3$ in G, where $e_i = v_i v_{i+1}$ for $1 \leq i \leq \ell - 1$ and $e_\ell = v_\ell v_1$. Let $E(G) - E(C_\ell) = \{f_1, f_2, \ldots, f_{m-\ell}\}$. We consider two cases.

Case 1. $\ell = 2k+1$, where $k \geq 1$. Define an edge coloring $c' : E(G) \to \mathbb{N}$ by $c'(e_1) = 1$, $c'(e_i) = 2$ if $2 \leq i \leq \ell$ and i is even, $c'(e_i) = 3$ if $3 \leq i \leq \ell$ and i is odd, and $c'(f_i) = i + 3$ for each i with $1 \leq i \leq m - \ell$. Since c' is a resolving edge coloring with $m - \ell + 3$ colors, $\chi_{re}(G) \leq m - \ell + 3$, as desired.

Case 2. $\ell = 2k$, where $k \geq 2$. Define an edge coloring $c : E(G) \to \mathbb{N}$ by $c(e_i) = 1$ if $1 \leq i \leq \ell$ and i is odd, $c(e_i) = 2$ if $2 \leq i \leq \ell$ and i is even, and $c(f_i) = i + 2$ for each i with $1 \leq i \leq m - \ell$. Since c is a resolving edge coloring with $m - \ell + 2$ colors, $\chi_{re}(G) \leq m - \ell + 2$, as desired. ∎

It can be verified that the upper bounds in Theorem 2.5 are sharp.

3 Graphs with Prescribed Resolving Edge Chromatic Number

We have seen that if G is a connected graph of size $m \geq 3$, then $3 \leq \chi_{re}(G) \leq m$. In this section, we first determine all connected graphs whose resolving edge chromatic number is one of these extremes. We begin by determining all connected graphs G with $\chi_{re}(G) = 3$. In order to do this, we first present some preliminary results.

Lemma 3.1 *If T is a tree with $\Delta(T) = 3$ and having exactly one vertex of degree 3, then $\chi_{re}(T) = 3$.*

Proof. Since $\Delta(T) = 3$, it follows by Observation 1.1 that $\chi_{re}(T) \geq 3$. Thus, it remains to show that $\chi_{re}(T) \leq 3$. Suppose that x is the only vertex of degree 3 in T. Then we may assume that T is the graph obtained from the paths

$$P_{k_1} : u_1, u_2, \ldots, u_{k_1}, \quad P_{k_2} : v_1, v_2, \ldots, v_{k_2}, \quad P_{k_3} : w_1, w_2, \ldots, w_{k_3}$$

by adding the vertex x and three edges xu_1, xv_1, xw_1, where k_1, k_2, and k_3 are positive integers. Define an edge coloring c of T by assigning (1) the color 1 to the edges xu_1, $u_i u_{i+1}$ for even i with $2 \leq i \leq k_1 - 1$, and $v_j v_{j+1}$ for odd j with $1 \leq j \leq k_2 - 1$, (2) the color 2 to the edges xv_1, $u_i u_{i+1}$ for odd i with $1 \le i \le k_1 - 1$, v_jv_{j+1} for even j with $2 \le j \le k_2 - 1$, and $w_\ell w_{\ell+1}$ for odd ℓ with $1 \le \ell \le k_3 - 1$, and (3) the color 3 to the edges xw_1, $w_\ell w_{\ell+1}$ for even ℓ with $2 \le \ell \le k_3 - 1$. Since c is a resolving edge coloring of T using three colors, $\chi_{re}(T) \le 3$. Therefore, $\chi_{re}(T) = 3$. ■

A connected graph containing exactly one cycle is a *unicyclic* graph. For brevity, we omit the proof of the following lemma.

Lemma 3.2 *Let G be a unicyclic graph with $\Delta(G) = 3$ and exactly one vertex of degree 3.*

(a) *If G contains an even cycle, then $\chi_{re}(G) = 3$;*

(b) *If G contains an odd cycle, then $\chi_{re}(G) = 4$.*

Let $\mathcal{T}$ be set of all trees T with $\Delta(T) = 3$ having exactly one vertex of degree 3 and let $\mathcal{U}$ be set of all unicyclic graphs G with $\Delta(G) = 3$ whose cycle is even and containing exactly one vertex of degree 3. The following corollary is a consequence of Lemmas 3.1 and 3.2.

Corollary 3.3 *Let G be a connected graph with $\Delta(G) = 3$ such that G contains exactly one vertex of degree 3. Then $\chi_{re}(G) = 3$ if and only if $G \in \mathcal{T} \cup \mathcal{U}$.*

We are now prepared to present a characterization of connected graphs G of order $n \ge 4$ with $\chi_{re}(G) = 3$.

Theorem 3.4 *Let G be a connected graph of order $n \ge 4$. Then $\chi_{re}(G) = 3$ if and only if*

(a) *$G = P_n$,*

(b) *$G = C_n$, where n is odd, or*

(c) *$G \in \mathcal{T} \cup \mathcal{U}$.*

Proof. By Proposition 2.1 and Corollary 3.3, each of the graphs described in (a), (b), and (c) has resolving edge chromatic number 3. For the converse, assume that G is a connected graph of order $n \ge 4$ with $\chi_{re}(G) = 3$. Let $\mathcal{D} = \{C_1, C_2, C_3\}$ be a decomposition of G resulting from a minimum resolving edge coloring c of G. Since $\chi_{re}(G) = 3$, it follows by Observation 1.1 that $\Delta(G) = 2$ or $\Delta(G) = 3$. If $\Delta(G) = 2$, then $G = P_n$, where $n \ge 4$, or $G = C_n$, where $n \ge 4$ is odd, and so (a) or (b) holds by Proposition 2.1. Thus we may assume that $\Delta(G) = 3$. In fact, G contains exactly one vertex of degree 3. To see this, suppose that G contains two distinct vertices u and v of degree 3. Then both u and v are incident with edges of all three colors. Thus, in order that all edges incident with u and v have distinct codes, all three edges incident with u must also be incident with v, producing a contradiction. Therefore, G contains exactly one vertex of degree 3. It then follows by Corollary 3.3 that $G \in \mathcal{T} \cup \mathcal{U}$ and so (c) holds. ■

Next, we establish a characterization of connected graphs of size m having resolving edge chromatic number m.

Theorem 3.5 *Let G be a connected graph of order $n \geq 3$ and size m.*

(a) *If $n = 3$ or $n = 4$, then $\chi_{re}(G) = m$.*

(b) *If $n \geq 5$, then $\chi_{re}(G) = m$ if and only if $G = K_{1,n-1}$.*

Proof. First, we verify (a). If $n = 3$, then $G \in \{P_3, K_3\}$. If $n = 4$, then

$$G \in \{P_4,\ K_{1,3},\ \overline{P_3 \cup K_1},\ C_4,\ K_4 - e,\ K_4\}.$$

It is straightforward to verify that each graph involved in this case has resolving edge chromatic number equal to its size and so (a) holds.

Next we verify (b). By Proposition 2.1, $\chi_{re}(K_{1,n-1}) = n - 1$. Thus it remains to show that if G is a connected graph of order $n \geq 5$ and size m that is not a star, then $\chi_{re}(G) \leq m - 1$. Since $G \neq K_{1,n-1}$, where $n \geq 5$, it follows that G contains a path $P : v_1, v_2, v_3, v_4$ of order 4. Let $e_i = v_i v_{i+1}$ for $i = 1, 2, 3$. Since $n \geq 5$ and G is connected, there exists $v \in V(G) - V(P)$ such that v is adjacent to at least one vertex of P. Consequently, some edge of $E(G) - E(P)$ is adjacent to exactly one of the two edges $v_1 v_2$ and $v_3 v_4$. By assigning the same color to $v_1 v_2$ and $v_3 v_4$ and distinct colors to each of the remaining edges in G, we produce a resolving edge coloring of G using $m - 1$ colors, as desired. Therefore, the star $K_{1,n-1}$ is the only connected graph of order $n \geq 5$ and size m with resolving edge chromatic number m and so (b) holds. ■

Combining (a) and (b) in Theorem 3.5, we have the following.

Corollary 3.6 *Let G be a connected graph of order $n \geq 3$ and size $m \geq 2$. Then $\chi_{re}(G) = m$ if and only if G is one the graphs in Figure 3.*

By Corollary 2.2, if G is a connected graph of size $m \geq 3$, then $3 \leq \chi_{re}(G) \leq m$. Indeed, the proof of the following result is straightforward.

Theorem 3.7 *For each pair k, m of integers with $3 \leq k \leq m$, there exists a connected graph G of size m with $\chi_{re}(G) = k$.*

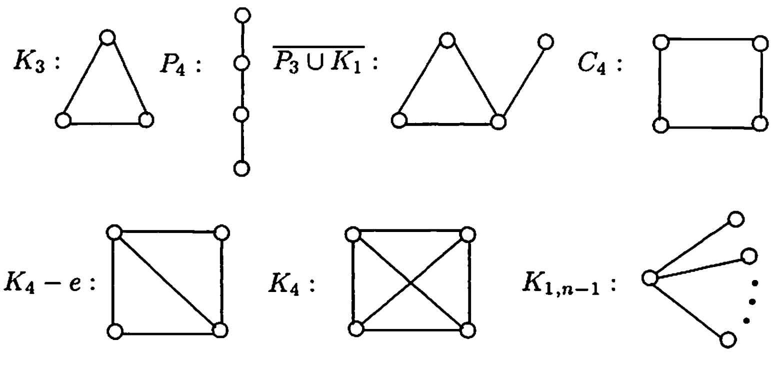

Figure 3: The graphs described in Corollary 3.6

4 On Resolving Edge Chromatic Numbers of Complete Graphs

In this section, we provide bounds for the resolving edge chromatic number of a complete graph. We have seen that $\chi_e(K_3) = \chi_{re}(K_3) = 3$ and $\chi_{re}(K_n) \geq \chi_e(K_n)$ for $n \geq 4$. In fact, $\chi_{re}(K_n) \geq \chi_e(K_n) + 1$ for $n \geq 4$, as we show next. It is known that $\chi_e(K_n) = n - 1$ if n is even and $\chi_e(K_n) = n$ if n is odd. Moreover, the edge independence number $\beta_1(K_n) = \lfloor n/2 \rfloor$ for each $n \geq 3$.

Proposition 4.1 *For every integer $n \geq 4$,*

$$\chi_{re}(K_n) \geq \chi_e(K_n) + 1 = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Assume, to the contrary, that $\chi_{re}(K_n) = \chi_e(K_n)$. Let c be a minimum resolving edge coloring of K_n and let $\mathcal{D} = \{G_1, G_2, \ldots, G_{\chi_e(K_n)}\}$ be the decomposition of K_n resulting from c. We consider two cases.

Case 1. *n is even.* Then $\chi_e(K_n) = n - 1$. Assume, to the contrary, that there is a resolving edge coloring of K_n with $n - 1$ colors. Then every color is present at every vertex of K_n. Consequently, every pair of edges that share the same color have the same color code, which is a contradiction.

Case 2. *n is odd.* Then $\chi_e(K_n) = n$. Since the size of K_n is $(n-1)n/2$ and $\beta_1(K_n) = (n-1)/2$, it follows that $|E(G_i)| = (n-1)/2 \geq 2$ for $1 \leq i \leq n$. Let $e, f \in E(G_1)$. If each of e and f is adjacent to at least one edge in G_i for all i with $2 \leq i \leq n$, then $c_{\mathcal{D}}(e) = c_{\mathcal{D}}(f) = (0, 1, 1, \ldots, 1)$, a contradiction. Thus we may assume that at least one of e and f is not adjacent to any edges in G_i for some i with $2 \leq i \leq n$, say e is not adjacent to any edges in G_2. Let $e = uv$. Then $G_2 \subseteq K_n - \{u, v\} = K_{n-2}$. Since $\beta_1(K_{n-2}) = (n-3)/2$, it follows that $|E(G_2)| \leq (n-3)/2$, which is a contradiction. ■

Since $\chi_{re}(K_4) = 6$ by Lemma 3.5, strict inequality in Proposition 4.1 holds for $n = 4$. On the other hand, we have equality in Proposition 4.1 for $n = 5$. Let $V(K_5) = \{v_1, v_2, \ldots, v_5\}$ and define an edge coloring c of K_5 by assigning the color 1 to v_1v_2 and v_3v_5, the color 2 to v_1v_3 and v_2v_4, the color 3 to v_1v_4, the color 4 to v_1v_5 and v_2v_3, the color 5 to v_2v_5 and v_3v_4, and the color 6 to v_4v_5. Let $\mathcal{D} = \{C_1, C_2, \ldots, C_6\}$ be the decomposition of K_5 resulting from c. Then $d(v_1v_2, C_6) = 2$, $d(v_3v_5, C_6) = 1$, $d(v_1v_3, C_6) = 2$, $d(v_2v_4, C_6) = 1$, $d(v_1v_5, C_6) = 1$, $d(v_2v_3, C_6) = 2$, $d(v_2v_5, C_3) = 2$, and $d(v_3v_4, C_3) = 1$. Thus the color codes $c_{\mathcal{D}}(e)$, $e \in E(K_5)$, are distinct and so c is a resolving edge coloring of K_5 using 6 colors. Thus $\chi_{re}(K_5) \leq 6$. It follows by Proposition 4.1 that $\chi_{re}(K_5) = 6$. Therefore, $\chi_{re}(K_5) = \chi_e(K_5) + 1$ and so we have equality in Proposition 4.1 for $n = 5$.

Next, we present an upper bound for $\chi_{re}(K_n)$ in terms of n for $n \geq 3$. In order to this, we need some additional definitions. Let $\mathcal{D}$ be a decomposition of a connected graph G. Then a decomposition $\mathcal{D}^*$ of G is called a *refinement* of $\mathcal{D}$ if every element in $\mathcal{D}^*$ is a subgraph of some element of $\mathcal{D}$. We state a fact about the refinements of a resolving decomposition of a graph without a proof.

Lemma 4.2 *Let $\mathcal{D}$ be a resolving decomposition of a connected graph G. If $\mathcal{D}^*$ is a refinement of $\mathcal{D}$, then $\mathcal{D}^*$ is also a resolving decomposition of G.*

Theorem 4.3 *For every integer $n \geq 3$,*

$$\chi_{re}(K_n) \leq \lceil (5n-3)/3 \rceil = \begin{cases} 2n/3 + (n-1) & \text{if } n \equiv 0 \pmod 3 \\ (2n+1)/3 + (n-1) & \text{if } n \equiv 1 \pmod 3 \\ (2n+2)/3 + (n-1) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. Since $\chi_{re}(K_3) = 3$, $\chi_{re}(K_4) = 6$, and $\chi_{re}(K_5) = 6$, the result is true for $3 \leq n \leq 5$. For $n \geq 6$, let $V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\}$. We define a decomposition $\mathcal{D}$ of K_n based on whether of n is congruent to 0, 1, or 2 modulo 3. Thus, there are three cases.

Case 1. $n \equiv 0 \pmod 3$. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_{(2n/3)+1}\}$, where

$$E(G_{2i+1}) = \{v_{3i}v_{3i+1}\} \text{ and } E(G_{2i+2}) = \{v_{3i+1}v_{3i+2}\}$$

for $0 \leq i \leq n/3 - 1$, and let $G_{(2n/3)+1}$ consist of the remaining edges of K_n that do not belong to any other elements in $\mathcal{D}$. Observe that $\{G_1, G_2, \cdots, G_{2n/3}\}$ is actually a K_2-decomposition of the factor $(n/3)P_3$ of K_n.

We show that $\mathcal{D}$ is a resolving decomposition of K_n. Observe that every edge of $G_{(2n/3)+1}$ either joins (a) a vertex that belongs to G_i only and a vertex that belongs to G_j only, where $1 \leq i \neq j \leq 2n/3$, (b) a vertex that belongs to G_i only and a vertex that belongs to both G_j and G_{j+1} for some i and j with $\{i\} \cap \{j, j+1\} = \emptyset$ or (c) a vertex that belongs to both G_i and G_{i+1} and a vertex that belongs to both G_j and G_{j+1} for some i and j with $\{i, i+1\} \cap \{j, j+1\} = \emptyset$. Furthermore, each edge of $G_{(2n/3)+1}$ satisfies exactly one of (a), (b), and (c). Thus $\mathcal{D}$ is a resolving decomposition of K_n. Define an independent refinement $\mathcal{D}^*$ of $\mathcal{D}$ by (1) retaining each subgraph G_i in $\mathcal{D}$ for $1 \leq i \leq 2n/3$ and (2) decomposing $G_{(2n/3)+1}$ into $n-1$ independent subgraphs. Notice that (2) is possible since $\Delta(G_{(2n/3)+1}) = n-2$ and so $\chi_e(G_{(2n/3)+1}) \leq n-1$. Thus

$$\mathcal{D}^* = \{G_1, G_2, \ldots, G_{2n/3}, H_1, H_2, \cdots, H_{n-1}\},$$

where $\{H_1, H_2, \cdots, H_{n-1}\}$ is an independent decomposition $G_{(2n/3)+1}$. Since D^* is a refinement of the resolving decomposition D of K_n, it then follows by Lemma 4.2 that $\mathcal{D}^*$ is a resolving independent decomposition of K_n and so $\chi_{re}(K_n) \leq |\mathcal{D}^*| = 2n/3 + (n-1)$.

Case 2. $n \equiv 1 \pmod 3$. We proceed as in Case 1 with $\mathcal{D} = \{G_1, G_2, \cdots, G_{(2n+1)/3}, G_{(2n+4)/3}\}$, where $\{G_1, G_2, \ldots, G_{(2n+1)/3}\}$ is a K_2-decomposition of the factor $[(n-4)/3]P_3 \cup K_{1,3}$. An argument similar to that of Case 1 shows that $\mathcal{D}$ is a resolving decomposition. Define an independent refinement $\mathcal{D}^*$ of $\mathcal{D}$ by (1) retaining each subgraph G_i in $\mathcal{D}$ for $1 \leq i \leq (2n+1)/3$ and (2) decomposing $G_{(2n+4)/3}$ into $n-1$ independent subgraphs. Then $\mathcal{D}^*$ is a resolving independent decomposition of K_n and so $\chi_{re}(K_n) \leq |\mathcal{D}^*| = (2n+1)/3 + (n-1)$.

Case 3. $n \equiv 2 \pmod 3$. Again we proceed as in the previous two cases, where $\mathcal{D} = \{G_1, G_2, \ldots, G_{(2n+2)/3}, G_{(2n+5)/3}\}$ such that $\{G_1, G_2, \ldots, G_{(2n+2)/3}\}$ is a K_2-decomposition of the factor $[(n-5)/3]P_3 \cup K_{1,4}$. Again, an argument similar to that used in Case 1 shows $\mathcal{D}$ is a resolving decomposition. Then we define an independent refinement $\mathcal{D}^*$ of $\mathcal{D}$ by (1) retaining each subgraph G_i in $\mathcal{D}$ for $1 \leq i \leq (2n+2)/3$ and (2) decomposing $G_{(2n+5)/3}$ into $n-1$ independent subgraphs. Thus $\chi_{re}(K_n) \leq |\mathcal{D}^*| = (2n+2)/3 + (n-1)$. ∎

If $n = 4$, then $\lceil (5n-3)/3 \rceil = 6$ and $\chi_{re}(K_4) = 6$ as well. Thus the upper bound in Theorem 4.3 is attained when $n = 4$. Since $\chi_{re}(K_5) = 6$ and $\lceil (5n-3)/3 \rceil = 8$ for $n = 5$, we have strict inequality in Theorem 4.3 when $n = 5$.

There are several open questions here:

(1) Just how good are the bounds for $\chi_{re}(K_n)$ given in Proposition 4.1 and Theorem 4.3?

(2) Is there, in fact, a formula for $\chi_{re}(K_n)$?

(3) One might think that $\chi_{re}(K_n) \leq \chi_{re}(K_{n+1})$ for all $n \geq 3$. Is this the case?

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